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SOME REMARKS ON THE DARBOUX PROPERTY FOR MULTIVALUED FUNCTIONS

The present paper deals with certain properties of multivalued functions which coincides with the Darboux property in the case of a single valued function. The results contained here generalize that of Joanna Czarnowska and Grażyna Kwiecińska [1] which were established in the case of real multivalued functions.

Notations and definitions

Let X, Y be Hausdorff spaces, $P(Y)$ be the family of the nonempty subsets of Y and $C(Y)$ the family of the nonempty and closed subsets of Y. A function $F : X \to P(Y)$ is called a multivalued function and for such a function and any set $A \subset X$ and $B \subset Y$, we denote by

$$P(A) = \bigcup\{F(x) \mid x \in A\},$$
$$F^+(B) = \{x \in X \mid F(x) \subset B\},$$
$$F^-(B) = \{x \in X \mid F(x) \cap B \neq \emptyset\}.$$

If $F : X \to P(Y)$ is a multivalued function, $E \subset X$ is a subset of X and $x \in \overline{E}$, we say that a point $y \in Y$ is a limit point of F with respect to the set E and the point $x \in \overline{E}$ if for every $V \in V(y)$ and $U \in V(x)$, there exists $x' \neq x$, $x' \in U \cap E$ and $y' \in F(x')$ such that $y' \in V$, and we write $y \in L_F(E, x)$. If X is locally arcwise connected, we denote by $L_F(x) = \bigcap L_F(E, x)$, where the intersection is taken over all arcs $E \subset X$ such that x is an endpoint of E.

As in [2], we say that the multivalued function $F : X \to P(Y)$ has the Darboux property (briefly has property D) if for every connected set $E \subset X$, it results that $F(E)$ is connected in Y.

If X is a Hausdorff space, a continuous injective map $\gamma : (0,1) \to X$ is called an open arc. The points x_1, x_2 from X are the endpoints of γ if $x_1 = \lim_{t\to 0} \gamma(t)$ and $x_2 = \lim_{t \to 1} \gamma(t)$.

If A_1, A_2, B are mutually disjoint subsets from X, we say that A_1 and A_2 are separated in X by B if $X \setminus B \subset Q_1 \bigcup Q_2$, where Q_1, Q_2 are open disjoint sets from X and $A_1 \subset Q_1$, $A_2 \subset Q_2$.

We introduce now the following definitions concerning various generalizations of the Darboux property for multivalued functions:

A multivalued function $F : X \to P(Y)$ has property D_0 if we cannot find $E \subset X$ connected, $x_1, x_2 \in \overline{E}$, $y_1 \in F(x_1)$, $y_2 \in F(x_2)$ and A_1, A_2 open disjoint sets from Y such that $F(E) \subset A_1 \bigcup A_2$ and $y_1 \in A_1$, $y_2 \in A_2$.

A multivalued function $F : X \to P(Y)$ has property D_1 if we cannot find $E \subset X$ connected, $x_1, x_2 \in \overline{E}$ and $y_1 \in F(x_1)$ such that for every $y \in F(x_2)$, there exists A_y, B_y open disjoint sets from Y such that $F(E) \subset A_y \bigcup B_y$ and $y_1 \in A_y, y \in B_y$.

A multivalued function $F : X \to P(Y)$ has property D_2 if we cannot find an open arc $\gamma : (0,1) \to X$ with endpoints x_1 and x_2 and $y_1 \in F(x_1)$ such that for every $y \in F(x_2)$, there exists A_y, B_y open disjoint sets from Y such that $F(E) \subset A_y \bigcup B_y$ and $y_1 \in A_y$, $y \in B_y$, where $E = \operatorname{Im} \gamma$.

A multivalued function $F : X \to P(Y)$ has property D_3 if we cannot find $E \subset X$ connected, $x_1, x_2 \in \overline{E}$, $y_1 \in F(x_1)$ and A_1, A_2 open disjoint sets from Y such that $F(E) \subset A_1 \cup A_2$ and $y_1 \in A_1, F(x_2) \subset A_2$.

A multivalued function $F : X \to P(Y)$ has property D_4 if we cannot find an open arc $\gamma : (0,1) \to X$ with endpoints x_1 and $x_2, y_1 \in F(x_1)$ and A_1, A_2 open disjoint sets from Y such that $F(\operatorname{Im} \gamma) \subset A_1 \cup A_2$ and $y_1 \in A_1$, $F(x_2) \subset A_2$.

A multivalued function $F : X \to P(Y)$ has property D_0' if we cannot find $E \subset X$ connected, $x_1, x_2 \in E$, $y_1 \in F(x_1), y_2 \in F(x_2)$ and A_1, A_2 open disjoint sets from Y such that $F(E) \subset A_1 \cup A_2, y_1 \in A_1, y_2 \in A_2$. In other words, F has property D_0' if we cannot find $E \subset X$ connected, $x_1, x_2 \in E, y_1 \in F(x_1)$ and $y_2 \in F(x_2)$ such that y_1 and y_2 are separated by $Y \setminus F(E)$ in Y.

A multivalued function $F : X \to P(Y)$ has property D_1' if we cannot find $E \subset X$ connected, $x_1, x_2 \in E$ and $y_1 \in F(x_1)$ such that for every $y \in F(x_2)$, y_1 and y are separated by $Y \setminus F(E)$ in Y.

A multivalued function $F : X \to P(Y)$ has property D_2' if we cannot find an open arc $\gamma : (0,1) \to X$ with endpoints x_1 and x_2 and $y_1 \in F(x_1)$ such that for every $y \in F(x_2)$, y_1 and y are separated by $Y \setminus F(E)$ in Y, where $E = \operatorname{Im} \gamma$.

A multivalued function $F : X \to P(Y)$ has property D_3' if we cannot find $E \subset X$ connected, $x_1, x_2 \in E$ and $y_1 \in F(x_1)$ such that y_1 and $F(x_2)$ are separated by $Y \setminus F(E)$ in Y.

A multivalued function $F : X \to P(Y)$ has property D_4' if we cannot find an open arc $\gamma : (0,1) \to X$ with endpoints x_1 and x_2 and $y_1 \in F(x_1)$ such that y_1 and $F(x_2)$ are separated by $Y \setminus F(E)$ in Y, where $E = \operatorname{Im} \gamma$.

Remark 1. It is obvious that property D'_0 is equivalent with property D and also that properties D_0 implies properties D. We have the following implications:

$$D_i \Rightarrow D'_i \quad \text{for } i = 0,1,2,3,4,$$
$$D_0 \Rightarrow D_1 \Rightarrow D_3 \Rightarrow D_4,$$
$$D_0 \Rightarrow D_1 \Rightarrow D_2 \Rightarrow D_4,$$
$$D'_0 \Rightarrow D'_1 \Rightarrow D'_3 \Rightarrow D'_4,$$
$$D'_0 \Rightarrow D'_1 \Rightarrow D'_2 \Rightarrow D'_4.$$

EXAMPLE 1. Let $F : [0,1] \to C(R)$, defined by $F(x) = x$ for $x \in (0,1)$ and $F(0) = [-1,0]$, $F(1) = [1,2]$. Then F has property D'_i but not D_i for $i = 0,1,2,3,4$ and F is upper semicontinuous.

THEOREM 1. *Let $E \subset X$ be connected and $F : X \to P(Y)$ a continuous multivalued function and suppose that there exists $x \in E$ such that $F(x)$ is a connected set. Then $F(E)$ is connected.*

Proof. Suppose that $F(E)$ is not connected and let A_1, A_2 be open disjoint sets from Y such that $F(E) \subset A_1 \cup A_2$ and $F(E) \cap A_i \neq \emptyset$ for $i = 1,2$. Suppose that $F(x) \cap A_2 \neq \emptyset$. Then, since $F(x)$ is connected, it means that $F(x) \cap A_1 = \emptyset$. Using the fact that $F(x) \subset F(E) \subset A_1 \cup A_2$, we deduce that $F(x) \subset A_2$, i.e. $F^+(A_2) \cap E \neq \emptyset$. Since $F(E) \cap A_1 \neq \emptyset$, this implies that $E \cap F^-(A_1) \neq \emptyset$. We will show that $E \subset F^-(A_1) \cap F^+(A_2)$. Indeed, if $z \in E$, then $F(z) \subset A_1 \cap A_2$. In the case $F(z) \cup A_1 \neq \emptyset$, we have $z \in E \cap F^-(A_1)$ and if $F(z) \cap A_1 = \emptyset$, we obtain that $F(z) \subset A_2$, hence $z \in E \cap F^+(A_2)$. Now we will show that $E \cap F^-(A_1) \cap F^+(A_2) = \emptyset$. Indeed, if there is $z \in E \cap F^-(A_1) \cap F^+(A_2)$, then $F(z) \cap A_1 \neq \emptyset$ and $F(z) \subset A_2$, which is a contradiction, since $A_1 \cap A_2 = \emptyset$. We showed that $E \subset F^-(A_1) \cup F^+(A_2)$, $F^-(A_1) \cap E \neq \emptyset$, $F^+(A_2) \cap E \neq \emptyset$, $E \cap F^-(A_1) \cap F^+(A_2) = \emptyset$ and $F^-(A_1)$, $F^+(A_2)$ are open in X, which gives a contradiction, since E is connected. This ends the proof.

THEOREM 2. *Let $F : X \to P(Y)$ be a continuous multivalued function. Then F has property D_3.*

Proof. Suppose that F does not satisfy property D_3. Therefore we can find a connected set $E \subset X$, $x_1, x_2 \subset \overline{E}$, $y_1 \in F(x_1)$ and A_1, A_2 open disjoint sets from Y such that $F(E) \subset A_1 \cup A_2$, $y_1 \in A_1$ and $F(x_2) \subset A_2$. Now $x_2 \in F^+(A_2)$ is an open set from X, hence, using the fact that $x_2 \in \overline{E}$, we conclude that $E \cap F^+(A_2) \neq \emptyset$. Also $x_1 \in F^-(A_1)$ and $F^-(A_1)$ is an open set, hence we deduce that $E \cap F^-(A_1) \neq \emptyset$. As in Theorem 1, we see that $E \subset F^-(A_1) \cap F^+(A_2)$ and that $E \cap F^-(A_1) \cap F^+(A_2) = \emptyset$. This contradicts the assumption that E is connected and proves our claim.

EXAMPLE 2. Let $F:[0,1]\to P(R)$,

$$F(x)=\begin{cases}\left\{\frac{1}{n}\right\}_{n\in N} & \text{for } x\in(0,1)\\ \left\{\frac{1}{n}\right\}_{n\in N}\cup\{0\} & \text{for } x=0,1.\end{cases}$$

Then F is continuous but F has not property D_2.

THEOREM 3. *Let $F:X\to P(Y)$ be a lower semicontinuous multivalued function such that $F(x)$ is connected for every $x\in X$. Then F has property D_0.*

Proof. Suppose that F does not satisfy property D_0. Then we can find connected $E\subset X$, $x_1,x_2\in\overline{E}$, $y_1\in F(x_1)$, $y_2\in F(x_2)$ and A_1,A_2 open disjoint sets from Y such that $F(E)\subset A_1\cup A_2$, $y_1\in A_1$, $y_2\in A_2$. Since for every $x\in E$ $F(x)\subset A_1\cup A_2$ and $F(x)$ are connected, we have $F^-(A_i)\cap E=F^+(A_i)\cap E$ for $i=1,2$. Since $x_i\in F^-(A_i)\cap\overline{E}$, it results that $F^-(A_i)\cap E\neq\emptyset$ for $i=1,2$. As before, we have $E\subset F^-(A_1)\cup F^+(A_2)$ and $E\cap F^-(A_1)\cap F^+(A_2)=\emptyset$. Now, $E=(F^-(A_1)\cap E)\cup(F^+(A_2)\cap E)=(F^-(A_1)\cap E)\cup(F^-(A_2)\cap E)$ and $F^-(A_1)\cap F^-(A_2)\cap E=F^-(A_1)\cap F^+(A_2)\cap E=\emptyset$, which represents a contradiction, since E is connected. This ends the proof.

Similarly we can prove:

THEOREM 4. *Let $F:X\to P(Y)$ be an upper semicontinuous multivalued function such that $F(x)$ is connected for every $x\in X$. Then F has property D.*

Using now Theorem 3, Theorem 4 and Remark 1, we obtain:

THEOREM 5. *Let $F:X\to P(Y)$ be an upper (lower) semicontinuous multivalued function such that $F(x)$ is connected for every $x\in X$. Then F has property D.*

THEOREM 6. *Let $F:X\to P(Y)$ be a multivalued function such that Y is a regular topological space. Then, if X is arcwise connected and F has property D_4, then we have $F(x)\subset\overline{L_F(x)}$ for every $x\in X$. If F has property D_3, then $F(\overline{E})\subset\overline{F(E)}$ for every connected $E\subset X$, and for every closed set $B\subset Y$, the connected components of $F^+(B)$ are closed.*

Proof. Suppose that F has property D_3. We shall prove first that for every connected set $E\subset X$, we have $F(\overline{E})\subset\overline{F(E)}$. It is obvious, if E reduces to a point, in the other case, let $x\in\overline{E}$. If there exists $y\in F(x)$ such that $y\notin\overline{F(E)}$, let $V\in V(y)$ be such that $\overline{V}\cap\overline{F(E)}=\emptyset$. Taking

$A_1 = V$, $A_2 = C\overline{V}$ we see that $x \in \overline{E}$, $y \in F(x) \cap A_1$, so for every $x' \neq x$, $x' \in E$, we have $F(x') \subset F(E) \subset A_2$. Since F has property D_3, we obtained a contradiction. We finally proved that $F(\overline{E}) \subset \overline{F(E)}$ if F has property D_3 and it is obvious that if F has property D_4, then $F(\overline{E}) \subset \overline{F(E)}$ for every arc $E \subset X$.

Now, if $B \subset Y$ is closed and E is a component of $F^+(B)$, it results that $F(E) \subset B$, hence $F(\overline{E}) \subset \overline{F(E)} \subset \overline{B} = B$, which implies that $\overline{E} \subset F^+(B)$ and hence that $E = \overline{E}$, q.e.d.

Suppose now that F has property D_4 and let $x \in X$. Let γ be an arc in X such that x is an endpoint of γ and let E be a subarc of γ with an endpoint x as well. Then $F(x) \subset F(\overline{E}) \subset \overline{F(E)}$ and since E may be arbitrarily chosen, it follows that $F(x) \subset L_F(\gamma, x)$ for every arc γ such that x is an endpoint of γ. We finally obtained that if F has property D_4 then $F(x) \subset L_F(x)$, q.e.d.

Remark 2. Let $F : [0,1] \to C(R)$ be as in Example 1. We see that F is upper semicontinuous and has property D_i' for $i = 0,1,2,3,4$, but $F^+([\frac{-1}{2}, \frac{3}{2}]) = (0,1)$ which is not closed in $[0,1]$. Also, $L_F(1) = \{1\}$ and $F(1) = [1,2]$, hence $F(1) \not\subset L_F(1)$.

THEOREM 7. *Let $F : X \to P(Y)$ be lower and upper first class mapping such that for every closed set $B \subset Y$ and every closed arc $\gamma : [0,1] \to X$, both the sets $F^+(B) \cap \operatorname{Im}\gamma$ and $F^-(B) \cap \operatorname{Im}\gamma$ have closed components. Then F has property D_4.*

Proof. Suppose that F does not satisfy property D_4. Then we can find an open arc $\gamma : (0,1) \to X$ with endpoints x_1 and x_2, $y_1 \in F(x_1)$ and A_1, A_2 open disjoint sets from Y such that $F(\operatorname{Im}\gamma) \subset A_1 \cup A_2$, $y_1 \in A_1$ and $F(x_2) \subset A_2$. Now, $x_1 \in F^-(A_1) \cap \overline{\operatorname{Im}\gamma}$ and $x_1 \notin F^+(A_2) \cap \overline{\operatorname{Im}\gamma}$, $x_2 \in (F^+(A_2) \cap \overline{\operatorname{Im}\gamma}) \setminus (F^-(A_1) \cap \overline{\operatorname{Im}\gamma})$, $\operatorname{Im}\gamma \subset F^-(A_1) \cup F^+(A_2)$ and $F^-(A_1) \cap F^+(A_2) = \emptyset$. Since $\operatorname{Fr} A_1 \cap A_2 = \emptyset$, $\operatorname{Fr} A_2 \cap A_1 = \emptyset$, we obtain that $F^-(\overline{A}_1) \cap \operatorname{Im}\gamma = F^-(A_1) \cap \operatorname{Im}\gamma$ and $F^+(\overline{A}_2) \cap \operatorname{Im}\gamma = F^+(A_2) \cap \operatorname{Im}\gamma$. We also see that $(F^-(\overline{A}_1) \cap \operatorname{Im}\gamma) \cup \{x_1\} = F^-(\overline{A}_1) \cap \overline{\operatorname{Im}\gamma}$, $F^+(\overline{A}_2) \cap \overline{\operatorname{Im}\gamma} = (F^+(\overline{A}_2) \cap \operatorname{Im}\gamma) \cup \{x_2\}$, hence we obtain that $F^-(\overline{A}_1) \cap \overline{\operatorname{Im}\gamma} = (F^-(\overline{A}_1) \cap \operatorname{Im}\gamma) \cup \{x_1\} = (F^-(A_1) \cap \operatorname{Im}\gamma) \cup \{x_1\} = F^-(A_1) \cap \overline{\operatorname{Im}\gamma}$ and that $F^+(\overline{A}_2) \cap \overline{\operatorname{Im}\gamma} = (F^+(\overline{A}_2) \cap \operatorname{Im}\gamma) \cup \{x_2\} = (F^+(A_2) \cap \operatorname{Im}\gamma) \cup \{x_2\} = F^+(A_2) \cap \overline{\operatorname{Im}\gamma}$. If we denote by $B_1 = F^-(A_1) \cap \overline{\operatorname{Im}\gamma}$ and $B_2 = F^+(A_2) \cap \overline{\operatorname{Im}\gamma}$, then $\overline{\operatorname{Im}\gamma} = B_1 \cup B_2$, $B_1 \cap B_2 = \emptyset$, B_1 and B_2 are F_σ-sets and all their components are compact, which contradicts with Lemma 2 from [1].

Remark 3. If $I \subset R$ is an interval and $F : I \to P(R)$ is a real multivalued function, we see that F has property D_1 or D_2 if and only if the following condition holds: "For every $x_1, x_2 \in I$ and $y_1 \in F(x_1)$, there exists $y_2 \in F(x_2)$ such that $(y_1, y_2) \subset F((x_1, x_2))$". We also have that $F : I \to P(R)$ has property D_1' or D_2' if and only if the following condition holds: "For every $x_1, x_2 \in I$ and $y_1 \in F(x_1)$, there exists $y_2 \in F(x_2)$ such that $[y_1, y_2] \subset F([x_1, x_2])$".

THEOREM 8. *Let $F : X \to P(Y)$ be a multivalued function. Then if $F(x)$ are compact for every $x \in X$ property of D_3 implies D_1 and property D_4 implies D_2.*

If $F(x)$ are compact or they have finitely many connected components for every $x \in X$, then property D_3' implies D_1' and property D_4' implies property D_2'.

Proof. Suppose that $F(x)$ are compact for every $x \in X$ and that F has property D_3 but no property D_1. Then we can find $E \subset X$ connected, $x_1, x_2 \in \overline{E}$ and $y_1 \in F(x_1)$ such that for every $y \in F(x_2)$, we can find A_y, B_y open disjoint sets from Y such that $F(E) \subset A_y \cup B_y$, $y_1 \in A_y$ and $y \in B_y$. Since $F(x_2)$ is compact and $F(x_2) \subset \bigcup_{y \in F(x_2)} B_y$, we can find $z_1, z_2, \ldots, z_j \in F(x_2)$ such that $F(x_2) \subset \bigcup_{k=1}^{j} B_{z_k}$. Taking

$$A_1 = \bigcap_{k=1}^{j} A_{z_k}, \quad A_2 = \bigcup_{k=1}^{j} B_{z_k}$$

we see that A_1, A_2 are open and disjoint sets, $F(E) \subset A_1 \cup A_2$ and $y_1 \in A_1$, $F(x_2) \subset A_2$, which gives a contradiction, since F has property D_3.

In the same way we prove that if $F(x)$ is compact for every $x \in X$ and F has property D_4 then F has property D_2.

Suppose now that $F(x)$ are compact or they have finitely many connected components for every $x \in X$. If F has property D_3' but no property D_1', then we can find $E \subset X$ connected, $x_1, x_2 \in E$ and $y_1 \in F(x_1)$ such that for every $y \in F(x_2)$ there are A_y, B_y open disjoint sets such that $F(E) \subset A_y \cup B_y, y_1 \in A_y$ and $y \in B_y$. Now, if $F(x_2)$ is compact, we obtain as before a contradiction. If $F(x_2)$ has finitely many components, say $C_1, C_2, \ldots, C_j$, we choose $z_k \in C_k$ for $k = 1, 2, \ldots, j$. Since $C_k \subset F(x_2) \subset F(E) \subset A_{z_k} \cup B_{z_k}$, C_k is connected and $z_k \in C_k \cap B_{z_k}$ for $k = 1, 2, \ldots, j$, it follows that $C_k \subset B_{z_k}$ for $k = 1, 2, \ldots, j$. Taking again $A_1 = \bigcap_{k=1}^{j} A_{z_k}$, $A_2 = \bigcup_{k=1}^{j} B_{z_k}$ we see that A_1, A_2 are open and disjoint sets with $F(E) \subset A_1 \cup A_2$, $y_1 \in A_1$ and $F(x_2) \subset A_2$, what contradicts with property D_3' on F.

In the same way we prove that if $F(x)$ are compact or they have finitely many components for every $x \in X$ then if F has property D_4' it has property D_2' as well.

Using now Theorem 2 and Theorem 8, we obtain:

THEOREM 9. *Let $F : X \to P(Y)$ be a continuous multivalued function. Then, if $F(x)$ are compact for every $x \in X$, F has property D_1 and if $F(x)$ is compact or it has finitely many components for every $x \in X$, F has property D_1'.*

Remark 4. If $F : X \to C(R)$ is such that F has property D_3 then it posesses property D_1.

Indeed, suppose that F does not posess property D_1. Then we can find connected $E \subset X$, $x_1, x_2 \in \overline{E}$ and $y_1 \in F(x_1)$ such that for every $y \in F(x_2)$, there exists $c_y \in (y_1, y) \setminus F(E)$. We see that $y_1 \notin F(x_2)$ so we may suppose that $F(x_2) \cap (y_1, \infty) \neq \emptyset$. Since $F(x_2)$ is closed, we can find a point $y_2 \in F(x_2)$ such that $y_1 < y_2$ and $(y_1, y_2) \cap F(x_2) = \emptyset$, hence we can pick a point $b \in (y_1, y_2) \setminus F(E)$. Now, if $F(x_2) \cap (-\infty, y_1) = \emptyset$, we take $A_1 = (-\infty, b), A_2 = (b, \infty)$ and we see that $F(E) \subset A_1 \cup A_2$, $y_1 \in A_1$, $F(x_2) \subset A_2$, which is a contradiction, since F has property D_3. If $F(x_2) \cap (-\infty, y_1) \neq \emptyset$, then using again the fact that $F(x_2)$ is closed, we can find $y_3 \in F(x_2)$ such that $y_3 < y_1$ and $(y_3, y_1) \cap F(x_2) = \emptyset$. Hence we can pick $a \in (y_3, y_1) \setminus F(E)$. Taking $A_1 = (a, b)$ and $A_2 = (-\infty, a) \cap (b, \infty)$ we see that $F(E) \subset A_1 \cup A_2$, $y_1 \in A_1$ and $F(x_2) \subset A_2$, which is a contradiction, since F has property D_3. Now, if $F(x_2) \cap (y_1, \infty) = \emptyset$, it results that $F(x_2) \cap (-\infty, y_1) \neq \emptyset$. Using again the fact that $F(x_2)$ is a closed set, we can find $y_4 \in F(x_2)$ such that $y_4 < y_1$ and $(y_4, y_1) \cap F(x_2) = \emptyset$, hence we can pick $c \in (y_4, y_1) \setminus F(E)$. We can now take $A_1 = (c, \infty)$ and $A_2 = (-\infty, c)$ and we see that $F(E) \subset A_1 \cup A_2$, $y_1 \in A_1$ and $F(x_2) \subset A_2$, which is a contradiction, since F has property D_3.

In the same way we prove that if $F : X \to C(R)$ is a multivalued function with property D_4, then F has property D_2.

Using Theorem 2, we obtain:

THEOREM 10. *Let $F : X \to C(R)$ be a continuous multivalued function. Then F has property D_1.*

Remark 5. Using Remark 3, we see that a continuous real multivalued function $F : I \to C(R)$, where $I \subset R$ is an interval, satisfies the following condition:

"For every $x_1, x_2 \in I$ and $y_1 \in F(x_1)$, there exists $y_2 \in F(x_2)$ such that $(y_1, y_2) \subset F((x_2, x_2))$".

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