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Magdalena Chrapek, Wiesław Dziubdziela

ON THE LIMIT DISTRIBUTIONS OF ORDER STATISTICS IN A SEQUENCE OF SOME 1-DEPENDENT RANDOM VARIABLES

1. Introduction

We present necessary and sufficient conditions for the weak convergence of the distributions of the kth order statistics from some sequence of 1-dependent random variables. Limit laws are represented in terms of a compound Poisson distribution.

Let $\{\zeta_n;\ n \geq 0\}$ be a sequence of i.i.d. random variables with values in a measurable space (E, A) and with a common probability distribution π.

Let $\{\xi_n(x,y); n \geq 1\}$, $(x,y) \in E \times E$, be a sequence of i.i.d. $A \times A$ — measurable random functions.

Assume that the sequences $\{\zeta_n;\ n \geq 0\}$ and $\{\xi_n(x,y);\ n \geq 1\}$ are independent.

Consider the sequence of random variables:

$$(1) \qquad X_1 = \xi_1(\zeta_0, \zeta_1),\ X_2 = \xi_2(\zeta_1, \zeta_2),\ \ldots,\ X_n = \xi_n(\zeta_{n-1}, \zeta_n), \ldots$$

Denote by

$$\min(X_1, \ldots, X_n) = M_n^{(n)} \leq M_n^{(n-1)} \leq \ldots \leq M_n^{(1)} = \max(X_1, \ldots, X_n)$$

the order statistics of the sequence $X_1, \ldots, X_n$.

M. O. Djnaid in [1] has obtained limit laws for $M_n^{(1)}$. In this paper we shall be concerned with conditions under which, for suitable normalizing constants $\{a_n > 0,\ b_n;\ n \geq 1\}$,

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$$P\{M_n^{(k)} \le a_n x + b_n\} \xrightarrow{w} G^{(k)}(x), \quad k = 1,2,\ldots, \tag{2}$$

where $G^{(k)}$ is the distribution function and the notation $\xrightarrow{w}$ denotes the weak convergence of the distribution functions. We shall also be interested in forms of limit distribution $G^{(k)}$.

2. Main results

While obtaining the necessary and sufficient conditions for convergence (2) we use the results from the paper by W. N. Hudson et al. [3].

Let $I_{nj}(x) = I_{\{X_j > a_n x + b_n\}}$, $n = 1,2,\ldots$, $j = 1,2,\ldots,n$, where I_A denotes the indicator function of the set A and let

$$J_n(x) = \sum_{j=1}^{n} I_{nj}(x), \quad n = 1,2,\ldots.$$

It is well known that, for $k = 1,2,\ldots$,

$$P\{M_n^{(k)} \le a_n x + b_n\} = P\{J_n(x) < k\}, \quad -\infty < x < \infty. \tag{3}$$

Now, we shall prove the following results:

THEOREM 1. *The convergence* (2) *holds if and only if*

$$n \int_E \int_E (1 - F(x_0, x_1; a_n x + b_n))\pi(dx_0)\pi(dx_1) \text{ converges}$$

and

$$n \int_E\int_E\int_E\int_E F(x_0, x_1; a_n x + b_n)(1 - F(x_1, x_2; a_n x + b_n))\cdot F(x_2, x_3; a_n x + b_n)\,\pi(dx_0)\pi(dx_1)\pi(dx_2)\pi(dx_3) \to \lambda_1(x), \tag{4}$$

$$n \int_E\int_E\int_E\int_E F(x_0, x_1; a_n x + b_n)(1 - F(x_1, x_2; a_n x + b_n))\cdot(1 - F(x_2, x_3; a_n x + b_n))\,\pi(dx_0)\pi(dx_1)\pi(dx_2)\pi(dx_3) \to \lambda_2(x), \tag{5}$$

where $F(x, y; u) = P\{\xi_n(x, y) \le u\}$.

The limit distribution functions $G^{(k)}(x)$ *are given by the formula:*

$$G^{(k)}(x) = \sum_{s=0}^{k-1} P\{s, \lambda_1(x), \lambda_2(x)\},$$

for x such that $0 < G^{(k)}(x) < 1$, where

$$P\{s, \lambda_1, \lambda_2\} = \begin{cases} \exp(-(\lambda_1 + \lambda_2)), & \text{if } s = 0 \\ \exp(-(\lambda_1 + \lambda_2)) \sum\limits_{\substack{k_1+2k_2=s, \\ k_1,k_2 \geq 0}} \dfrac{\lambda_1^{k_1}\lambda_2^{k_2}}{k_1!k_2!}, & \text{if } s = 1, 2, \ldots \end{cases}$$

(Here we take $0^0 = 1$).

Proof. It is clear that the sequence $\{X_n;\ n \geq 1\}$ defined by (1) is strictly stationary and 1-dependent. Hence, the random variables $\{I_{nj}(x);\ n \geq 1\}$ are strictly stationary and 1-dependent in each row. It follows from Theorem 2 in [3] that the sequence $\{J_n(x);\ n \geq 1\}$ weakly converges (as $n \to \infty$) if and only if

$$nP\{I_{n1} = 1\} = n \iint\limits_{E\,E} (1 - F(x_0, x_1; a_n x + b_n))\, \pi(dx_0)\, \pi(dx_1)$$

converges and the conditions (4) and (5) hold. Moreover, the limit distribution of $I_n(x)$ has a characteristic function given by the formula

$$\phi(t) = \exp\Big(\sum_{n=1}^{2} \lambda_n(x)(e^{itn} - 1)\Big).$$

Note that it is the characteristic function of the compound Poisson distribution (see eg. [2]). Using (3) we complete the proof.

Now, we give two examples illustrating the main result.

EXAMPLE 1. Let $\{\zeta_n;\ n \geq 0\}$ be a sequence of i.i.d. random variables with the exponential distribution with the parameter 1. Let $\xi_n(x, y) = \max(x, y)$, $x, y \in R$, $n \geq 1$.

It is easy to compute that for $a_n = 1$, $b_n = \ln(n)$, $n = 1, 2, \ldots$, we have $\lambda_1(x) = 0$, $\lambda_2(x) = e^{-x}$.

EXAMPLE 2. Let $\{\zeta_n;\ n \geq 0\}$ and $\{a_n, b_n;\ n \geq 1\}$ be the same as in Example 1. Let $\{N_n;\ n \geq 1\}$ be a sequence of i.i.d. Bernoulli random variables with a parameter $a = P\{N_1 = 1\} = 1 - P\{N_1 = 0\}$, $0 < a < 1$. Assume that $\{\zeta_n;\ n \geq 0\}$ and $\{N_n;\ n \geq 1\}$ are independent.

Define

$$X_n = \begin{cases} \zeta_{n-1}, & \text{if } N_n = 1 \\ \zeta_n, & \text{if } N_n = 0. \end{cases}$$

By Theorem 1 we obtain $\lambda_1(x) = 2e^{-x}(2a^2 - 2a + 1)$, $\lambda_2(x) = 2e^{-x}(a - a^2)$.

As an easy consequence of Theorem 1 we have the following corollary:

COROLLARY 1. *Let $\{X_n;\ n \geq 1\}$ defined by* (1) *satisfy the following assumption:*

Assumption A. There exist normalizing constants $\{a_n > 0, b_n;\ n \geq 1\}$ and a nondegenerate distribution function G such that

$$\lim_{n\to\infty} n \int_E \int_E (1 - F(x_0, x_1; a_n x + b_n))\, \pi(dx_0)\, \pi(dx_1) =$$

$$= \begin{cases} \infty, & \text{if } G(x) = 0 \\ -\log G(x), & \text{if } G(x) > 0. \end{cases}$$

Moreover, assume that

$$\lim_{n\to\infty} \sup_x \Big(1 - \int_E F(x, y; a_n x + b_n)\, \pi(dy)\Big) = 0.$$

Then,

$$P\{M_n^{(k)} \leq a_n x + b_n\} \xrightarrow{w} \begin{cases} 0, & \text{if } G(x) = 0 \\ \sum_{s=0}^{k-1} \frac{1}{s!}(-\log G(x))^s, & \text{if } 0 < G(x) < 1 \\ 1, & \text{if } G(x) = 1, \end{cases}$$

for $k = 1, 2, \ldots$.

Proof. By Theorem 1 it is sufficient to show that $\lambda_1(x) = -\log G(x)$, $\lambda_2(x) = 0$, where $\lambda_1(x)$ and $\lambda_2(x)$ are given by the formulas (4) and (5). By stationarity we have

$$\int_E\int_E\int_E\int_E F(x_0, x_1; a_n x + b_n)(1 - F(x_1, x_2; a_n x + b_n))(1 - F(x_2, x_3; a_n x + b_n))\cdot$$

$$\cdot \pi(dx_0)\, \pi(dx_1)\, \pi(dx_2)\, \pi(dx_3) \leq$$

$$\leq \int_E\int_E\int_E (1 - F(x_0, x_1; a_n x + b_n))(1 - F(x_1, x_2; a_n x + b_n))\pi(dx_0)\pi(dx_1)\pi(dx_2).$$

Since

$$\int_E\int_E \Big[(1 - F(x_0, x_1; a_n x + b_n))\cdot$$

$$\cdot\Big(1 - \int_E F(x_1, x_2; a_n x + b_n)\, \pi(dx_2)\Big)\Big]\, \pi(dx_0)\, \pi(dx_1) \leq$$

$$\leq \Big(\int_E\int_E \pi(dx_0)\, \pi(dx_1) - \int_E\int_E F(x_0, x_1; a_n x + b_n)\, \pi(dx_0)\, \pi(dx_1)\Big)\cdot$$

$$\cdot \sup_{x_1} \Big(1 - \int_E F(x_1, x_2; a_n x + b_n)\, \pi(dx_2)\Big)$$

so, by the assumptions of Corollary 1, we obtain that $\lambda_2(x) = 0$.

Furthermore, we have:

$$n\int\int\int\int_{EEEE} F(x_0,x_1;a_nx+b_n)(1-F(x_1,x_2;a_nx+b_n))\cdot F(x_2,x_3;a_nx+b_n)\cdot$$
$$\cdot\pi(dx_0)\,\pi(dx_1)\,\pi(dx_2)\,\pi(dx_3)=$$
$$=n\int\int_{EE}(1-F(x_0,x_1;a_nx+b_n))\,\pi(dx_0)\,\pi(dx_1)-$$
$$-n\int\int\int_{EEE}(1-F(x_0,x_1;a_nx+b_n))(1-F(x_1,x_2;a_nx+b_n))\,\pi(dx_0)\,\pi(dx_1)\,\pi(dx_2)-$$
$$-n\int\int\int\int_{EEEE} F(x_0,x_1;a_nx+b_n)(1-F(x_1,x_2;a_nx+b_n))\cdot$$
$$\cdot(1-F(x_2,x_3;a_nx+b_n))\pi(dx_0)\,\pi(dx_1)\,\pi(dx_2)\,\pi(dx_3).$$

Thus, by the assumptions, the minuend tends to $-\log G(x)$ and the subtrahends tend to 0, so that $\lambda_1(x)=-\log G(x)$.

Remark 1. For $k=1$ from Corollary 1 we obtain the result of [1].

Remark 2. There is another way to obtain some results which are contained in Theorem 1. Namely, the following theorem may be obtained as a consequence of Theorem 4 of [2].

THEOREM 2. *Let $\{X_n;\ n\geq 1\}$ defined by* (1) *satisfy Assumption A and let*

$$G^{(k)}(x)=\begin{cases}0, & \text{if } G(x)=0\\ \sum_{s=0}^{k-1} H(s;G(x),d), & \text{if } 0<G(x)<1\\ 1, & \text{if } G(x)=1,\ k=1,2,\ldots,\end{cases}$$

where

$$H(s;G(x),d)=\begin{cases}(G(x))^d, & \text{if } s=0\\ (G(x))^d\sum\limits_{\substack{k_1+2k_2=s,\\ k_1,k_2\geq 0}}\frac{(-\log G(x))^{k_1+k_2}(2d-1)^{k_1}(1-d)^{k_2}}{k_1!k_2!} & \text{if } =1,2,\ldots\end{cases}$$

and $0.5\leq d\leq 1$.

The convergence (2) *holds if and only if*

$$\lim_{n\to\infty} n\int\int\int_{EEE}(1-F(x_0,x_1;a_nx+b_n))\cdot$$
$$\cdot(1-F(x_1,x_2;a_nx+b_n))\,\pi(dx_0)\,\pi(dx_1)\,\pi(dx_2)=-(1-d)\log G(x),$$

for all x such that $0<G(x)<1$.

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Magdalena Chrapek
INSTITUTE OF MATHEMATICS
PEDAGOGICAL UNIVERSITY
M. Konopnickiej 21
25-406 KIELCE, POLAND

Wiesław Dziubdziela
INSTITUTE OF MATHEMATICS AND INFORMATICS
TECHNICAL UNIVERSITY
Dąbrowskiego 73
42-201 CZĘSTOCHOWA, POLAND

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