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REFINEMENTS OF JESSEN'S INEQUALITY

1. Jessen's inequality

Let f be a real convex function defined on $[a, b]$. The classical Hermite–Hadamard's inequality (see [9]) asserts that:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2}. \tag{1}$$

This inequality was generalized (see [1], [7] and [10]) for an arbitrary isotonic linear functional, i.e., a functional $A : C[a, b] \to \mathbb{R}$ with the properties:

(i) $A(tf + sg) = tA(f) + sA(g)$ for $t, s \in \mathbb{R}$, $f, g \in C[a, b]$;
(ii) $A(f) \geq 0$ if $f(x) \geq 0$ for all $x \in [a, b]$.

The result from [7] is: if f is convex and A is an isotonic linear functional with $A(1) = 1$, then

$$f(A(e)) \leq A(f(e)) \leq [(b - A(e))f(a) + (A(e) - a)f(b)]/(b - a) \tag{2}$$

where $e(x) = x$ for $x \in [a, b]$.

Note that taking in (2)

$$A(f) := \frac{1}{b-a} \int_a^b f(x)\,dx, \tag{3}$$

we get (1), so the inequality (2) generalizes, for isotonic linear functionals, the well known Jessen's inequality.

In turn, the inequality (2) was generalized in [1] where the function e was replaced by an arbitrary one.

2. Some refinements

The following lemma is proved in [10]:

LEMMA 1. *Let X be a real linear space and $C \subset X$ be a convex subset. If $f : C \to \mathbb{R}$ is convex then for all $x, y \in C$ the mapping $g_{x,y}(t) := f(tx + (1-t)y)$ is convex on $[0,1]$.*

Using this result, the authors proved a generalization of (2) for functions defined on an arbitrary linear space.

Another result of this type was established in [6].

Analogously we can prove the following lemma:

LEMMA 2. *If $f : [a,b] \to \mathbb{R}$ is convex, then for every $t \in [0,1]$ and every $y \in [a,b]$, the function $g_{t,y} : [a,b] \to \mathbb{R}$ given by $g_{t,y}(x) := f(tx + (1-t)y)$ is convex.*

Further on we will use the following convention:
if the functional A acts on the function

$$g(x_i) = f(x_1, \ldots, x_i, \ldots, x_n),$$

where all the variables except for x_i are fixed, then we denote

$$A(g) = a_{x_i}(f(x_1, \ldots, x_i, \ldots, x_n)).$$

Applying the inequality (2) to the convex function $g_{t,y}$ from Lemma 2, we get:

THEOREM 1. *Let $f : [a,b] \to \mathbb{R}$ be a continuous convex function and A be an isotonic linear functional with $A(1) = 1$. Then for every $t \in [0,1]$ and for every $y \in [a,b]$ the inequalities*

$$(4) \qquad f(tA(e) + (1-t)y) \le A_x(f(tx + (1-t)y)) \le$$
$$\le [(b - A(e))f(ta + (1-t)y) + (A(e) - a)f(tb + (1-t)y)]/(b-a)$$

hold.

We can obtain another variant of (4) generalizing the method used in [3] (see also [5]).

LEMMA 3. *Assume that the function $f : [a,b] \to \mathbb{R}$ is continuous convex and the functional A is linear and isotonic. Then the function $H_y : [0,1] \to \mathbb{R}$ defined by*

$$H_y(t) := A_x[f(tx + (1-t)y)], \qquad y \in [a,b]$$

is convex on $[0,1]$.

Proof. Let $x, y \in [a,b]$ and $t, s, u, v \in [0,1]$ and $u + v = 1$, then we have

$$f((ut + vs)x + (1 - ut - vs)y) = f(u(tx + (1-t)y) + v(sx + (1-s)y)) \le$$
$$\le uf(tx + (1-t)y) + vf(sx + (1-s)y),$$

because f is convex. Because the functional A is linear and isotonic it is increasing and so

$$H_y(ut+vs) \le uH_y(t)+vH_y(s).$$

Now we prove

THEOREM 2. *If the function* $f:[a,b]\to\mathbb{R}$ *is continuous convex and the functional* A *is linear, isotonic with* $A(\mathbf{1})=1$, *then the function* $H_0:[0,1]\to\mathbb{R}$ *defined by*

$$H_0(t):=A_x(f(tx+(1-t)A(e))$$

has the following properties:

(i) H_0 *is convex on* $[0,1]$;

(ii) *it has the bounds*

$$\sup_{t\in[0,1]} H_0(t)=H_0(1)=A(f(e))$$

and

$$\inf_{t\in[0,1]} H_0(t)=H_0(0)=f(A(e));$$

(iii) H_0 *is nondecreasing on* $[0,1]$.

Proof. (i) It follows from Lemma 3 by taking $y=A(e)$.

In order to get (ii) let us notice that

$$f(tx+(1-t)A(e))\le tf(x)+(1-t)f(A(e)),\quad t\in[0,1]$$

and so

$$H_0(t)\le tA(f)+(1-t)f(A(e))\le A(f)=H_0(1)$$

because from (2) we have $f(A(e))\le A(f)$. On the other hand the function $h:[a,b]\to\mathbb{R}$, given by $h(x):=f(tx+(1-t)A(e))$, is convex for every fixed $t\in[0,1]$ and so, again by (2)

$$H_0(t)=A(h)\ge h(A(e))=f(A(e))=H_0(0)$$

what gives (ii).

(iii) Let $0<t_1<t_2<1$. Then by the convexity argument for H_0 and by (ii) one has:

$$[H_0(t_2)-H_0(t_1)]/(t_2-t_1)\ge[H_0(t_1)-H_0(0)]/t_1\ge 0$$

what shows that H_0 is increasing on $(0,1)$ and by (ii) also in $[0,1]$.

Remark 1. Obviously the above theorem gives a generalization of the result from [3] (see also [5]). On the other hand the statement (ii) can be written as:

$$f(A(e)) \le A_x[f(tx+(1-t)A(e))] \le A(f) \tag{5}$$

which represent a refinement of Jessen's inequality.

APPLICATION. If the function $f:[a,b]\to\mathbb{R}$ is convex, $x_1,\dots,x_n\in[a,b]$ and $p_1,\dots,p_n$ are strictly positive weights, then denoting

$$m := \sum_{k=1}^{n} p_k x_k \Big/ \sum_{k=1}^{n} p_k,$$

we have the inequality

$$\sum_{k=1}^{n} p_k f(x_k) \ge \sum_{k=1}^{n} p_k f((x_k+m)/2).$$

Indeed, it follows from (5) for $A(f) := \sum_{k=1}^{n} p_k f(x_k) / \sum_{k=1}^{n} p_k$ and $t=1/2$.

Remark 2. Notice that, this inequality follows also from an inequality of Fuchs (see also [8]), so we get another proof of it.

3. Iteration of Jessen's inequality

We will start with the following lemma:

LEMMA 4. *If the function $f:[a,b]\to\mathbb{R}$ is continuous convex and the functional A is linear and isotonic, then the function $G_t:[a,b]\to\mathbb{R}$ given by*

$$G_t(x) := A_y[f(tx+(1-t)y)]$$

is convex for all $t\in[0,1]$.

The proof is similar to that one of Lemma 3 and we will omit the details.

THEOREM 3. *Let $f:[a,b]\to\mathbb{R}$ be a continuous convex functions and A,B are two isotonic linear functionals with $A(1)=1$ and $B(1)=1$. Then one has the inequalities*

$$\begin{aligned} f(tA(e)+(1-t)B(e)) &\le B_y(f(tA(e)+(1-t)y)) \le \\ &\le B_y(A_x(f(tx+(1-t)y)) \le tA(f)+(1-t)B(f) \le \\ \le [(b-B(e))f(a)&+(B(e)-a)f(b)]/(b-a)+ \\ &+t(B(e)-A(e))(f(a)-f(b))/(b-a). \end{aligned} \tag{6}$$

Proof. Applying the inequality (2) to the convex functions given by the previous lemmas we have:

$$A_x(f(tx+(1-t)y)) \ge f(tA(e)+(1-t)y)$$

and then

$$B_y(A_x(f(tx+(1-t)y))) \geq B_y(f(tA(e)+(1-t)y)) \geq f(tA(e)+(1-t)B(e)).$$

Thus, we get the first and the second inequality in (6).

Further on, from

$$f(tx+(1-t)y) \leq tf(x)+(1-t)f(y)$$

we deduce succesively that

$$A_x(f(tx+(1-t)y)) \leq tA(f)+(1-t)f(y)$$

and

$$B_y(A_x(f(tx+(1-t)y))) \leq tA(f)+(1-t)B(f)$$

getting so the second inequality from (2).

COROLLARY. *If $f:[a,b]\to\mathbb{R}$ is a continuous convex function and A an isotonic linear functional with $A(1)=1$, then*

$$f(A(e)) \leq A_y(f(tA(e)+(1-t)y)) \leq A_y(A_x(f(tx+(1-t)y)) \leq$$
$$\leq A(f) \leq [(b-A(e))f(a)+(A(e)-a)f(b)]/(b-a)$$

for all $t\in[0,1]$.

Remark 3. These inequalities also give a refinement of Jessen's inequality. They generalize some results from [1–5], given for the mapping from (3).

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