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ON SUMMATION FORMULAS INDUCED BY FUNCTIONAL SHIFTS OF RIGHT INVERTIBLE OPERATORS

0. Let X be a linear space over the field $\mathbb{C}$ of the complex numbers. Denote by $L(X)$ the set of all linear operators with domains and ranges in X and by $L_0(X)$ the set of those operators from $L(X)$ which are defined on the whole space X. An operator $D \in L(X)$ is said to be *right invertible* if there exists an operator $R \in L(X)$ such that $DR = I$. The set of all right invertible operators belonging to $L(X)$ will be denoted by $R(X)$. For a given $D \in R(X)$ we denote by $\mathcal{R}_D$, $\mathcal{F}_D$ the set of all its right inverses , initial operators, respectively. The theory of right invertible operators and its applications is presented by D. Przeworska-Rolewicz in [14].

We admit here and in the sequel that $\mathcal{R}_D \subset L_0(X)$, $\dim\ker D > 0$, i.e. D is right invertible but not invertible and $0^0 := 1$. We also write: $\mathbb{N}$ for the set of all positive integers and $\mathbb{N}_0 := \{0\} \cup \mathbb{N}$.

For a given operator $D \in R(X)$ we shall write (cf. [12], [15], [20]):

$$D_\infty := \bigcap_{k\in\mathbb{N}_0} D_k, \tag{0.1}$$

where $D_0 = X$, $D_k = \operatorname{dom} D^k$ $(k \in \mathbb{N})$,

$$S := \bigcup_{i=1}^{\infty} \ker D^i, \tag{0.2}$$

$$E := \bigcup_{\lambda\in\mathbb{C}} \ker(D - \lambda I). \tag{0.3}$$

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Elements of the set D_∞ are said to be *smooth elements*. If $R \in \mathcal{R}_D$ then the set S is equal to the linear span $P(R)$ of all D-monomials , i.e.

$$(0.4) \qquad S = P(R) := \operatorname{lin}\{R^k z : z \in \ker D,\ k \in \mathbb{N}_0\}.$$

Evidently, the set $P(R)$ is independent of the choice of the right inverse R.

1. The purpose of the present section is to give an analogue of the Euler–Maclaurin Formmula (cf. [11], [18]) induced by functional shifts (cf. [4], [5]). We shall present some examples of applications of formulas obtained for some elementary functions.

In this section, K will stand either for a disk $K_\rho := \{h \in \mathbb{C} : |h| < \rho\}$, $0 < \rho < \infty$, or for the complex plane $\mathbb{C}$. Denote by $H(K)$ the class of all functions analytic on the set $K \subseteq \mathbb{C}$. Suppose that a function $f \in H(K)$ has the following expansion

$$(1.1) \qquad f(h) = \sum_{k=0}^{\infty} a_k h^k \quad \text{for all } h \in K.$$

DEFINITION 1.1. Suppose that $D \in R(X)$. A family $T_{f,K} = \{T_{f,h}\}_{h \in K} \subset L_0(X)$ is said to be a family of *functional shifts* for the operator D induced by the function f if

$$(1.2) \qquad T_{f,h}x = [f(hD)]x := \sum_{k=0}^{\infty} a_k h^k D^k x \quad \text{for all } h \in K;\ x \in S,$$

where S is defined by Formula (0.2).

We should point out that by definition of the set S, the last sum has only a finite number of members different than zero.

It is well-known that the set $H(K)$ is a commutative ring with the following algebraic operations:

$$(f+g)(h) = f(h) + g(h), \quad (\alpha g)(h) = \alpha g(h), (fg)(h) = f(h)g(h),$$

where $f, g \in H(K); \alpha \in \mathbb{C}$, $h \in K$.

Let $T(K)$ be the set of all families of functional shifts for an operator $D \in R(X)$ induced by the members of $H(K)$, i.e.

$$(1.3) \qquad T(K) := \{T_{g,K} : g \in H(K)\}.$$

Define the following operations

$$(1.4) \qquad T_{f,K} + T_{g,K} = T_{f+g,K}, \quad \alpha T_{g,K} = T_{\alpha g,K}, T_{fg,K} = T_{f,K}T_{g,K},$$

where $f, g \in H(\Omega)$; $\alpha \in \mathbb{C}$.

A special role in our considerations in this section will be played by the following:

THEOREM 1.1 (cf. [5]). *Suppose that $D \in R(X)$ and $T(K)$ is defined by Formula* (1.3). *Let $T_S(K) := T(K)|_S$, where S is defined by Formula* (0.2). *Then*

(i) *The set $T_S(K)$ is a commutative ring with the operations defined by Formulas* (1.4);

(ii) *The rings $H(K)$ and $T_S(K)$ are isomorphic. The mapping $T : f \Rightarrow T_{f,K}|_S$ is a ring isomorphism from $H(K)$ onto $T_S(K)$.*

Suppose that $D \in R(X)$ and $T_{f,K} = \{T_{f,h}\}_{h \in K}$ is a family of functional shifts for D induced by the function $f \in H(K)$. We consider the so-called **Pommiez** type operators $\mathfrak{P}$, P defined as follows:

$$\mathfrak{P}T_{f,h} := \begin{cases} h^{-1}(T_{f,h} - T_{f,0}) & \text{for } 0 \neq h \in K \\ DT_{f^{(1)},0} & \text{for } h = 0 \end{cases} \quad \text{on } X, \tag{1.5}$$

$$(Pf)(h) := \begin{cases} h^{-1}(f(h) - f(0)) & \text{for } 0 \neq h \in K \\ f^{(1)}(0) & \text{for } h = 0 \end{cases}, \tag{1.6}$$

where, as usual, $f^{(n)} = d^n f/dh^n$, $n \in \mathbb{N}$.

Observe that the function $Pf \in H(K)$ and has the following expansion

$$(Pf)(h) = \sum_{k=0}^{\infty} a_{k+1} h^k \quad \text{for all } h \in K. \tag{1.7}$$

This implies

$$P \in L_0(H(K)). \tag{1.8}$$

Note, for the function $g = \exp$ the operator $(Pg)(hD)$ is called the **Bernoulli operator** for D (cf. [18]).

With a help of the author's result (see [4], Proposition 1.3.) it is easy to prove

PROPOSITION 1.1. *Suppose that $D \in R(X)$ and $T_{f,K} = \{T_{f,h}\}_{h \in K}$ is a family of functional shifts for D induced by the function $f \in H(K)$. Then on the set S we have*

$$\mathfrak{P}T_{f,h} = (Pf)(hD)D, \quad \textit{for all } h \in K. \tag{1.9}$$

The fundamental result of this section is

THEOREM 1.2. *Suppose that all assumptions of Proposition* 1.1 *are satisfied, $f^{(1)}(0) \neq 0$ and $f(h) \neq f(0)$ for $0 \neq h \in K$. Then the operator $(Pf)(hD)$ is invertible on the set S for all $0 \neq h \in K$ and*

$$[(Pf)(hD)]^{-1} = g(hD),$$

where $g(h) = \sum_{k=0}^{\infty} b_k h^k \in H(K)$, *the coefficients* b_k $(k \in \mathbb{N}_0)$ *are determined by the following recursion relation*

$$b_0 = a_1^{-1},$$

$$b_n = -a_1^{-1} \sum_{k=0}^{n-1} b_k a_{n-k+1} \quad (n \in \mathbb{N}). \tag{1.10}$$

Proof. Our assumptions imply that the function

$$g(h) = \begin{cases} h[f(h) - f(0)]^{-1} & \text{for } 0 \neq h \in K, \\ [f^{(1)}(0)]^{-1} & \text{for } h = 0 \end{cases} \tag{1.11}$$

is a member of $H(K)$ and

$$(Pf)(h)g(h) = 1 \quad \text{for all } h \in K. \tag{1.12}$$

Theorem 1.1 implies that the operator $(Pf)(hD), (h \in K)$ is invertible on S and $[(Pf)(hD)]^{-1} = g(hD)$. Suppose that the function $g(h) = \sum_{k=0}^{\infty} b_k h^k$ for $h \in K$. Formula (1.7) and Formula (1.12) together imply

$$1 = (Pf)(h)g(h) = \Big(\sum_{k=0}^{\infty} a_{k+1} h^k\Big)\Big(\sum_{k=0}^{\infty} b_k h^k\Big) = \sum_{k=0}^{\infty} c_k h^k \quad \text{for all } h \in K,$$

where $c_m = \sum_{k=0}^{\infty} b_k a_{m-k+1}$ for $m \in \mathbb{N}_0$. This implies Formula (1.10).

Theorem 1.2 and Formula (1.7) together imply

THEOREM 1.3. *Suppose that all assumptions of Theorem* 1.2 *are satisfied. Then* $T_{Pf,K}$ *is invertible in the ring* $T_S(K)$ *and its inverse* $[T_{Pf,K}]^{-1} = T_{g,K} \in T_S(K)$, *where the function* g *is defined by Formula* (1.11).

We are now ready to give the main theorem of this section.

THEOREM 1.4. *Suppose that all assumptions of Theorem* 1.2. *are satisfied. Then the following* **Euler–Maclaurin** *type formulas hold on the set* S

$$I = b_0(Pf)(hD) + \sum_{n=1}^{\infty} b_n h^n (Pf)(hD) D^n, \tag{1.13.a}$$

$$I = b_0(Pf)(hD) + \sum_{n=1}^{\infty} b_n h^n \mathfrak{P} T_{f,h} D^{n-1}, \tag{1.13.b}$$

where $0 \neq h \in K$, b_k $(k \in \mathbb{N}_0)$ *are determined by Formula* (1.10).

Proof. This follows directly from Theorem 1.3. and Formula (1.9).

EXAMPLE 1.1 (cf. [18]). Let $K = \mathbb{C}$ and $f = \exp$. Then Euler–Maclaurin Formula (1.13.b) has the following form

$$I = (P\exp)(hD) + \sum_{n=1}^{\infty} B_n h^{n-1} (n!)^{-1} (e^{hD} - I) D^{n-1} \quad \text{for } 0 \neq h \in K,$$

where B_n are Bernoulli numbers.

EXAMPLE 1.2. (The expansions of elementary functions which are used here can be found in [9].)

a) Let $K = K_{\Pi/2}$, f be the tangent function, tan. Then for $0 \neq h \in K$

$$\tan(h) = \sum_{n=1}^{\infty} \frac{2^{2n}(2^{2n-1}-1)}{(2n)!}|B_{2n}|h^{2n-1}, \quad (P\tan)(h) = h^{-1}\tan(h),$$

$$g(h) = 1/(P\tan)(h) = h[\tan(h)]^{-1} = 1 - \sum_{n=1}^{\infty} \frac{2^{2n}}{(2n)!}|B_{2n}|h^{2n},$$

where B_{2n}, as before, are Bernoulli numbers.

In this case Formula (1.13.b) has the form

$$I = \frac{\tan(hD)}{hD} - \sum_{n=1}^{\infty} \frac{2^{2n}}{(2n)!}|B_{2n}|h^{2n-1}\tan(hD)D^{2n-1}, \quad 0 \neq h \in K,$$

b) Let $K = K_{\Pi/2}$, f be the hyperbolic tangent function, tanh. Then for $0 \neq s \in K$

$$\tanh(s) = \sum_{n=1}^{\infty} \frac{2^{2n}(2^{2n-1}-1)}{(2n)!}B_{2n}s^{2n-1},$$

$$g(s) = 1/(P\tanh)(s) = s\coth(s) = 1 + \sum_{n=1}^{\infty} \frac{2^{2n}}{(2n)!}B_{2n}s^{2n},$$

where B_{2n} are Bernoulli numbers.

In this case Formula (1.13.b) has the form

$$I = \frac{\tanh(sD)}{sD} + \sum_{n=1}^{\infty} \frac{2^{2n}}{(2n)!}B_{2n}s^{2n-1}\tanh(sD)D^{2n-1}, \quad 0 \neq s \in K.$$

c) Let $K = K_{\Pi}$, f be the sine function, sin. Then for $0 \neq h \in K$

$$(P\sin)(h) = h^{-1}\sin(h),$$

$$g(h) = h/\sin(h) = h\operatorname{cosec}(h) = 1 + \sum_{n=1}^{\infty} \frac{2(2^{2n-1}-1)}{(2n)!}|B_{2n}|h^{2n},$$

where as before B_{2n} are Bernoulli numbers.

In this case Formula (1.13.b) has the form

$$I = \frac{\sin(hD)}{hD} + \sum_{n=1}^{\infty} \frac{2(2^{2n-1}-1)}{(2n)!}|B_{2n}|h^{2n-1}\sin(hD)D^{2n-1}$$

for all $0 \neq h \in K$,

d) Let $K = K_{\Pi}$, f be the hyperbolic sine function, sinh. Then for $0 \neq h \in K$

$$(P\sinh)(t) = t^{-1}\sinh(t),$$

$$g(t) = t/\sinh(t) = t\operatorname{cosech}(t) = 1 - \sum_{n=1}^{\infty} \frac{2(2^{2n-1}-1)}{(2n)!} B_{2n} t^{2n},$$

where B_{2n} are Bernoulli numbers.

In this case Formula (1.13.b) has the form

$$I = \frac{\sinh(tD)}{tD} - \sum_{n=1}^{\infty} \frac{2(2^{2n-1}-1)}{(2n)!} B_{2n} t^{2n-1} \sinh(tD) D^{2n-1}$$

for all $0 \neq t \in K$.

Clearly, Theorem 1.1 implies

Remark 1.1. Suppose that all assumptions of Proposition 1.1. are satisfied and $f(h) \neq 0$ for $h \in K$. Then the following formula holds

$$(1.14) \qquad I = \sum_{k=0}^{\infty} c_k h^k f(hD) D^n \quad \text{on } S,$$

where $h \in K \backslash \{0\}$, c_k are determined by the recursion relation

$$c_0 = a_0^{-1}, \quad c_n = -a_0^{-1} \sum_{k=0}^{n-1} c_k a_{n-k}, \quad n \in \mathbb{N}.$$

2. In the present section, Euler–Maclaurin type Formulas (1.13) for linear complete metric spaces, induced by functional shifts (cf. [7]) are established.

In this section, we assume that X is a complete linear metric space. In the sequel, for reader's convenience only K will stand either for the unit disk K or for the complex plane $\mathbb{C}$, the function $f \in H(K)$ has the expansion

$$(2.1) \qquad f(h) = \sum_{k=0}^{\infty} a_k h^k \quad \text{for all } h \in K.$$

For an operator $D \in R(X)$ we define the sets $S_f^{(n)}(D)$ $(n \in \mathbb{N}_0)$, $S_f(D)$, $S_f^{\infty}(D)$, $S_K(D)$ as follows

$$(2.2) \quad S_f^{(n)}(D) := \Big\{ x \in D_{\infty} : \sum_{k=0}^{\infty} a_k h^k D^{k+n} x \text{ is convergent for all } h \in K \Big\}, \quad n \in \mathbb{N},$$

$$(2.3) \qquad S_f(D) := S_f^{(0)}(D),$$

$$(2.4) \qquad S_f^\infty(D) := \bigcap_{n\in\mathbb{N}_0} S_f^{(n)}(D),$$

$$(2.5) \qquad S_K(D) := \bigcap_{g\in H(K)} S_g^\infty(D),$$

where D_∞ is the set of smooth elements defined by Formula (0.1).

PROPOSITION 2.1 (cf. [7]). *Suppose that $D \in R(X)$. Then*

(i) $S \subset S_f^\infty(D) \subset S_f(D) \subset D_\infty \subset \operatorname{dom} D$,
(ii) $E \subset S_f^\infty(D)$, $E \subset S_K(D)$, *for* $K = \mathbb{C}$,
where the sets S and E are defined by Formulas (0.2), (0.3), *respectively.*

As in Section 1, we take

DEFINITION 2.1. A family $T_{f,K} = \{T_{f,h}\}_{h\in K} \subset L_0(X)$ is said to be a family of *functional shifts* for an operator $D \in R(X)$ induced by the function f if

$$(2.6) \qquad T_{f,h}x = [f(hD)]x \quad \text{for all } h \in K;\ x \in S_f(D),$$

where the operator $f(hD)$ is defined by Formula (1.2), the set $S_f(D)$ is defined by Formula (2.3).

We need (cf. [7])

PROPOSITION 2.2. *Suppose that $D \in R(X)$ and $T_{f,K} = \{T_{f,h}\}_{h\in K}$ is a family of functional shifts for D induced by the function $f \in H(K)$. Let $E_\lambda = \ker(D - \lambda I) \neq \{0\}$ for a $\lambda \in K$. Then for all $h \in K$*

$$(2.7) \qquad T_{f,h}x = f(\lambda h)x \quad \textit{for all } x \in E_\lambda;$$

THEOREM 2.1 (cf. [7]). *Let $D \in R(X)$ and $T(K)$ be defined by Formula* (1.3). *Let $T_{E_\lambda}(K) := T(K)|_{E_\lambda}$, where $E_\lambda = \ker(D - \lambda I) \neq \{0\}$ and $\lambda \in K$. Then*

(i) *The set $T_{E_\lambda}(K)$ is a commutative ring with the operations defined by Formulas* (1.4);

(ii) *If $\lambda \neq 0$ then the rings $H(K)$ and $T_{E_\lambda}(K)$ are isomorphic. The mapping $T : f \Rightarrow T_{f,K}|_{E_\lambda}$ is an isomorphism of $H(K)$ onto $T_{E_\lambda}(K)$.*

We note that by our assumptions there always exists a number $\lambda \in K$ such that $E_\lambda \neq \{0\}$, for example $0 = \lambda \in K$.

Let $D \in R(X)$ and $E_\lambda \neq \{0\}$ for $0 \neq \lambda \in K$. We introduce, in a similar way as in Section 1 (cf.Formula (1.5)) the **Pommiez** type operator defined as follows

$$(2.8) \qquad \mathfrak{P}T_{f,h} := \begin{cases} \frac{f(hD)-f(0)I}{h} & \text{for } 0 \neq h \in K \\ f^{(1)}(0)D & \text{for } h = 0. \end{cases}$$

The equality $Dx = \lambda x$ for $x \in E_\lambda$ and Formula (2.7) together imply that on the set E_λ we have

$$\mathfrak{P}T_{f,h} = \begin{cases} \frac{f(\lambda h)-f(0)}{h} I & \text{for } 0 \neq h \in K \\ \lambda f^{(1)}(0) I & \text{for } h = 0. \end{cases} \tag{2.9}$$

Clearly, Formula (2.9) implies that the operator D commutes with $\mathfrak{P}$ on the set $T_{E_\lambda}(K)$.

We have a similar result to Proposition 1.1.

PROPOSITION 2.3. *Suppose that all assumptions of Proposition 2.2 are satisfied. Then on the set E_λ we have the formula*

$$\mathfrak{P}T_{f,h} = (Pf)(hD)D = (DPf)(hD) \quad \textit{for all } h \in K, \tag{2.10}$$

where the Pommiez operator $P \in L_0(H(K))$ is defined by Formula (1.6).

Proof. From the beginning, note that Formula (1.8) and Proposition 2.1(ii) together imply that $E_\lambda \subset S_{Pf}(K)$. By definition, and Formula (1.7) we get on E_λ

$$(DPf)(hD) = (Pf)(hD)D = \Big(\sum_{n=1}^{\infty} a_n h^{n-1} D^{n-1}\Big)D = h^{-1}\sum_{n=1}^{\infty} a_n h^n D^n$$

$$= h^{-1}(f(hD) - f(0)I) = h^{-1}(f(\lambda h)I - f(0)I) = \mathfrak{P}T_{f,h} \text{ for all } 0 \neq h \in K.$$

By definition, Formula (2.10) holds on E_λ for $h = 0$, also.

LEMMA 2.1. *Suppose that $D \in R(X)$, $R \in \mathcal{R}_D$ and $T_{f,K} = \{T_{f,h}\}_{h\in K}$ is a family of functional shifts for D induced by the function $f \in H(K)$. Then on the set $S_{Pf}(D)$ the equality*

$$(Pf)(hD) = \mathfrak{P}T_{f,h}R \tag{2.11}$$

holds for all $h \in K$, where $S_{Pf}(D)$ is determined by Formula (2.3).

Proof. Let $x \in S_{Pf}(D)$ and $0 \neq h \in K$ be arbitrarily fixed. Definition and Formula (1.7.) together imply

$$(Pf)(hD)x = \sum_{k=0}^{\infty} a_{k+1}h^k D^k x = \sum_{k=0}^{\infty} a_{k+1}h^k D^k DRx$$

$$= h^{-1}\sum_{k=0}^{\infty} a_{k+1}h^{k+1}D^{k+1}Rx = h^{-1}(f(hD)Rx - a_0 Rx)$$

$$= h^{-1}(f(hD) - f(0)I)Rx = \mathfrak{P}T_{f,h}Rx.$$

Evidently, Formula (2.11) holds for $h = 0$, also.

Note, the last equality implies that if $x \in S_{Pf}(D)$ then $Rx \in S_f(D)$. Clearly and conversly, if $Rx \in S_f(D)$ then $x \in S_{Pf}(D)$.

Lemma 2.1. implies

PROPOSITION 2.4. *Suppose that all assumptions of Lemma 2.1 are satisfied. Then Formula (2.11) holds on the set $S_K(D)$, where $S_K(D)$ is determined by Formula (2.5).*

Obviously, Proposition 2.1(ii) and the definition together imply

$$E_\lambda \subset S_K(D) \subset S_{Pf}(D).$$

Using Theorem 2.1, in a similar way as Theorem 1.2. (cf. Theorem 1.3) we prove

THEOREM 2.2. *Suppose that all assumptions of Proposition 2.2 are satisfied, $f^{(1)}(0) \neq 0$ and $f(h) \neq f(0)$ for $0 \neq h \in K$. Then $T_{Pf,K}$ is invertible in the ring $T_{E_\lambda}(K) = T(K)|_{E_\lambda}$ and its inverse $[T_{Pf,K}]^{-1} = T_{g,K} \in T_{E_\lambda}(K)$, where the function g is defined by Formula (1.11).*

Theorem 2.2, Proposition 2.3 together imply

THEOREM 2.3. *Suppose that all assumptions of Theorem 2.2 are satisfied. Then Formulas (1.13.a), (1.13.b) hold on the set E_λ.*

Evidently, Theorem 2.1. implies

Remark 2.1. Suppose that all assumptions of Proposition 2.2. are satisfied and $f(h) \neq 0$ for $h \in K$. Then Formula (1.14) holds on E_λ.

COROLLARY 2.1. *Suppose that all assumptions of Proposition 2.2 are satisfied and $\lambda \in K$. Let an operator $R \in \mathcal{R}_D$ be such that the operator $I - \lambda R$ is invertible. Then the set*

$$E_\lambda(R) := (I - \lambda R)^{-1}(\ker D) \subset E_\lambda,$$

(cf. [14]). This implies that Theorem 2.2, Theorem 2.3, Remark 2.1 also hold for $E_\lambda(R)$. If $K = \mathbb{C}$ then these same hold for the set E defined by Formula (0.3).

Evidently, the above corollary holds in the case when the operator D has a Volterra right inverse R, (cf. [14]).

Here, we shall show that Euler–Maclaurin type formulas hold on the set $S_K(D)$.

In [7] was proved the following

PROPOSITION 2.5. *Suppose that all assumptions of Lemma 2.1 are satisfied and D is closed. Then for all $h \in K$ $T_{f,h}$ commute on the set $S_f(D) \cap S_f^{(1)}(D)$ with the operator D, where $S_f(D)$, $S_f^{(1)}(D)$ are determined by Formulas (2.3), (2.2), respectively.*

THEOREM 2.4. *Suppose that all assumptions of Proposition 2.5 are satisfied. Let $T_{S_K(D)} = T(K)|_{S_K(D)}$, where $T(K)$ is determined by Formula (1.3). Then*

(i) *the set* $T_{S_K(D)}$ *is a commutative ring with the operations defined by Formulas* (1.4);

(ii) *the rings* $H(K)$ *and* $T_{S_K(D)}$ *are isomorphic. The mapping* $T : f \Rightarrow T_{f,K}|_{S_K(D)}$ *is a ring isomorphism from* $H(K)$ *onto* $T_{S_K(D)}$.

Definition of the set $S_K(D)$, Formula (1.8) and Proposition 2.5. imply

PROPOSITION 2.6. *Suppose that all assumptions of Proposition* 2.5 *are satisfied. Then Formula* (2.10) *holds on the set* $S_K(D)$.

Using Theorem 2.4, in a similar way as Theorem 1.2 (cf. Theorem 1.3) we prove

THEOREM 2.5. *Suppose that all assumptions of Proposition* 2.5 *are satisfied,* $f^{(1)}(0) \neq 0$ *and* $f(h) \neq f(0)$ *for* $0 \neq h \in K$. *Then* $T_{Pf,K}$ *is invertible in the ring* $T_{S_K(D)}$ *and its inverse* $[T_{Pf,K}]^{-1} = T_{g,K} \in T_{S_K(D)}$, *where the function* g *is defined by Formula* (1.11).

Theorem 2.5. and Proposition 2.6. together imply

THEOREM 2.6. *Suppose that all assumptions of Theorem* 2.5 *are satisfied. Then Formulas* (1.13.a), (1.13.b) *hold on the set* $S_K(D)$.

Clearly, Theorem 2.4 implies

Remark 2.1. Suppose that all assumptions of Proposition 2.5. are satisfied and $f(h) \neq 0$ for $h \in K$. Then Formula (1.14) holds on $S_K(D)$.

3. In the present section, an isomorphism of a ring of analytic functions onto a ring of continuous shifts is established. As conseqeunces, Formulas (1.13) for a subset of the space of D-analytic elements (cf. [14], [15]) are given.

In this section we still assume that X is a complete linear metric space. As before,the function $f \in H(K)$ has expansion (2.1), where $K = K_\rho$, $0 < \rho \leq +\infty$. Let $D \in R(X)$ and F be an initial operator for D corresponding to an $R \in \mathcal{R}_D$. Write

$$A_R(D) := \left\{ x \in D_\infty : x = \sum_{n=0}^{\infty} R^n F D^n x \right\},$$

$$A(D) := \bigcup_{R \in \mathcal{R}_D} A_R(D).$$

The set $A(D)$ is said to be the space of D-analytic elements (cf. [15]).

It is obvious that

$$S \subset A(D) \subset D_\infty.$$

Denote by $T^c(K) \subset T(K)$ the set of all continuous functional shifts for $D \in R(X)$ induced by the set $H(K)$.

THEOREM 3.1 (cf. [7]). *Suppose that an operator $D \in R(X)$ and $T_{f,K} = \{T_{f,h}\}_{h\in K} \subset T^c(K)$ is a family of functional shifts for D induced by the function $f \in H(K)$. If an operator $R \in \mathcal{R}_D$ is continuous then $A_R(D) \subset S_f(D)$.*

The following theorem (similar to Theorems 1.1, 2.1, 2.6) for the set $T^c(K)|_{A_R(D)}$, where $R \in \mathcal{R}_D$ is continuous holds

THEOREM 3.2. *Let $D \in R(X)$ and $T^c_{A_R(D)} := T^c(K)|_{A_R(D)}$, where $R \in \mathcal{R}_D$ is continuous. Then*

(i) *the set $T^c_{A_R(D)}$ is a commutative ring with the operations defined by Formulas* (1.4);

(ii) *the rings $H(K)$ and $T^c_{A_R(D)}$ are isomorphic. The mapping $T : f \Rightarrow T^c_{f,K}|_{A_R(D)}$ is a ring isomorphism from $H(K)$ onto $T^c_{A_R(D)}$.*

Proof. (i) Evidently,it is enough to show that the multiplication of two continuous families of functional shifts for D which are restricted to the set $A_R(D)$ is well defined. Suppose that we are given families $T_{f,K}, T_{g,K} \in T^c(K)$, where $f, g \in H(K)$ are arbitrarily fixed and

$$f(h) = \sum_{k=0}^{\infty} a_k h^k, \quad g(h) = \sum_{k=0}^{\infty} b_k h^k \quad \text{for all } h \in K.$$

Theorem 3.1 implies that $A_R(D) \subset S_f(D), S_g(D), S_{fg}(D)$. Let F be an initial operator for D corresponding to the continuous operator $R \in \mathcal{R}_D$. Let $0 \neq x \in A_R(D)$ and $h \in K$ be arbitrarily fixed. Our assumptions and Proposition 1.1 imply

$$\begin{aligned}
[T_{f,h}T_{g,h}]x &= T_{f,h}[T_{g,h}x] = T_{f,h}\Big[T_{g,h}\sum_{n=0}^{\infty} R^n F D^n x\Big] \\
&= T_{f,h}\Big[\sum_{n=0}^{\infty} T_{g,h} R^n F D^n x\Big] = T_{f,h}\Big[\sum_{n=0}^{\infty}\sum_{k=0}^{n} b_{n-k}h^{n-k}R^k F D^n x\Big] \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{n} b_{n-k}h^{n-k} T_{f,h} R^k F D^n x \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{n} b_{n-k}h^{n-k} \sum_{j=0}^{k} a_{k-j}h^{k-j} R^j F D^n x \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{j=0}^{k} a_{k-j} b_{n-k} h^{n-j} R^j F D^n x
\end{aligned}$$

$$= \sum_{n=0}^{\infty}\sum_{j=0}^{n}\Big(\sum_{k=j}^{n} a_{k-j}b_{n-k}\Big)h^{n-j}R^{j}FD^{n}x$$

$$= \sum_{n=0}^{\infty}\sum_{j=0}^{n} c_{n-j}h^{n-j}R^{j}FD^{n}x, \text{ where } c_q = \sum_{p=0}^{q} a_p b_{q-p} \ (q \in \mathbb{N}_0).$$

Clearly, if $\mathbf{a} = \{a_q\}$, $\mathbf{b} = \{b_q\}$ then the sequence $\mathbf{c} = \{c_q\}$ is a convolution of $\mathbf{a}$ and $\mathbf{b}$, i.e. $\mathbf{c} = \mathbf{a} * \mathbf{b} = (\sum_{p=0}^{q} a_p b_{q-p})$.

Take $w(h) = f(h)g(h)$ then $w \in H(K)$ and $w(h) = \sum_{n=0}^{\infty} c_n h^n$, $h \in K$. This follows from the Cauchy theorem about multiplication of two series.

Hence,

$$[T_{f,h}T_{g,h}]x = \sum_{n=0}^{\infty} T_{w,h}R^{n}FD^{n}x = T_{w,h}\Big(\sum_{n=0}^{\infty} R^{n}FD^{n}x\Big) = T_{w,h}x$$

for $T_{w,K} = T_{fg,K} \in T^{c}(K)$.

(ii) The proof is along going the same lines as the proofs of the mentioned theorems which concern the sets S, E_λ and $S_K(K)$ (cf. [5], [7]).

PROPOSITION 3.1. *Suppose that all assumptions of Theorem 3.1 are satisfied and $R \in \mathcal{R}_D$ is continuous. Then following formula*

$$\mathfrak{P}T_{f,h} = (Pf)(hD)D \tag{3.1}$$

holds on the set $A_R(D)$ for all $h \in K$.

Proof. Fix $x \in A_R(D)$ and take $y = Dx$. The continuity of $R \in \mathcal{R}_D$ implies that

$$A_R(D) \subset S_{Pf}(D), \quad D(A_R(D)) \subset A_R(D).$$

First inclusion follows from Theorem 3.1. and Formula (1.8) , second from [15] (Theorem 3.2 p. 22). This and Formula (1.7) together imply

$$[(Pf)(hD)D]x = (Pf)(hD)y = \sum_{n=1}^{\infty} a_n h^{n-1}D^{n-1}y = \sum_{n=1}^{\infty} a_n h^{n-1}D^{n-1}Dx$$

$$= h^{-1}\sum_{n=1}^{\infty} a_n h^{n}D^{n}x = h^{-1}(f(hD) - f(0)I) = \mathfrak{P}T_{f,h} \quad \text{for all } 0 \neq h \in K.$$

For $h = 0$ Formula (3.1) follows from definition.

THEOREM 3.3. *Suppose that all assumptions of Proposition 3.1 are satisfied, $f^{(1)}(0) \neq 0$ and $f(h) \neq f(0)$ for $0 \neq h \in K$. Then $T^{c}_{Pf,K}$ is invertible in the ring $T^{c}_{A_R(D)}$ and its inverse $[T_{Pf,K}]^{-1} = T^{c}_{g,K} \in T^{c}_{A_R(D)}$, where the function g is defined by Formula (1.11).*

Theorem 3.3.and Proposition 3.1. together imply

THEOREM 3.4. *Suppose that all assumptions of Theorem* 3.3 *are satisfied. Then Formulas* (1.13.a), (1.13.b) *hold on the set* $A_R(D)$.

Clearly, Theorem 3.3 implies

Remark 3.1. Suppose that all assumptions of Proposition 3.1. are satisfied and $f(h) \neq 0$ for $h \in K$. Then Formula (1.14) holds on $A_R(D)$.

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