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Sławomir Krzyśka

ON THE EXISTENCE OF CONTINUOUS SOLUTIONS OF URYSOHN AND VOLTERRA INTEGRAL EQUATIONS IN BANACH SPACES

1. Introduction

In this paper using measure of weak noncompactness developed by de Blasi [5] we prove some existence theorems for the Urysohn integral equation

$$x(t) = p(t) + \lambda \int_I f(t, s, x(s))\, ds, \tag{1}$$

and for the Volterra integral equation

$$x(t) = p(t) + \int_0^t f(t, s, x(s))\, ds, \tag{2}$$

where $I = [0, d]$ is a compact interval in R, f, p and x are functions with values in a Banach space E and the integrals are Pettis integrals (for the definitions see [8], [15], [1]).

There have appeared a lot of papers using the measure of weak noncompactness in proving existence theorems for ordinary differential equations.

For the weak solutions if f is only assamed weakly-weakly continuous, it has been shown that weak weak continuity of the right side is insufficient for the existence of weak solutions [6].

DEFINITION. Let A be a bounded nonvoid subset of E. The measure of weak noncompactness $\beta(A)$ is defined by

$$\beta(A) = \inf\{t > 0 : \text{there exists } C e K^w \text{ such that } A \subset C + tB_0\},$$

where K^w is the set of weakly compact subset of E and B_0 is the unit ball.

The properties of measure of weak noncompactness β are analogous to the properties of Kuratowski measure of noncompactness (see [5], [12]).

In this paper we will apply the following theorems.

THEOREM 1 [11]. *Let E be a metrizable locally convex topological vektor space and let D be a closed convex subset of E, and let F be a weakly sequentially continuous map of D into itself. If for some $x \in D$ the implication*

(*) $\overline{V} = \overline{\text{conv}}(\{x\} \cup F(V)) \Rightarrow V$ *is relatively weakly compact, holds for every subset V of D, then F has a fixed point.*

THEOREM 2 [12]. *Let H be a bounded, equicontinuous subset of $C(I, E)$. Then $\beta(H) = \sup_{t \in I} \beta(H(t)) = \beta(H(I))$.*

2. The Uryshon integral equation

Consider the integral equation (1) with the following assumptions:

(1^0) p is a continuous function from I into E;

(2^0) $(t, s, x) \to f(t, s, x)$ is a function from $I^2 \times E$ into E

which satisfies the following conditions:

(i) for each $(t, s) \in I^2$, $f(t, s, \cdot)$ is weakly-weakly sequentially continuous,

(ii) for each strongly continuous function $x : I \to E$, $f(\cdot, \cdot, x(\cdot))$ is Pettis-integrable on I,

(iii) for any $h > 0$ there exists a mesaurable function $m_h : I^2 \times R_+$, such that $\|f(t, s, x)\| \leq m_h(t, s)$ $(t, s \in I, \|x\| \leq h)$ and $\int_I m_h(t, s)\, ds \leq a(h) < \infty$,

(iv) for any $h > 0$ there is a function $d_h : I^3 \to R_+$ such that $\|f(t, s, x) - f(\tau, s, x)\| \leq d_h(\tau, t, s)$ $(\tau, t, s \in I, \|x\| \leq h)$ and $\lim_{t \in \tau} \int_I d_h(\tau, t, s)\, ds = 0$.

THEOREM 3. *Assume, in addition to 1^0 and 2^0, that there exists an integrable function $k : I^2 \times R_+$ such that for every $t \in I$, $\varepsilon > 0$ and for every bounded subset X of E there exists a closed subset I_ε of I such that* $\text{mes}(I \setminus I_\varepsilon) < \varepsilon$ *and*

$$\beta(t, T \times X) \leq \sup_{s \in T} k(t, s) \beta(x) \tag{3}$$

for any compact subset T of I_ε.

Then there exists $\varrho > 0$ such that for each λ, $0 \leq \lambda \leq \varrho$ there exists at least one continuous solution of (1).

P r o o f. Denote by C the Banach space of continuous functions $u : I \to E$ with the usual supremum norm $\| \ \|_C$. Let $r(K)$ be the spectral radius of the integral operator K defined by

$$Ku(t) = \int_I k(t,s)u(s)\,ds \quad (u \in C,\ t \in I).$$

Put

$$\varrho = \min\left(\sup \frac{h - \|p\|_C}{a(h)}, \frac{1}{r(K)}, \frac{1}{d}\right).$$

For fixed $\lambda \in R$, $0 \le \lambda < \varrho$, choose $b > 0$ in such a way that

$$\|p\|_C + \lambda a(b) \le b. \tag{4}$$

Put $B = \{x \in C : \|x\|_C \le b\}$. Consider the operator G defined by

$$G(x)(t) = p(t) + \lambda \int_I f(t,s,x(s))\,ds \quad (x \in B,\ t \in I).$$

Because for $x^* \in E^*$ with $\|x^*\| \le 1$ and $x \in B$ by (4) we have

$$|x^*(G(x)(t))| \le |x^*(p(t))| + |\lambda| \int_I |x^*(f(t,s,x(s))|\,ds \le$$
$$\le \|p\|_C + |\lambda| \int_I \|f(t,s,x(s))\|\,ds \le$$
$$\le \|p\|_C + |\lambda| \int_I m_b(t,s)\,ds \le \|p\|_C + |\lambda| a(b) \le b.$$

Consequently

$$\sup\{|x^*(G(x)(t))| : x^* \in E^*,\ \|x^*\| \le 1\} = \|G(x)(t)\| \le b. \tag{5}$$

Also

$$|x^*(G(x)(t) - G(x)(\tau))|$$
$$\le |x^*(p(t) - p(\tau))| + \lambda \int_I |x^*[f(t,s,x(s)) - f(\tau,s,x(s))]|\,ds \le$$
$$\le \|p(t) - p(\tau)\| + \lambda \int_I \|f(t,s,x(s)) - f(\tau,s,x(s))\|\,ds \le$$
$$\le \|p(t) - p(\tau)\| + \lambda \int_I d_h(\tau,t,s)\,ds.$$

This implies that

$$\|G(x)(t) - G(x)(\tau)\| \leq \|p(t) - p(\tau)\| + \lambda \int_I d_h(\tau, t, s)\, ds. \tag{6}$$

The assumptions $1^0, 2^0$ and (5), (6) imply that G is a continuous mapping from B into itself and G(B) is strongly equicontinuous subset of B.

Since $F(t, s, \cdot)$ is weakly-weakly sequentially continuous, by using the Lebesgue dominated convergence theorem, for each $x^* \in E^*$.

$$x^*(G(x_n)(t)) \to x^*(G(x)(t)) \quad \text{whenever } x_n \to x \text{ in } (C(I, E), \omega).$$

So by Lemma 1.9 [12] G is weakly-weakly sequentially continuous.

Let $\nabla = \overline{\text{conv}}\,(G(V) \cup \{0\})$. Obviously

$$V(t) \subset \overline{\text{conv}}(G(V)(t) \cup \{0\}) \quad \text{for } t \in D.$$

Since V is equicontinuous, the function $t \to V(t) = \beta(V(t))$ is continuous on I.

Fix $t \in D$ and $\varepsilon > 0$. By (3) and the Lusin theorem there exists a compact subset I_ε of I such that $\text{mes}(I \setminus I_\varepsilon) < \varepsilon$ and $\beta(f(t, T \times X) \leq \sup_{s \in T} k(t, s)\beta(X)$ for any compact subset T of D_ε, while the function $s - k(t, s)$ is continuous and

$$\lambda \int_{I \setminus I_\varepsilon} m_b(t, s)\, ds < \frac{\varepsilon}{2}.$$

We devide the interval $I = [0, d]$ into n parts $0 = d_0 < d_1 < \ldots < d_n = d$ in such a way that

$$|k(t, s)V(r) - k(t, u)V(z)| < \varepsilon \quad \text{for } s, r, u, z \in T_i = D_i \cap D_\varepsilon,$$

where $D_i = [d_{i-1}, d_i]$ $(i = 1, \ldots, n)$.

Set $V_i = \{u(s) : u \in V,\ s \in D_i\}$, then

$$\begin{aligned} &\beta\Big(p(t) + \lambda \int_I f(t, s, V(s))\, ds\Big) \\ &\leq \beta\Big(p(t) + \lambda \int_{I_\varepsilon} f(t, s, V(s))\, ds + \lambda \int_{I \setminus I_\varepsilon} f(t, s, V(s))\, ds\Big) \leq \\ &\leq \beta\Big(\lambda \int_{I_\varepsilon} f(t, s, V(s))\, ds\Big) + \varepsilon. \end{aligned} \tag{7}$$

Let us observe that

$$\lambda \int_{I_t} f(t,s,V(s))\,ds \subset \sum_{i=1}^{n} \lambda \int_{T_i} f(t,s,V(s))\,ds \subset$$

$$\subset \lambda \sum_{i=1}^{n} \operatorname{mes} T_i \, \overline{\operatorname{conv}}\, f(t, T_i \times V_i).$$

By the properties of measure of weak noncompactness we have

$$\beta\Big(\lambda \int_{I_t} f(t,s,V(s))\Big)\,ds \le \beta\Big(\lambda \sum_{i=1}^{n} \operatorname{mes} T_i \, \overline{\operatorname{conv}}\, f(t, T_i \times V_i)\Big) \le$$

$$\le \lambda \sum_{i=1}^{n} \operatorname{mes} T_i \beta(f(t, T_i \times V_i)) \le$$

$$\le \lambda \sum_{i=1}^{n} \operatorname{mes} T_i \sup_{s \in T_i} k(t,s) \beta(V_i) =$$

$$= \lambda \sum_{i=1}^{n} \operatorname{mes} T_i k(t,q_i) V(s_i),$$

where $q_i \in T_i$, $s_i \in D_i$. Moreover, as

$$|k(t,s)V(s) - k(t,q_i)V(s_i)| < \varepsilon$$

for $s \in T_i$, we have

$$\operatorname{mes} T_i k(t,q_i) V(s_i) \le \int_{T_i} k(t,s) V(s)\,ds + \varepsilon \operatorname{mes} T_i.$$

Thus

$$\beta\Big(\lambda \int_{I_t} f(t,s,V(s))\,ds\Big) \le \lambda \int_{I_t} k(t,s)\,ds + \lambda\varepsilon \operatorname{mes} T_\varepsilon.$$

As ε is arbitrarily small, from this and (7) we deduce that

$$\beta\Big(p(t) + \lambda \int_{I} f(t,s,V(s)\,ds\Big) \le \lambda \int_{I} k(t,s) V(s)\,ds$$

and therefore

$$\beta(V(t)) \le \lambda \int_{I} k(t,s) V(s)\,ds.$$

Because this inequality holds for every $t \in I$ and $\lambda r(K) < 1$, by applying the theorem on integral inequalities, we conclude that $\beta(V(t)) = 0$ for $t \in I$.

By Theorem 2, V is relatively weakly compact in $C(I, E)$. Applying now Theorem 1 we conclude that G has a fixed point, which is a solution of the equation (1).

3. The Volterra integral equation

Consider now the integral equation (2) assuming that p and f satisfy 1^0 and 2^0. Choose $b > 0$ in such a way that $b > 2\sup_{t\in I}\|p(t)\|$. From 2^0(iii) it follows that there is number a, $0 < a \leq d$ such that

$$\int_0^t m_b(t,s)\,ds \leq \frac{b}{2} \quad \text{for } 0 \leq t \leq a.$$

Let $J = [0, a]$. Put $B = \{u \in C(J, E) : \|u\|_C \leq b\}$ and

$$F(x)(t) = p(t) + \int_0^t f(t, s, x(s))\,ds \quad \text{for } x \in B,\ t \in J.$$

Similarly to the Urysohn integral equation, we can show that F is a weakly-weakly sequentially continuous mapping and the set $F(B)$ is equiuniformly continuous.

Further, let $P = \{(t, s, z) \in R^3 : 0 \leq s \leq t \leq l,\ |z| < C\}$, where $l > a$, $c > 2b$. Assume that a nonnegative real-valued function $(t, s, z) \to h(t, s, z)$ defined on P is a Kamke function, i.e. h satisfies the Caratheodory conditions and 2^0(iii)–(iv), and

(v) for each fixed (t, s) the function $z \to h(t, s, z)$ is nondecreasing,

(vi) for each q, $0 \leq q \leq l$, the zero function is the unique continuous solution of the equation

$$z(t) = \int_0^t h(t, s, z(s))\,ds \quad \text{defined on } [0, q].$$

THEOREM 3. *Assme that for any $\varepsilon > 0$, bounded $X \subset E$ and $t \in J$ there exists a closed subset I_ε of $[0, t]$ such that* $\text{mes}([0, t] \setminus I_\varepsilon) < \varepsilon$ *and*

$$\beta(f(t, TxX)) \leq \sup_{s\in T} h(t, s, \beta(X)) \tag{8}$$

for each closed subset T of I_ε. Then the equation (2) *has at least one continuous solution on J.*

Proof. Let $V \subset B$ be such that $\nabla = \overline{\text{conv}}(F(V) \cup \{0\})$. Let as fix $t \in J$, $\varepsilon > 0$.

By the Scorza Dragoni theorem there exists a closed subset D_ε of J such that $\text{mes}(J \setminus D_\varepsilon) < \varepsilon$ and the function h is uniformly continuous on

$D_\varepsilon \times [0, b_1]$, where $b_1 = b\beta(K(B(J)))$. Analogously as in [7] we prove that

$$\beta(V(t)) \leq \int_0^t h(t, s, V(s))\, ds \quad \text{for } t \in J.$$

From the property of Kamke functions and the theorem on integral inequalities, we conclude that

$$\beta(V(t)) = 0 \quad \text{for } t \in J.$$

Now as in the proof of Theorem 2 we conclude that F has a fixed point.

Remark. An analogous theorem can be proved for axiomatic measures of weak noncompactness (see [2], [7]).

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INSTITUTE OF MATHEMATICS
ADAM MICKIEWICZ UNIVERSITY
ul. Matejki 48/49
60-769 POZNAŃ, POLAND

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