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Antoni Chronowski

ON SOME RELATIONSHIPS BETWEEN AFFINE SPACES AND TERNARY SEMIGROUPS OF AFFINE MAPPINGS

1. Introduction

In the present paper we introduce the notion of a ternary semigroup of affine mappings. This ternary semigroup is the counterpart of the semigroup of affine endomorphisms. In the main theorem of this paper we give necessary and sufficient conditions for a certain characterization of affine spaces by means of ternary semigroups of affine mappings. The paper is a continuation of some ideas contained in the papers [1], [2], [3].

2. Basic definitions

DEFINITION 2.1 (cf. [1]). A ternary semigroup is an algebraic structure (A, f) such that A is a nonempty set and $f : A^3 \to A$ is a ternary operation satisfying the following associative law: $f(f(x_1, x_2, x_3), x_4, x_5) = f(x_1, f(x_2, x_3, x_4), x_5) = f(x_1, x_2, f(x_3, x_4, x_5))$ for all $x_1, \ldots, x_5 \in A$.

DEFINITION 2.2 (cf. [1]). A nonempty subset $I \subset A$ is called an ideal of a ternary semigroup (A, f) if $f(I, A, A) \subset I$, $f(A, I, A) \subset I$, $f(A, A, I) \subset I$.

DEFINITION 2.3. An element $x_0 \in A$ is said to be a left zero of a ternary semigroup (A, f) if $f(x_0, x_1, x_2) = x_0$ for all $x_1, x_2 \in A$.

Throughout this paper the letter f will be reserved to denote the ternary operation in ternary semigroups.

Let $(X, V(X), w)$ and $(Y, V(Y), w)$ be affine spaces over a field K. Let $A(X, Y)$ be the set of all affine mappings from $(X, V(X), w)$ to $(Y, V(Y, w)$. Put $A[X, Y] = A(X, Y) \times A(Y, X)$. Define the ternary operation $f : A[X, Y]^3 \to A[X, Y]$ by the rule:

$$f((p_1, q_1), (p_2, q_2), (p_3, q_3)) = (p_1 \circ q_2 \circ p_3, q_1 \circ p_2 \circ q_3)$$

for all $(p_i, q_i) \in A[X, Y]$, where $i = 1, 2, 3$.

The algebraic structure $(A[X, Y], f)$ is a ternary semigroup.

DEFINITION 2.4. The ternary semigroup $(A[X,Y],f)$ is called the ternary semigroup of affine mappings of the affine spaces X and Y.

For characterizing two affine spaces X and Y by means of affine mappings of these spaces we should consider affine mappings from X into Y, and conversely. The ternary semigroup $A[X,Y]$ meets the above requirements, and it is useful to achieve the desirable aim.

3. Main result

Let $(X,V(X),w)$ and $(Y,V(Y),w)$ be affine spaces over a field K. Let $A[X,Y]$ be the ternary semigroup of affine mappings of the affine spaces X and Y.

Consider the following sets:

$$A_c(X,Y) = \{p \in A(X,Y) : \exists y_0 \in Y \forall x \in X : p(x) = y_0\},$$
$$A_c(Y,X) = \{q \in A(Y,X) : \exists x_0 \in X \forall y \in Y : q(y) = x_0\}.$$

The affine mappings $p \in A_c(X,Y)$ and $q \in A_c(Y,X)$ such that their single values are $y_0 \in Y$ and $x_0 \in X$ we denote by p_{y_0} and q_{x_0}, respectively. Put $A_c[X,Y] = A_c(X,Y) \times A_c(Y,X)$. It is easy to notice that $A_c[X,Y]$ is a ternary subsemigroup of $A[X,Y]$.

Define an affine space $(A_c(X,Y),V(Y),w)$ over K by the rule:

$$w(p_{y_1},p_{y_2}) = w(y_1,y_2)$$

for all $p_{y_1},p_{y_2} \in A_c(X,Y)$.

Define an affine space $(A_c(Y,X),V(X),w)$ over K by the rule:

$$w(q_{x_1},q_{x_2}) = w(x_1,x_2)$$

for all $q_{x_1},q_{x_2} \in A_c(Y,X)$.

Consider the set

$$M(K) = \{(a_1,\ldots,a_k) : k \in N, a_1,\ldots,a_k \in K, a_1+\ldots+a_k = 1\}.$$

Assume that $p_{y_1},\ldots,p_{y_k}, p_y \in A_c(X,Y)$ and $(a_1,\ldots,a_k) \in M(K)$. Then $a_1p_{y_1}+\ldots+a_mp_{y_m} = p_y$ iff $a_1y_1+\ldots+a_my_m = y$. Similarly, if $q_{x_1},\ldots,q_{x_k}$, $q_x \in A_c(Y,X)$, then $a_1q_{x_1}+\ldots+a_kq_{x_k} = q_x$ iff $a_1x_1+\ldots+a_kx_k = x$.

LEMMA 3.1. *Let X and Y be affine spaces over a field K. A pair of affine mappings (p,q) is a left zero of the ternary semigroup $A[X,Y]$ if and only if $(p,q) \in A_c[X,Y]$.*

Proof. Let (p,q) be a left zero of $A[X,Y]$. By Definition 2.3 we have $f((p,q),(p_1,q_1),(p_2,q_2)) = (p,q)$ for all $(p_1,q_1),(p_2,q_2) \in A[X,Y]$. Put $(p_1,q_1) = (p_{y_0},q_{x_0})$ for some $x_0 \in X$, $y_0 \in Y$. Hence

$$f((p,q),(p_{y_0},q_{x_0}),(p_2,q_2)) = (p,q) \quad \text{and}$$
$$p = p \circ q_{x_0} \circ p_2, \quad q = q \circ p_{y_0} \circ q_2.$$

Therefore, $\forall x \in X : p(x) = p(x_0)$ and $\forall y \in Y : q(y) = q(y_0)$, and so $(p,q) \in A_c[X,Y]$.

Conversely, suppose that $(p,q) \in A_c[X,Y]$. Consequently $p = p_{y_0}$ and $q = q_{x_0}$ for some $x_0 \in X$, $y_0 \in Y$. For any $(p_1,q_1),(p_2,q_2) \in A[X,Y]$ we obtain $f((p,q),(p_1,q_1),(p_2,q_2)) = f((p_{y_0},q_{x_0}),(p_1,q_1),(p_2,q_2)) = (p_{y_0} \circ q_1 \circ p_2, q_{x_0} \circ p_1 \circ q_2) = (p_{y_0},q_{x_0}) = (p,q)$. Thus, (p,q) is a left zero of $A[X,Y]$.

PROPOSITION 3.1. *The set $A_c[X,Y]$ is the smallest ideal of the ternary semigroup $A[X,Y]$.*

Proof. It is easy to check that $A_c[X,Y]$ is an ideal of $A[X,Y]$. Put $I_c = A_c[X,Y]$. Let $I \subset A[X,Y]$ be an ideal of $A[X,Y]$. By Lemma 3.1 $f(I_c,I,I) = I_c$. On the other hand, $f(I_c,I,I) \subset I$. Hence $I_c \subset I$.

LEMMA 3.2. *Let X_i and Y_i for $i = 1,2$ be affine spaces over a field K. Let $F : A[X_1,Y_1] \to A[X_2,Y_2]$ be an epimorphism of the ternary semigroups $A[X_1,Y_1]$ and $A[X_2,Y_2]$. Then $F(A_c[X_1,Y_1]) = A_c[X_2,Y_2]$.*

Proof. Suppose that $(p,q) \in A_c[X_1,Y_1]$. By Lemma 3.1 we have $f((p,q),(p_1,q_1),(p_2,q_2)) = (p,q)$ for all $(p_1,q_1),(p_2,q_2) \in A[X_1,Y_1]$. Therefore, $f(F(p,q),F(p_1,q_1),F(p_2,q_2)) = F(p,q)$ for all $(p_1,q_1),(p_2,q_2) \in A[X_1,Y_1]$. Again by Lemma 3.1 $F(p,q) \in A_c[X_2,Y_2]$. Conversely, suppose that $(r,s) \in A_c[X_2,Y_2]$. This implies that $r = r_{y_2}$ and $s = s_{x_2}$ for some $x_2 \in X_2$, $y_2 \in Y_2$. There exists a pair $(p',q') \in A[X_1,Y_1]$ such that $F(p',q') = (r_{y_2},s_{x_2})$. Assume that $(p_{y_1'},q_{x_1'}) \in A_c[X_1,Y_1]$ is an arbitrary fixed pair and take $(p_1,q_1) \in A[X_1,Y_1]$. Put $(p,q) = f((p',q'),(p_{y_1'},q_{x_1'}),(p_1,q_1))$. Hence $p = p' \circ q_{x_1'} \circ p_1$ and $q = q' \circ p_{y_1'} \circ q_1$. Set $y_1 = p'(x_1')$ and $x_1 = q'(y_1')$. Thus $p = p_{y_1}$ and $q = q_{x_1}$, hence $(p,q) \in A_c[X_1,Y_1]$. We have $F(p,q) = f(F(p',q'),F(p_{y_1'},q_{x_1'}),F(p_1,q_1)) = f((r_{y_2},s_{x_2}),F(p_{y_1'},q_{x_1'}),F(p_1,q_1)) = (r_{y_2},s_{x_2}) = (r,s)$. Therefore, there exists a pair $(p,q) \in A_c[X_1,Y_1]$ such that $F(p,q) = (r,s)$.

Remark. Notice that a mapping $F_0 : A_c[X_1,Y_1] \to A_c[X_2,Y_2]$ is an isomorphism of the ternary semigroups $A_c[X_1,Y_1]$ and $A_c[X_2,Y_2]$ if and only if F_0 is a bijection.

Let X_i and Y_i for $i = 1,2$ be affine spaces over a field K. Suppose that $f_1 : X_1 \to X_2$ and $f_2 : Y_1 \to Y_2$ are affine isomorphisms. Define the mapping $F : A[X_1,Y_1] \to A[X_2,Y_2]$ by the rule:

$$F(p,q) = (f_2 \circ p \circ f_1^{-1}, f_1 \circ q \circ f_2^{-1}) \tag{1}$$

for every $(p,q) \in A[X_1,Y_1]$. It is easy to check that F is an isomorphism of the ternary semigroups $A[X_1,Y_1]$ and $A[X_2,Y_2]$.

DEFINITION 3.1. The mapping F defined by the formula (1) is called the isomorphism of the ternary semigroups $A[X_1,Y_1]$ and $A[X_2,Y_2]$ induced by the pair of affine isomorphisms (f_1, f_2).

An isomorphism F of the ternary semigroups $A[X_1, Y_1]$ and $A[X_2, Y_2]$ need not imply the existence of isomorphisms $f_1 : X_1 \to X_2$ and $f_2 : Y_1 \to Y_2$ of the affine spaces X_1, X_2, Y_1, Y_2.

The following example illustrates the above statement.

EXAMPLE. Assume that $X_1 = R$, $X_2 = R^2$, $Y_1 = \{y_1\}$, $Y_2 = \{y_2\}$. Thus X_1 and X_2 are affine spaces over the field of real numbers R. The sets Y_1 and Y_2 are one-element affine spaces over R. We have $A(X_1, Y_1) = \{p_{y_1}\}$, $A(Y_1, X_1) = \{q_{x_1} : x_1 \in X_1\}$, $A(X_2, Y_2) = \{r_{y_2}\}$, $A(Y_2, X_2) = \{s_{x_2} : x_2 \in X_2\}$. Notice that $A(X_1, Y_1) = A_c(X_1, Y_1)$, $A(Y_1, X_1) = A_c(Y_1, X_1)$, $A(X_2, Y_2) = A_c(X_2, Y_2)$, $A(Y_2, X_2) = A_c(Y_2, X_2)$. Therefore, $A[X_1, Y_1] = A_c[X_1, Y_1]$ and $A[X_2, Y_2] = A_c[X_2, Y_2]$. Additionally, $A_c[X_1, Y_1]$ and $A_c[X_2, Y_2]$ are the sets of the power of the continuum. Consequently, there exists a bijection $F : A_c[X_1, Y_1] \to A_c[X_2, Y_2]$ of the set $A_c[X_1, Y_1]$ onto the set $A_c[X_2, Y_2]$. According to Remark the ternary semigroups $A[X_1, Y_1]$ and $A[X_2, Y_2]$ are isomorphic. However, the affine spaces X_1 and X_2 are not isomorphic.

Let X_i and Y_i for $i = 1, 2$ be affine spaces over K. Let $F : A[X_1, Y_1] \to A[X_2, Y_2]$ be an isomorphism of the ternary semigroups $A[X_1, Y_1]$ and $A[X_2, Y_2]$ induced by a pair of affine isomorphisms (f_1, f_2). Assume that $(a_1, a_2, a_3) \in M(K)$. Suppose that $p_{y_1}, p_{y_2}, p_{y_3} \in A_c(X_1, Y_1)$ and $q_1, q_2, q_3, q_4 \in A_c(Y_1, X_1)$. Put $a_1 p_{y_1} + a_2 p_{y_2} + a_3 p_{y_3} = p_{y_4}$ for $p_{y_4} \in A_c(X_1, Y_1)$. Thus we have $F(p_{y_1}, q_i) = (f_2 \circ p_{y_i} \circ f_1^{-1}, f_1 \circ q_i \circ f_2^{-1})$ for $i = 1, \ldots, 4$. Notice that $f_2 \circ p_{y_i} \circ f_1^{-1} = r_{f_2(y_i)}$ for $r_{f_2(y_i)} \in A_c(X_2, Y_2)$, where $i = 1, \ldots, 4$. Assume that $a_1 r_{f_2(y_1)} + a_2 r_{f_2(y_2)} + a_3 r_{f_2(y_3)} = r_y$ for some $r_y \in A_c(X_2, Y_2)$. Thus $a_1 f_2(y_1) + a_2 f_2(y_2) + a_3 f_2(y_3) = y$, and so $f_2(a_1 y_1 + a_2 y_2 + a_3 y_3) = y$. Since $a_1 y_1 + a_2 y_2 + a_3 y_3 = y_4$, it follows that $a_1 r_{f_2(y_1)} + a_2 r_{f_2(y_2)} + a_3 r_{f_2(y_3)} = r_{f_2(y_4)}$.

Conversely, assume that $a_1 r_{f_2(y_1)} + a_2 r_{f_2(y_2)} + a_3 r_{f_2(y_3)} = r_{f_2(y_4)}$. Hence $a_1 f_2(y_1) + a_2 f_2(y_2) + a_3 f_2(y_3) = f_2(y_4)$, this means that $f_2(a_1 y_1 + a_2 y_2 + a_3 y_3) = f_2(y_4)$. Therefore, $a_1 y_1 + a_2 y_2 + a_3 y_3 = y_4$ and so $a_1 p_{y_1} + a_2 p_{y_2} + a_3 p_{y_3} = p_{y_4}$.

Let us denote by π_1 and π_2 the projections of Cartesian product. From the foregoing we have obtained the following condition:

$$(W_1) \quad \forall (a_1, a_2, a_3) \in M(K) \forall p_1, \ldots, p_4 \in A_c(X, Y) \forall q_1, \ldots, q_4 \in A_c(Y, X)$$
$$[a_1 p_1 + a_2 p_2 + a_3 p_3 = p_4 \Leftrightarrow a_1 \pi_1(F(p_1, q_1)) + a_2 \pi_1(F(p_2, q_2)) + a_3 \pi_1(F(p_3, q_3)) = \pi_1(F(p_4, q_4))].$$

A similar argument yields the following condition:

$$(W_2) \quad \forall (a_1, a_2, a_3) \in M(K) \forall p_1, \ldots, p_4 \in A_c(X, Y) \forall q_1, \ldots, q_4 \in A_c(Y, X)$$

$$[a_1q_1 + a_2q_2 + a_3q_3 = q_4 \Leftrightarrow a_1\pi_2(F(p_1, q_1)) + a_2\pi_2(F(p_2, q_2)) + a_3\pi_2(F(p_3, q_3)) = \pi_2(F(p_4, q_4))].$$

Let X_i and Y_i for $i = 1,2$ be affine spaces over K. Assume that $F : A[X_1,Y_1] \to A[X_2,Y_2]$ is an isomorphism of the ternary semigroups $A[X_1,Y_1]$ and $A[X_2,Y_2]$ satisfying the conditions (W_1) and (W_2). From the condition (W_1) it follows that

(2) $$\forall p_1, p_2 \in A_c(X,Y) \forall q_1, q_2 \in A_c(Y,X) \quad [p_1 = p_2 \Leftrightarrow \pi_1(F(p_1, q_1)) = \pi_1(F(p_2, q_2)].$$

Similarly, the condition (W_2) yields

(3) $$\forall p_1, p_2 \in A_c(X,Y) \forall q_1, q_2 \in A_c(Y,X) \quad [q_1 = q_2 \Leftrightarrow \pi_2(F(p_1, q_1)) = \pi_2(F(p_2, q_2)].$$

THEOREM 3.1. *Let X_i and Y_i for $i = 1,2$ be affine spaces over a field K. An isomorphism $F : A[X_1, Y_1] \to A[X_2, Y_2]$ of the ternary semigroups $A[X_1, Y_1]$ and $A[X_2, Y_2]$ is induced by a pair of affine isomorhisms (f_1, f_2) if and only if the isomorphism F satisfies the conditions (W_1) and (W_2).*

Proof. We have proved that the isomorphism F induced by the pair of affine isomorphisms (f_1, f_2) satisfies the conditions (W_1) and (W_2).

Let us assume that $F : A[X_1, Y_1] \to A[X_2, Y_2]$ is an isomorphism of the ternary semigroups $A[X_1, Y_1]$ and $A[X_2, Y_2]$ such that the conditions (W_1) and (W_2) are satisfied. In view of Lemma 3.2 we can define the mapping $F^* : X_1 \times Y_1 \to X_2 \times Y_2$ by the formula:

(4) $$F^*(x_1, y_1) = (x_2, y_2) \Leftrightarrow F(p_{y_1}, q_{x_1}) = (r_{y_2}, s_{x_2})$$

for $(x_1, y_1) \in X_1 \times Y_1$ and $(x_2, y_2) \in X_2 \times Y_2$. It is easy to notice that F^* is a bijection. Let $y_0 \in Y_1$ be an arbitrary fixed element. We define the mapping $f_1 : X_1 \to X_2$ by the rule:

(5) $$f_1(x_1) = x_2 \Leftrightarrow \pi_1(F^*(x_1, y_0)) = x_2$$

for $x_1 \in X_1$, $x_2 \in X_2$. We will prove that

(6) $$f_1(x_1) = x_2 \Leftrightarrow \forall y_1 \in Y_1 : \pi_1(F^*(x_1, y_1)) = x_2$$

for $x_1 \in X_1, x_2 \in X_2$. Suppose that $F^*(x_1, y_0) = (x_2, y_2)$ and $F^*(y_1, y_1) = (x_2', y_2')$ for an arbitrary fixed element $y_1 \in Y_1$. Thus we have

$$F^*(x_1, y_0) = (x_2, y_2) \Leftrightarrow F(p_{y_0}, q_{x_1}) = (r_{y_2}, s_{x_2}),$$
$$F^*(x_1, y_1) = (x_2', y_2') \Leftrightarrow F(p_{y_1}, q_{x_1}) = (r_{y_2'}, s_{x_2'}).$$

In view of the condition (3) we infer that $s_{x_2} = s_{x_2'}$, and so $x_2 = x_2'$ Hence we get (6).

It is easy to verify that

$$f_1(x_1) = x_2 \Leftrightarrow \exists y_1 \in Y_1 : \pi_1(F^*(x_1, y_1)) = x_2 \tag{7}$$

for $x_1 \in X_1, x_2 \in X_2$.

Next, we will prove that $f_1 : X_1 \to X_2$ is a bijection. Suppose that $x_2 \in X_2$. Let us take an arbitrary fixed element $y_2 \in Y_2$. Thus there exists a pair $(x_1, y_1) \in X_1 \times Y_2$ such that $F^*(x_1, y_1) = (x_2, y_2)$. Therefore using the condition (7) we obtain $f_1(x_1) = x_2$, and so f_1 is a surjection. Suppose that $f_1(x_1) = f_1(x_1')$ for $x_1, x_1' \in X_1$. Hence $f_1(x_1) = x_2$ and $f_1(x_1') = x_2$ for some $x_2 \in X_2$. By (5) it follows that $F^*(x_1, y_0) = (x_2, y_2)$ and $F^*(x_1', y_0) = (x_2, y_2')$ for some $y_2, y_2' \in Y_2$. Hence we have $F(p_{y_0}, q_{x_1}) = (r_{y_2}, s_{x_2})$ and $F(p_{y_0}, q_{x_1'}) = (r_{y_2'}, s_{x_2})$. Using the condition (3) we get $q_{x_1} = q_{x_1'}$, and so $x_1 = x_1'$. Therefore f_1 is an injection.

We will prove that $f_1 : X_1 \to X_2$ is an affine mapping. It is enough to prove that

$$\begin{aligned}\forall(a_1, a_2, a_3) \in M(K)\forall x_{11}, x_{12}, x_{13} \in X_1[f_1(a_1x_{11} + a_2x_{12} + a_3x_{13})\\ = a_1f_1(x_{11}) + a_2f_1(x_{12}) + a_3f_1(x_{13})].\end{aligned}$$

Assume that $a_1x_{11} + a_2x_{12} + a_3x_{13} = x_1$ for some $x_1 \in X_1$. Suppose that $f_1(x_{11}) = x_{21}$, $f_1(x_{12}) = x_{22}$, $f_1(x_{13}) = x_{23}$, $f_1(x_1) = x_2$ for some $x_{21}, x_{22}, x_{23}, x_2 \in X_2$. Thus,

$$\begin{aligned}f_1(x_{11}) = x_{21} &\Leftrightarrow \pi_1(F^*(x_{11}, y_0)) = x_{21},\\ f_1(x_{12}) = x_{22} &\Leftrightarrow \pi_1(F^*(x_{12}, y_0)) = x_{22},\\ f_1(x_{13}) = x_{23} &\Leftrightarrow \pi_1(F^*(x_{13}, y_0)) = x_{23},\\ f_1(x_1) = x_2 &\Leftrightarrow \pi_1(F^*(x_1, y_0)) = x_2.\end{aligned}$$

Assume that

$$\begin{aligned}F^*(x_{11}, y_0) &= (x_{21}, y_{21}),\\ F^*(x_{12}, y_0) &= (x_{22}, y_{22}),\\ F^*(x_{13}, y_0) &= (x_{23}, y_{23}),\\ F^*(x_1, y_0) &= (x_2, y_2)\end{aligned}$$

for some $y_{21}, y_{22}, y_{23}, y_2 \in Y_2$.

We have

$$\begin{aligned}F^*(x_{11}, y_0) = (x_{21}, y_{21}) &\Leftrightarrow F(p_{y_0}, q_{x_{11}}) = (r_{y_{21}}, s_{x_{21}}),\\ F^*(x_{12}, y_0) = (x_{22}, y_{22}) &\Leftrightarrow F(p_{y_0}, q_{x_{12}}) = (r_{y_{22}}, s_{x_{22}}),\\ F^*(x_{13}, y_0) = (x_{23}, y_{23}) &\Leftrightarrow F(p_{y_0}, q_{x_{13}}) = (r_{y_{23}}, s_{x_{23}}),\\ F^*(x_1, y_0) = (x_2, y_2) &\Leftrightarrow F(p_{y_0}, q_{x_1}) = (r_{y_2}, s_{x_2}).\end{aligned}$$

Since $a_1x_{11}+a_2x_{12}+a_3x_{13} = x_1$, it follows that $a_1q_{x_{11}}+a_2q_{x_{12}}+a_3q_{x_{13}} = q_{x_1}$. In view of the condition (W_2) we get $a_1s_{x_{21}}+a_2s_{x_{22}}+a_3s_{x_{23}} = s_{x_2}$. Therefore, $a_1x_{21} + a_2x_{22} + a_3x_{23} = x_2$, and so $a_1f_1(x_{11}) + a_2f_1(x_{12}) + a_3f_1(x_{13}) = f_1(a_1x_{11} + a_2x_{12} + a_3x_{13})$. Summarizing, the mapping $f_1 : X_1 \to X_2$ is an affine isomorphism of the affine spaces X_1 and X_2.

Let $x_0 \in X_1$ be an arbitrary fixed element. We define the mapping $f_2 : Y_1 \to Y_2$ by the formula:

$$(8) \qquad f_2(y_1) = y_2 \Leftrightarrow \pi_2(F^*(x_0, y_1) = y_2$$

for $y_1 \in Y_1$ and $y_2 \in Y_2$.

The analogous argument applied to the mapping f_2 allows to prove that

$$(9) \qquad f_2(y_1) = y_2 \Leftrightarrow \forall x_1 \in X_1 : \pi_2(F^*(x_1, y_1)) = y_2,$$

$$(10) \qquad f_2(y_1) = y_2 \Leftrightarrow \exists x_1 \in X_1 : \pi_2(F^*(x_1, y_1)) = y_2$$

for $y_1 \in Y_1$ and $y_2 \in Y_2$.

We can similarly show that the mapping $f_2 : Y_1 \to Y_2$ is an affine isomorphism of the affine spaces Y_1 and Y_2.

By the conditions (6) and (9) we get $F^*(x_1, y_1) = (\pi_1(F^*(x_1, y_1)), \pi_2(F^*(x_1, y_1))) = (f_1(x_1), f_2(y_1))$ for every $(x_1, y_1) \in X_1 \times Y_1$.

Consequently,

$$(11) \qquad F^* = (f_1, f_2).$$

We will prove that the isomorphism F is induced by the pair of affine isomorphisms (f_1, f_2). First, we will show that the following condition is satisfied:

$$(12) \quad \forall x_1 \in X_1 \forall y_1 \in Y_1 \forall (p,q) \in A[X_1, Y_1] : F(p,q)(f_1(x_1), f_2(y_1)) = (f_2(p(x_1)), f_1(q(y_1))).$$

Suppose that $x_1 \in X_1, y_1 \in Y_1$, and $(p,q),(p_1,q_1) \in A[X_1, Y_1]$. Hence $f((p,q),(p_{y_1}, q_{x_1}),(p_1,q_1)) = (p \circ q_{x_1} \circ p_1, q \circ p_{y_1} \circ q_1) = (p_{p(x_1)}, q_{q(y_1)})$. We have $F(p_{p(x_1)}, q_{q(y_1)}) = F(f((p,q),(p_{y_1}, q_{x_1}),(p_1,q_1))) = f(F(p,q), F(p_{y_1}, q_{x_1}), F(p_1,q_1))$. Set $F(p,q) = (r,s)$ and $F(p_1,q_1) = (r_1,s_1)$. By Lemma 3.2 we get $F(p_{y_1}, q_{x_1}) = (r_{y_2}, s_{x_2})$ for some $x_2 \in X_2$, $y_2 \in Y_2$. By (11) $F(p_{y_1}, q_{x_1}) = (r_{y_2}, s_{x_2}) \Leftrightarrow F^*(x_1, y_1) = (x_2, y_2) \Leftrightarrow (f_1(x_1), f_2(y_1)) = (x_2, y_2) \Leftrightarrow (x_2 = f_1(x_1) \wedge y_2 = f_2(y_1))$. Therefore, $F(p_{p(x_1)}, q_{q(y_1)}) = f((r,s),(r_{f_2(y_1)}, s_{f_1(x_1)}),(r_1,s_1)) = (r \circ s_{f_1(x_1)} \circ r_1, s \circ r_{f_2(y_1)} \circ s_1) = (r_{r(f_1(x_1))}, s_{s(f_2(y_1))})$. On the other hand, $F(p_{p(x_1)}, q_{q(y_1)}) = (r_{y_2}, s_{x_2})$ for some $x_2 \in X_2, y_2 \in Y_2$. By (11) $F(p_{p(x_1)}, q_{q(y_1)}) = (r_{y_2}, s_{x_2}) \Leftrightarrow F^*(q(y_1), p(x_1)) = (x_2, y_2) \Leftrightarrow (f_1(q(y_1)), f_2(p(x_1))) = (x_2, y_2) \Leftrightarrow (x_2 = f_1(q(y_1)) \wedge y_2 = f_2(p(x_1)))$. Therefore, $F(p_{p(x_1)}, q_{q(y_1)}) = (r_{f_2(p(x_1))}, s_{f_1(q(y_1))})$. Consequently, $r(f_1(x_1)) = f_2(p(x_1))$ and $s(f_2(y_1)) = f_1(q(y_1))$. Thus, $F(p,q)(f_1(x_1), f_2(y_1)) = (r,s)(f_1(x_1), f_2(y_1)) = (r(f_1(x_1)), s(f_2(y_1))) = (f_2(p(x_1)), f_1(q(y_1)))$. Therefore, we have obtained the formula (12).

For $x_2 \in X_2$ and $y_2 \in Y_2$ there exist $x_1 \in X_1$ and $y_1 \in Y_1$ such that $f_1(x_1) = x_2$ and $f_2(y_1) = y_2$. Hence $x_1 = f_1^{-1}(x_2)$ and $y_1 = f_2^{-1}(y_2)$. Using the formula (12) we obtain $F(p,q)(x_2,y_2) = ((f_2 \circ p \circ f_1^{-1})(x_2), (f_1 \circ q \circ f_2^{-1})(y_2)) = (f_2 \circ p \circ f_1^{-1}, f_1 \circ q \circ f_2^{-1})(x_2, y_2)$ for any pair $(p,q) \in A[X_1, Y_1]$. Therefore,

$$F(p,q) = (f_2 \circ p \circ f_1^{-1}, f_1 \circ q \circ f_2^{-1})$$

for every $(p,q) \in A[X_1, Y_1]$.

Finally, we conclude that the isomorphism F is induced by the pair of affine isomorphisms (f_1, f_2) defined by the formulas (5) and (8). The proof of Theorem 3.1 is completed.

DEFINITION 3.2. Let X_i and Y_i for $i = 1,2$ be affine spaces over K. The ternary semigroups $A[X_1, Y_1]$ and $A[X_2, Y_2]$ are called W-isomorphic if there exists an isomorphism $F : A[X_1, Y_1] \to A[X_2, Y_2]$ of the ternary semigroups $A[X_1, Y_1]$ and $A[X_2, Y_2]$ fulfilling the conditions (W_1) and (W_2).

From Theorem 3.1 we deduce the following two corollaries.

COROLLARY 3.1. *Let X_i and Y_i for $i = 1,2$ be affine spaces over a field K. The ternary semigroups $A[X_1, Y_1]$ and $A[X_2, Y_2]$ are W-isomorphic if and only if the affine spaces X_1 and X_2 are isomorphic and the affine spaces Y_1 and Y_2 are isomorphic.*

COROLLARY 3.2. *Let X_i and Y_i for $i = 1,2$ be affine spaces over a field K. Let $G : A[X_1, Y_1] \to A[X_2, Y_2]$ be an isomorphism of the ternary semigroups $A[X_1, Y_1]$ and $A[X_2, Y_2]$. The affine spaces X_1 and X_2 are isomorphic and the affine spaces Y_1 and Y_2 are isomorphic if and only if there exists an automorphism μ of the ternary semigroup $A[X_1, Y_1]$ such that the isomorphism $F = G \circ \mu$ satisfies the conditions (W_1) and (W_2).*

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INSTITUTE OF MATHEMATICS
PEDAGOGICAL UNIVERSITY
Podchorążych 2
30-084 KRAKÓW, POLAND

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