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SOME PROPERTIES OF TANGENCY RELATIONS

The relation $T_l(a,b,k,p)$ of the tangency of the sets in a metric space (E,ρ) is dependent on a real non-negative functions a and b. The paper gives a certain conditions towards to the real non-negative functions a and b if relation of the tangency of sets $T_l(a,b,k,p)$ is non-empty.

Let (E,ρ) be an arbitrary metric space.For any set $A \subset E$, we denote r-neighbourhood

$$A_r = \bigcup_{p\in A} K(p,r) \quad \text{for } r > 0, \tag{1.1}$$

or

$$A_r = A \quad \text{for } r = 0. \tag{1.2}$$

The $K(p,r)$ $(S(p.r))$ denotes the open ball (sphere) with the center at the point p and the radius r.

Let a and b be any non-negative real functions defined in a certain right-hand side neighbourhood of 0 such that

$$a(r) \to 0 \quad \text{and} \quad b(r) \to 0 \text{ for } r \to 0^+. \tag{2}$$

A pair (A,B) of subsets of E will be called (a,b)-clustered at the p if 0 is a cluster point of the set Q of all real numbers $r > 0$ such that the sets

$$A \cap S(p,r)_{a(r)} \quad \text{and} \quad B \cap S(p,r)_{b(r)}$$

are non-empty.

Here $S(p,r)_{a(r)}$ is an $a(r)$-neighbourhood of the set $S(p,r)$, (1.1) and (1.2).

Let l be a real non-negative function defined on the Cartesian product $E_0 \times E_0$, where E_0 is a family of all non empty subsets of the set E, satisfying the condition

$$l(\{x\},\ \{y\}) = \rho(x,y) \quad \text{for } x,y \in E. \tag{3}$$

The set A is (a,b)-tangent of order k at the point $p \in E$ to the set B if the pair (A,B) is (a,b)-clustered at the point p of the space (E,ρ) and

$$\frac{1}{r^k} l(A \cap S(p,r)_{a(r)}, B \cap S(p,r)_{b(r)}) \to 0 \quad \text{for } r \to 0^+, \tag{4}$$

where k is an arbitrary positive real number.

Let

$$T_l(a,b,k,p) = \Big\{(A,B) : A \cup B \subset E \text{ and } (A,B) \text{ is } (a,b)\text{-clustered at the point } p \text{ of the space } (E,\rho) \text{ and } \frac{1}{r^k} l(A \cap S(p,r)_{a(r)}, B \cap S(p,r)_{b(r)}) \xrightarrow[r \to 0^+]{} 0\Big\}. \tag{5}$$

The relation (5) will be called the relation of the (a,b)-tangency of order k at the point p (or shortly relation of the tangency) of the sets in the metric space (E,ρ).

The metric ρ induces some functions l_i $(i = 1,2,\dots,6)$ defined by

$$l_1(A,B) = \inf\{\inf\{\rho(x,y) : y \in B\} : x \in A\}, \tag{6.1}$$

$$l_2(A,B) = \sup\{\inf\{\rho(x,y) : y \in B\} : x \in A\}, \tag{6.2}$$

$$l_3(A,B) = \inf\{\sup\{\rho(x,y) : y \in B\} : x \in A\}, \tag{6.3}$$

$$l_4(A,B) = \sup\{\sup\{\rho(x,y) : y \in B\} : x \in A\}, \tag{6.4}$$

$$l_5(A,B) = \max\{l_2(A,B), l_2(B,A)\}, \tag{6.5}$$

$$l_6(A,B) = \max\{l_3(A,B), l_3(B,A)\}, \tag{6.6}$$

for $A, B \in E_0$.

Let (E,ρ) be a metric space. We denote

$$Q_p(A) = \{r > 0 : A \cap S(p,r) \neq 0\}, \tag{7}$$

where $\emptyset \neq A \subset E$ and $p \in E$.

LEMMA 1. *If the set $Q_p(A)$ is dense in the $(0,s)$, for some number $s > 0$, then*

$$\sup\{\rho(p,x) : x \in A \cap S(p,r)_t\} = r + t$$

and

$$\inf\{\rho(p,x) : x \in A \cap S(p,r)_t\} = \max\{0, r - t\},$$

for $r + t < s$.

Proof. Let $x \in A \cap S(p,r)_t$. Then there exists $q \in S(p,r)$ such that $x \in K(q,t)$. From the triangle inequality we obtain

$$r - t < \rho(p,x) < r + t.$$

Consequently

$$\sup\{\rho(p,x) : x \in A \cap S(p,r)_t\} \leq r + t$$

and

$$\inf\{\rho(p,x) : x \in A \cap S(p,r)_t\} \geq \max\{0, r-t\}$$

The set $Q_p(A)$ is dense in the $(0,s)$ and

$$(\max\{0, r-t\}, r+t) \subset (0,s).$$

Hence

$$\sup\{\rho(p,x) : x \in A \cap S(p,r)_t\} = r+t$$

and

$$\inf\{\rho(p,x) : x \in A \cap S(p,r)_t\} = \max\{0, r-t\}.$$

We suppose that

$$\sup\{\rho(p,x) : x \in A \cap S(p,r)_t\} = u < t+r.$$

Then for an arbitrary $x \in A \cap S(p,r)_t$ we obtain $\rho(p,x) \leq u < r+t$, hence it follows that there exists a number $v \in (u, r+t)$ and certain a neighbourhood U such that $Q_p(A) \cap U = \emptyset$. That is contradictory. This ends the proof.

THEOREM 1. *If sets $Q_p(A)$, $Q_p(B)$ are dense in the $(0,s)$ for some number s and non-negative real functions a and b defined in the $(0,s)$ satisfying the conditions* (2) *and*

$$(*) \qquad \lim_{r\to 0^+} \sup \frac{r-a(r)}{r^k} = 0 \quad \textit{and} \quad \lim_{r\to 0^+} \sup \frac{r-b(r)}{r^k} = 0,$$

the pair (A,B) is (a,b)-clustered at the point p, then $(A,B) \in T_{l_1}(a,b,k,p)$ and $(A,B) \notin T_{l_4}(a,b,k,p)$, for arbitrary number $k > 0$.

Proof. Let $\varepsilon > 0$. From $(*)$ and from 2 it follows that there exists a real number $\delta > 0$ such that

$$r - a(r) < \frac{1}{2}\varepsilon r^k \quad \text{and} \quad r - b(r) < \frac{1}{2}\varepsilon r^k$$

for $r \in (0, \min\{\delta, s\})$. There exists points $x \in A \cap S(p,r)_{a(r)}$ and $y \in B \cap S(p,r)_{b(r)}$ such that

$$r - a(r) < \rho(p,x) < \frac{1}{2}\varepsilon r^k \quad \text{and} \quad r - b(r) < \rho(p,y) < \frac{1}{2}\varepsilon r^k.$$

Hence and from the triangle inequality we have

$$\rho(x,y) < \rho(p,x) + \rho(p,y) < \varepsilon r^k.$$

Therefore

$$\inf\{\inf\{\rho(x,y) : y \in B \cap S(p,r)_{b(r)}\} : x \in A \cap S(p,r)_{a(r)}\} < \varepsilon r^k,$$

then $(A,B) \in T_{l_1}(a,b,k,p)$.

Let $\varepsilon \in (0,s)$ and $\varepsilon < 1$. There exists number $0 < \delta < \varepsilon$ such that $r - a(r) < \frac{1}{2}\varepsilon r^k$, $r - b(r) < \frac{1}{2}\varepsilon r^k$, $r + a(r) < \varepsilon$ and $r + b(r) < \varepsilon$ for $r \in (0,\delta)$. From the triangle inequality, we obtain

$$|\rho(p,x) - \rho(p,y)| \le \rho(x,y),$$

for $x \in A \cap S(p,r)_{a(r)}$, $y \in B \cap S(p,r)_{b(r)}$. Hence

$$\sup\{\rho(p,x) : x \in A \cap S(p,r)_{a(r)}\} - \inf\{\rho(p,y) : y \in B \cap S(p,r)_{b(r)}\} \le$$
$$\le \sup\{\sup\{\rho(x,y) : y \in B \cap S(p,r)_{b(r)}\} : x \in A \cap S(p,r)_{a(r)}\}.$$

Hence and from Lemma 1 and (6.6) we obtain

$$a(r) + b(r) \le l_4(A \cap S(p,r)_{a(r)},\ B \cap S(p,r)_{b(r)})$$

or else

$$r + a(r) \le l_4(A \cap S(p,r)_{a(r)},\ B \cap S(p,r)_{b(r)}).$$

Therefore

$$\frac{2}{r^{k-1}} - \varepsilon \le \frac{1}{r^k} l_4(A \cap S(p,r)_{a(r)},\ B \cap S(p,r)_{b(r)}).$$

Because $\frac{2}{r^{k-1}} - \varepsilon > \frac{2}{r^{k-1}} - 1 > 1$, then

$$\frac{1}{r^k} l_4(A \cap S(p,r)_{a(r)},\ B \cap S(p,r)_{b(r)}) > 1.$$

Consequently $(A,B) \notin T_{l_4}(a,b,k,p)$. This ends the proof.

LEMMA 2. *If set $Q_p(A)$ is dense in the $(0,s)$ for some real number $s > 0$ and*

$$\frac{1}{r^k} l_4(A \cap S(p,r)_{a(r)},\ A \cap S(p,r)_{a(r)}) \to 0 \textit{ and}$$
$$a(r) \to 0 \quad \textit{for } r \to 0^+, \textit{ then}$$
$$\frac{a(r)}{r^k} \to 0 \quad \textit{for } r \to 0^+.$$

Proof. There exists $\delta \in (0, \frac{1}{2}s)$ such that $a(r) < \frac{1}{2}s$ for $r \in (0,\delta)$ and $r + a(r) < s$. Hence and from Lemma 1

$$\sup\{\rho(p,x) : x \in A \cap S(p,r)_{a(r)}\} = r + a(r)$$

and

$$\inf\{\rho(p,x) : x \in A \cap S(p,r)_{a(r)}\} = \max\{0, r - a(r)\}$$

for $r \in (0,\delta)$.

Let $x, y \in A \cap S(p,r)_{a(r)}$, from the triangle inequality we obtain

$$|\rho(p,x) - \rho(p,y)| \le \rho(x,y).$$

Therefore

$$\sup\{\rho(p,x) : x \in A \cap S(p,r)_{a(r)}\} - \inf\{\rho(p,y) : y \in A \cap S(p,r)_{a(r)}\} \leq$$
$$\leq l_4(A \cap S(p,r)_{a(r)},\ A \cap S(p,r)_{b(r)}).$$

Hence and from Lemma 1 we obtain

$$2a(r) \leq l_4(A \cap S(p,r)_{a(r)},\ A \cap S(p,r)_{a(r)}) \quad \text{for } r - a(r) \geq 0.$$

Consequently

$$0 \leq \frac{a(r)}{r^k} \leq \frac{1}{r^k} l_4(A \cap S(p,r)_{a(r)},\ A \cap S(p,r)_{a(r)}).$$

Hence $\frac{a(r)}{r^k} \to 0$ for $r \to 0^+$. This end the proof.

If the set $Q_p(A)$ is not dense set in any right-hand side neighbourhood of 0 then Lemma 2 may be not true.

Example. Let A be a subset of E and let p be a cluster point in a metric space (E,ρ). The point $p \in A$ is not a cluster point of set A. There exists number $t > 0$ such that $A \cap K(p,t) = \{p\}$. Let $a(r) \geq r$ in a certain right-hand side neighbourhood of 0 and $a(r) \to 0$ for $r \to 0^+$. There exists the number $\delta \in (0, \frac{1}{2}t)$ such that $r + a(r) < t$ for $r \in (0,\delta)$. Therefore $S(p,r)_{a(r)} \subset K(p,t)$. Hence $A \cap S(p,r)_{a(r)} = \{p\}$ for $r \in (0\delta)$, then

$$l_4(A \cap S(p,r)_{a(r)},\ A \cap S(p,r)_{a(r)}) = 0$$

and

$$\lim_{r \to 0^+} \sup \frac{a(r)}{r^k} \geq 1 \quad \text{for } k \geq 1.$$

THEOREM 2. *If there exists two dense sets $Q_p(A)$, $Q_p(B)$ in a interval $(0,s)$ for some real number $s > 0$ and $(A,B) \in T_{l_4}(a,b,k,p)$, then*

$$\frac{a(r)}{r^k} \to 0 \quad \textit{and} \quad \frac{b(r)}{r^k} \to 0 \textit{ for } r \to 0^+.$$

Proof. From the triangle inequality and from properties of supremum and infimum we obtain

$$l_4(A \cap S(p,r)_{a(r)},\ A \cap S(p,r)_{a(r)}) \leq 2l_4(A \cap S(p,r)_{a(r)},\ B \cap S(p,r)_{b(r)}),$$
$$l_4(B \cap S(p,r)_{b(r)},\ B \cap S(p,r)_{b(r)}) \leq 2l_4(A \cap S(p,r)_{a(r)},\ B \cap S(p,r)_{b(r)}).$$

Hence and from Lemma 2 we obtain

$$\frac{a(r)}{r^k} \to 0 \quad \text{and} \quad \frac{b(r)}{r^k} \to 0 \quad \text{for } r \to 0^+.$$

This ends the proof.

Analogously we prove the following theorems:

THEOREM 3. *If there exists the pair* $(A,B) \in T_{l_2}(a,b,k,p)$ *and two sets* $Q_p(A)$ *and* $Q_p(B)$ *are dense in the interval* $(0,s)$, *for some* $s > 0$, *and* $\frac{b(r)}{r^k} \xrightarrow[r\to 0^+]{} 0$, *then* $\frac{a(r)}{r^k} \to 0$ *for* $r \to 0^+$.

THEOREM 4. *If there exists the pair* $(A,B) \in T_{l_3}(a,b,k,p)$ *and the set* $Q_p(A)$ *is dense in the interval* $(0,s)$, *for some real number* $s > 0$, *then*

$$\frac{b(r)}{r^k} \to 0 \quad \textit{for } r \to 0^+.$$

THEOREM 5. *If there exists the pair* $(A,B) \in T_{l_6}(a,b,k,p)$ *and sets* $Q_p(A)$ *and* $Q_p(B)$ *are dense sets in the interval* $(0,s)$, *for some* $s > 0$, *then*

$$\frac{a(r)}{r^k} \to 0 \quad \textit{and} \quad \frac{b(r)}{r^k} \to 0 \quad \textit{for } r \to 0^+$$

THEOREM 6. *If there exists the pair* $(A,B) \in T_{l_5}(a,b,k,p)$ *and sets* $Q_p(A)$ *and* $Q_p(B)$ *are dense in the interval* $(0,s)$ *for some* $s > 0$, *then*

$$\frac{|a(r)-b(r)|}{r^k} \to 0 \quad \text{for } r \to 0^+.$$

Proof. Let $\varepsilon \in (0, \frac{1}{2}s)$. Then there exists a number $\delta > 0$ such that

$$l_5(A \cap S(p,r)_{a(r)},\ B \cap S(p,r)_{b(r)}) < \frac{1}{2}\varepsilon r^k,$$

$$b(r) < \frac{1}{2}\varepsilon r^k < \frac{1}{2}s \quad \text{and} \quad a(r) < \frac{1}{2}\varepsilon r^k < \frac{1}{2}s$$

for $r \in (0, \min\{\delta, \frac{1}{2}s\})$.

From definitions (6.2) and (6.5) we obtain $l_2(A,B) \le l_5(A,B)$ for $A, B \in E_0$.

Hence and from inequality

$$\sup\{\rho(p,x) : x \in A \cap S(p,r)_{a(r)}\} \le \sup\{\rho(p,y) : y \in B \cap S(p,r)_{b(r)}\} + \\ + l_2(A \cap S(p,r)_{a(r)},\ B \cap S(p,r)_{b(r)})$$

and symmetry of function l_5 we obtain

$$|\sup\{\rho(p,x) : x \in A \cap S(p,r)_{a(r)}\} - \sup\{\rho(p,y) : y \in B \cap S(p,r)_{b(r)}\}| \le \\ \le l_5(A \cap S(p,r)_{a(r)},\ B \cap S(p,r)_{b(r)}) < \varepsilon r^k.$$

From Lemma 1 we have $|a(r) - b(r)| < \varepsilon r^k$. Therefore

$$\frac{|a(r)-b(r)|}{r^k} \to 0 \quad \text{for } r \to 0^+.$$

This ends the proof.

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