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STRICT DISCONJUGACY CRITERIA FOR LINEAR VECTOR DIFFERENTIAL EQUATIONS WITH DELAYS

In [3], the author has introduced the notions of disconjugate and strictly disconjugate linear differential equations with delay (which are generalizations of a similar notion for ordinary differential equations without delay) as well as new types of multipoint boundary value problems for differential equations with delay (which are generalizations of de la Vallee Poussin's multipoint boundary value problem for ordinary differential equations without delay). It turns out that such equations are disconjugate (strictly disconjugate) iff each generalized boundary value problem has exactly one solution. The purpose of this paper is to derive the tests for strict disconjugacy of linear vector differential equations with delays on a compact interval.

Let us consider the n-th order linear vector differential equation with delays

$$(E_n^v) \qquad x^{(n)}+\sum_{i=1}^{n}\sum_{j=1}^{m}A_{ij}(t)x^{(n-i)}(t-\Delta_j(t))=\Theta,$$
$$x^{(0)}\equiv x, \qquad n\geq 2,$$

where the coefficients $A_{ij}(t)$, $(i=1,\dots,n;\ j=1,\dots,m)$ are real $v\times v$ matrix functions; x is the v-dimensional vector-column $x=(x_1,\dots,x_v)^T$ and $\Delta_j(t)\geq 0$, $j=1,\dots,m$ are delays; $t\in I=\langle a,b\rangle$ and $\Theta:=(0,\dots,0)$.

By a solution of (E_n^v) we shall mean any function with an absolutely continuous derivative of order $n-1$, satisfying (E_n^v) almost everywhere.

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For $t_0 \in \langle a,b)$ let us denote by $B'(E_n^v,t_0)$ the set of all solutions of (E_n^v) with constant initial vector functions

$$\phi_0(t),\phi_1(t),\dots,\phi_{n-1}(t)$$

defined on the initial set E_{t_0}.

A point $\xi \in \langle t_0,b)$ will be called a zero point of order p of a solution $x \in B'(E_n^v,t_0)$ iff

$$x(\xi)=\dots=x^{(p-1)}(\xi)=\Theta \quad \text{and} \quad x^{(p)}(\xi)\neq\Theta.$$

If $p=1$, such a point will be referred to as a simple zero point of x. The solution $x(t)=\Theta$, $t\in\langle t_0,b)$ (in this case $\phi_i(t)=\Theta$, $i=0,\dots,n-1$ for $t\in E_{t_0}$) will be called trivial.

Let $x\in B'(E_n^v,t_0)$, $x(t)\not\equiv\Theta$ on $\langle t_0,b)$. The n-th consecutive zero (including multiplicity) of $x(t)$ to the right of t_0 will be denoted by $\eta(x,t_0)$.

DEFINITION 1. Let $c\in I$. By the first adjoint point to the point c (with respect to (E_n^v)) we mean the point

$$\alpha_1(c):=\inf\{\eta(x,c): x\in B'(E_n^v,c), x(t)\not\equiv\Theta\}.$$

DEFINITION 2. The equation (E_n^v) is said to be strictly disconjugate on an interval J iff for each $c\in J$ the following implication holds

$$c\in J \Rightarrow \alpha_1(c)\notin J.$$

By $C^n(I,R)$ we denote the class of real-valued functions defined on an interval I with a continuous derivative of the n-th order.

Let $w\in C^n(\langle a,b\rangle,R)$ and $\Delta(t)\geq 0$ for $t\in\langle a,b\rangle$. We define $w^{(i)}(t-\Delta(t))$, $(i=0,1,\dots,n)^{t\in\langle a,b\rangle}$ as follows

$$w^{(i)}(t-\Delta(t)):=\begin{cases} w^{(i)}(t-\Delta(t)), & \text{iff } t-\Delta(t)\in\langle a,b\rangle \\ w^{(i)}(a) & \text{iff } t-\Delta(t)<a. \end{cases}$$

In the following we shall put $\sum_{k=1}^{0} a_k=0$.

Before establishing tests for disconjugacy, we give some lemmas which will be needed in the sequel.

LEMMA 1. *Let $a\leq t_1\leq\dots\leq t_n\leq b$, $q\geq 1$. Then*

$$\int_a^b |(t-t_1)\dots(t-t_n)|^q\,dt \leq \frac{(b-a)^{nq+1}}{nq+1}.$$

Proof. The above inequality follows from the proof of Lemma 2.1 in [8].

LEMMA 1$'$. *Let $a\leq t_1\leq\dots\leq t_n\leq b$, $q\geq 1$ and $\Delta(t)\in C(\langle a,b\rangle,R)$, $\Delta(t)\geq 0$, for $t\in\langle a,b\rangle$.*

Then

$$L := \Big[\int_a^b |(t - \Delta(t) - t_1)\dots(t - \Delta(t) - t_n)|^q \, dt\Big]^{1/q}$$

$$\le \frac{(b - a :^{n+1/q}}{(nq + 1)^{1/q}} + \sum_{k=1}^{n} \Big[\binom{n}{n-k} \frac{(b - a)^{n-k)2q+1}}{(n - k)2q + 1} \int_a^b \Delta^{k2q}(t)\, dt\Big]^{1/2q}.$$

P r o o f. Multiplying the terms in absolute value and then using Minkowski's Inequality we get

$$I \le \Big[\int_a^b |(t - t_1)\dots(t - t_n)|^q \, dt\Big]^{1/q}$$

$$+ \Big[\int_a^b \Delta^q(t)|(t - t_2)\dots(t - t_n) + \dots + (t - t_1)\dots(t - t_{n-1})|^q \, dt\Big]^{1/q}$$

$$+ \Big[\int_a^b \Delta^{2q}(t)|(t - t_3)\dots(t - t_n) + \dots + (t - t_1)\dots(t - t_{n-2})|^q \, dt\Big]^{1/q}$$

$$+ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$+ \Big[\int_a^b \Delta^{(n-1)q}(t)|(t - t_1) + \dots + (t - t_n)|^q \, dt\Big]^{1/q} + \Big[\int_a^b \Delta^{nq}(t)\, dt\Big]^{1/q}.$$

Now applying the Holder's Inequality (with $p = 2$) to the right hand members of the above inequality we get

$$I \le \Big[\int_a^b |(t - t_1)\dots(t - t_n)|^q \, dt\Big]^{1/q} + \Big[\int_a^b \Delta^{2q}(t)\, dt\Big]^{1/2q}$$

$$\cdot \Big[\int_a^b |(t - t_2)\dots(t - t_n) + \dots + (t - t_1)\dots(t - t_{n-1})|^{2q} \, dt\Big]^{1/2q}$$

$$+ \Big[\int_a^b \Delta^{2.2q}(t)\, dt\Big]^{1/2q}$$

$$\cdot \Big[\int_a^b |(t - t_3)\dots(t - t_n) + \dots + (t - t_1)\dots(t - t_{n-2})|^{2q} \, dt\Big]^{1/2q}$$

$$+ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$+ \Big[\int_a^b \Delta^{(n-1)2q}(t)\, dt\Big]^{1/2q} \Big[\int_a^b |(t - t_1) + \dots + (t - t_n)|^{2q} \, dt\Big]^{1/2q}$$

$$+\Big[\int_a^b \Delta^{n2q}(t)\,dt\Big]^{1/2q}\Big[\int_a^b dt\Big]^{1/2q}$$

and by Minkowski's Inequality we have

$$I \le \Big[\int_a^b |(t-t_1)\dots(t-t_n)|^q\,dt\Big]^{1/q} + \Big[\int_a^b \Delta^{2q}(t)\,dt\Big]^{1/2q}$$

$$\cdot\Big\{\Big[\int_a^b |(t-t_2)\dots(t-t_n)|^{2q}\,dt\Big]^{\frac{1}{2q}}+\dots+\Big[\int_a^b |(t-t_1)\dots(t-t_{n-1})|^{2q}dt\Big]^{\frac{1}{2q}}\Big\}$$

$$+\Big[\int_a^b \Delta^{2.2q}(t)dt\Big]^{1/2q}\Big\{\Big[\int_a^b |(t-t_3)\dots(t-t_n)|^{2q}\,dt\Big]^{\frac{1}{2q}}+\dots$$

$$+\Big[\int_a^b |(t-t_1)\dots(t-t_{n-2})|^{2q}\,dt\Big]^{\frac{1}{2q}}\Big\}$$

$$+ \;\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$+\Big[\int_a^b \Delta^{(n-1)2q}(t)\,dt\Big]^{1/2q}\cdot\Big\{\Big[\int_a^b |t-t_1|2q\,dt\Big]^{\frac{1}{2q}}+\dots+\Big[\int_a^b |t-t_n|^{2q}\,dt\Big]^{\frac{1}{2q}}\Big\}$$

$$+\Big[\int_a^b \Delta^{n2q}(t)\,dt\Big]^{1/2q}\Big[\int_a^b dt\Big]^{1/2q}.$$

Finally using Lemma 1 to evaluate the right hand members of the last inequality we get the inequality of Lemma 1′.

LEMMA 2. *Let the function $w \in C^n(\langle a,b\rangle, R)$ have at least n zeros (including multiplicity) in the interval $\langle a,b\rangle$, $q\ge 1$ and $\mu := \max_{t\in\langle a,b\rangle}|w^{(n)}(t)|$. Then*

$$I_1 := \Big[\int_a^b |w(t-\Delta(t))|^q\,dt\Big]^{1/q}$$

$$\le \frac{\mu}{n!}\Big\{\frac{(b-a)^{n+1/q}}{(nq+1)^{1/q}}+\sum_{k=1}^{n}\Big[\binom{n}{n-k}\frac{(b-a)^{(n-k)2q+1}}{(n-k)2q+1}\int_a^b \Delta^{k2q}(t)\,dt\Big]^{1/2q}\Big\}.$$

Proof. If $t_1,\dots,t_n$ denote the succesive zeros of $w(t)$ (where $a\le t_1\le \dots \le t_n \le b$) we have

$$(1)\qquad |w(t)| \le \frac{\mu}{n!}|(t-t_1)\dots(t-t_n)|,\qquad t\in\langle a,b\rangle$$

(see [7], p. 156). Moreover, the following inequality holds as well

$$(2)\quad |w(t-\Delta(t))| \le \frac{\mu}{n!}|(t-\Delta(t))-t_1)\dots(t-\Delta(t)-t_n)|, \qquad t\in\langle a,b\rangle.$$

In fact, if i) $t-\Delta(t)\in\langle a,b\rangle$, then (2) follows from (1),
if ii) $t-\Delta(t)<a$, then (see the definition of $w(t-\Delta(t))$

$$\begin{aligned}|w(t-\Delta(t))| = |w(a)| &\le \frac{\mu}{n!}|(a-t_1)\dots(a-t_n)|\\ &\le \frac{\mu}{n!}|(t-\Delta(t)-t_1)\dots(t-\Delta(t)-t_n)|.\end{aligned}$$

Lemma 2 now follows from the inequality (2) and Lemma 1'.

If $w\in C^n(\langle a,b\rangle,R)$ has at least n zeros (including multiplicity) in $\langle a,b\rangle$, then the function $w^{(n-i)}(t)$ $(i=1,\dots,n)$ belongs to $C^i(\langle a,b\rangle,R)$ and has at least i zeros in $\langle a,b\rangle$. Applying Lemma 2 to $w^{(n-i)}(t)$, we get

COROLLARY 1. *Let the function $w(t)$ satisfy the assumptions of Lemma 2. Then*

$$I_2 := \Big[\int_a^b |w^{(n-i)}(t-\Delta(t))|^q\,dt\Big]^{1/q}$$
$$\le \frac{\mu}{i!}\left\{\frac{(n-a)^{i+1/q}}{(iq+1)^{1/q}} + \sum_{k=1}^{i}\left[\binom{i}{i-k}\frac{(b-a)^{(i-k)2q+1}}{(i-k)2q+1}\int_a^b \Delta^{k2q}(t)\,dt\right]^{\frac{1}{2q}}\right\},$$

$i=0,1,\dots,n-1$.

Now let us denote by $\|x\| := \sum_{k=1}^{v}|x_k|$ the norm of an element $x\in R^v$ and by $\|A\| := max_{j=1,\dots,v}\sum_{k=1}^{v}|a_{kj}|$ the norm of the matrix $A=(a_{kj})$.

Let $z=(z_1,\dots,z_v)\in C^n(\langle a,b\rangle,R^v)$ (i.e. $z_k\in C^n(\langle a,b\rangle,R)$, $k=1,\dots,v$) and $\Delta(t)\ge 0$, $t\in\langle a,b\rangle$. Then by $z^{(i)}(t-\Delta(t))$, $(i=0,1,\dots,n)$ $t\in\langle a,b\rangle$ we shall mean

$$z^{(i)}(t-\Delta(t)) := \begin{cases} z^{(i)}(t-\Delta(t) & \text{iff } t-\Delta(t)\in\langle a,b\rangle\\ z^{(i)}(a) & \text{iff } t-\Delta(t)\le a.\end{cases}$$

LEMMA 3. *Let the function $z\in C^n(\langle a,b\rangle,R^v)$ have at least n zeros (including multiplicity) in $\langle a,b\rangle$, $q\ge 1$ and let $\mu := (\sum_{k=1}^{v}\mu_k^q)^{1/q}$, where*

$$\mu_k := max_{t\in\langle a,b\rangle}|z_k^{(n)}(t)|, \qquad k=1,\dots,v.$$

Then

$$(3)\quad I_3 := \Big[\int_a^b \|z(t-\Delta(t))\|^q\,dt\Big]^{1/q}$$
$$\le \mu\frac{v^{1-1/q}}{n!}\Big\{\frac{(b-a)^{n+1/q}}{(nq+1)^{1/q}} + \sum_{k=1}^{n}\Big[\binom{n}{n-k}\frac{(b-a)^{(n-k)2q+1}}{(n-k)2q+1}\int_a^b \Delta^{k2q}(t)\,dt\Big]^{\frac{1}{2q}}\Big\}.$$

Proof. Let $q=1$. Then

$$(4)\qquad \int_a^b \|z(t-\Delta(t))\|\,dt = \sum_{k=1}^{v}\int_a^b |z_k(t-\Delta(t))|\,dt.$$

Now to prove Lemma 3 it suffices to utilize Lemma 2 for the evaluation of the integrals on the right hand side of (4). Let $q>1$. Then, applying Hölder's Inequality,

$$I_3 := \Big[\int_a^b \|z(t-\Delta(t))\|^q\,dt\Big]^{1/q} \le \Big[v^{q-1}\sum_{k=1}^{v}\int_a^b |z_k(t-\Delta(t))|^q\,dt\Big]^{1/q}.$$

Using now Lemma 2 for the evaluation of the right hand term of this inequality we get (3).

LEMMA 4. *Let the assumptions of Lemma 3 be satisfied. Then*

$$I_4 := \Big[\int_a^b \|z^{(n-i)}(t-\Delta(t))\|^q\,dt]^{1/q}$$
$$\le \mu\frac{v^{1-1/q}}{i!}\Big\{\frac{(b-a)^{i+1/q}}{(iq+1)^{1/q}} + \sum_{k=1}^{i}\Big[\binom{i}{i-k}\frac{(b-a)^{(i-k)2q+1}}{(i-k)2q+1}\int_a^b \Delta^{k2q}(t)\,dt\Big]^{\frac{1}{2q}}\Big\},$$

$i=0,1,\dots,n-1$.

Proof. The proof of Lemma 4 proceeds analogically as that of Lemma 3 but Corollary 1 should be used instead of Lemma 2.

THEOREM 1. *Let the coefficients $A_{ij}(t)$, $(i=1,\dots,n; j=1,\dots,m)$ be measurable and bounded on $\langle a,b\rangle(\|A_{ij}(t)\| \le A_{ij} \in R, t\in\langle a,b\rangle)$, $\Delta_j(t)\ge 0$, $j=1,\dots,m$ be continuous on $\langle a,b\rangle$ and let the following inequality hold*

$$(5)\quad \sum_{i=1}^{n}\sum_{j=1}^{m}\frac{A_{ij}}{(i-1)!}$$
$$\times\Big\{\frac{(b-a)^i}{i} + \sum_{k=1}^{i-1}\Big[\binom{i-1}{i-1-k}\frac{(b-a)^{2(i-k)-1}}{2(i-k)-1}\int_a^b \Delta_j^{2k}(t)dt\Big]^{\frac{1}{2}}\Big\} < 1.$$

Then the differential equation (E_n^v) is strictly disconjugate on $\langle a,b\rangle$.

Proof. Suppose on the contrary that (E_n^v) is not strictly disconjugate on $\langle a,b\rangle$. Then there is a point $t_0 \in \langle a,b)$ and a nontrivial solution $x \in B'(E_n^v, t_0)$ with at least n zeros (including multiplicity) in $\langle t_0, b\rangle$. This fact by Rolle's Theorem yields the existence $\tau_k \in \langle t_0, b\rangle$, $k = 1,\dots,v$ such that

$$x_k^{(n-1)}(\tau_k) = 0.$$

Therefore

$$x_k^{(n-1)}(t) = \int_{\tau_k}^{t} x_k^{(n)}(s)\,ds, \qquad k = 1,\dots,v$$

from which we get

$$|x_k^{(n-1)}(t)| \le \int_{t_0}^{b} |x_k^{(n)}(s)|\,ds$$

and thus

$$\delta := \sum_{k=1}^{v} \max_{t\in\langle t_0,b\rangle} |x_k^{(n-1)}(t)| \le \int_{t_0}^{b} \Big(\sum_{k=1}^{v} |x_k^{(n)}(s)|\Big)\,ds = \int_{t_0}^{b} \|x^{(n)}(s)\|\,ds.$$

Since $x \in B'(E_n^v, t_0)$, we have

$$(6) \qquad \delta \le \sum_{i=1}^{n}\sum_{j=1}^{m} \int_{t_0}^{b} \|A_{ij}(t)\|\|x^{(n-i)}(t-\Delta_j(t))\|\,dt$$

$$\le \sum_{i=1}^{n}\sum_{j=1}^{m} A_{ij} \int_{t_0}^{b} \|x^{(n-i)}(t-\Delta_j(t))\|\,dt.$$

In view of Lemma 3 and Lemma 4 applied to $x \in C^{n-1}(\langle a,b\rangle, R^v)$ with $q = 1$, we see that

$$\delta \le \delta \sum_{i=1}^{n}\sum_{j=1}^{m} \frac{A_{ij}}{(i-1)!}$$

$$\times\left\{\frac{(b-a)^i}{i} + \sum_{k=1}^{i-1}\left[\binom{i-1}{i-1-k}\frac{(b-a)^{2(i-k)-1}}{2(i-k)-1}\int_a^b \Delta_j^{2k}(t)\,dt\right]^{\frac12}\right\}.$$

Since $x \not\equiv 0$ in $\langle t_0, b\rangle$, then $\delta > 0$. Dividing the above inequality by δ we obtain an inequality which is opposite to (5).

LEMMA 5. *For $x_i \ge 0$, $p > 1$ we have*

$$(7) \qquad \Big(\sum_{i=1}^{n} x_i^p\Big)^{1/p} \le \sum_{i=1}^{n} x_i.$$

Proof. Inequality (7) is an immediate consequence of Theorem I.19.5 of [1] which is formulated as follows: For $x_i \geq 0$, $p > 1$, we have

$$\Big(\sum_{i=1}^{n} x_i^p\Big)^{1/p} = \max_{R(y)} \sum_{i=1}^{n} x_i y_i$$

where $R(y)$ is defined by

$$\sum_{i=1}^{n} y_i^q = 1, \quad y_i \geq 0 \quad \text{and} \quad \frac{1}{p} + \frac{1}{q} = 1.$$

THEOREM 2. *Let the coefficients $A_{ij}(t)$, $(i = 1, \dots, n; j = 1, \dots, m)$ be such that $\|A_{ij}(t)\| \in L^p(\langle a, b\rangle)$, $\Delta_j(t) \geq 0$, $j = 1, \dots, m$ be continuous on $\langle a, b\rangle, p > 1$ and let the inequality*

$$\begin{aligned}(8)\quad & v^{1/p} \sum_{i=1}^{n} \sum_{j=1}^{m} \frac{1}{(i-1)!} \Big\{ \frac{(b-a)^{i-1+1/q}}{((i-1)q+1)^{1/q}} + \\ & + \sum_{k=1}^{i-1} \Big[\binom{i-1}{i-1-k} \frac{(b-a)^{(i-1-k)2q+1}}{(i-1-k)2q+1} \int_a^b \Delta_k^{k2q}(t)\, dt \Big]^{\frac{1}{2q}} \Big\} \times \\ & \times \|A_{ij}(t)\|_p < 1,\end{aligned}$$

(where $\|A_{ij}(t)\|_p := [\int_a^b \|A_{ij}(t)\|^p dt]^{1/p} \cdot \frac{1}{p} + \frac{1}{q} = 1$) hold.

Then the differential equation (E_n^v) is strictly disconjugate on $\langle a, b\rangle$.

Proof. The first part of the proof of Theorem 2 is the same as that of Theorem 1. Hence we shall start with the inequality (6) from which by Lemma 5 we have

$$\mu \leq \sum_{i=1}^{n} \sum_{j=1}^{m} \int_{t_0}^{b} \|A_{ij}(t)\| \|x^{(n-i)}(t - \Delta_j(t))\|\, dt,$$

where

$$\mu := \Big(\sum_{k=1}^{v} \mu_k^q\Big)^{1/q}, \quad \mu_k := \max_{t \in \langle t_0, b\rangle} |x_k^{(n-1)}(t)|, k = 1, \dots, v.$$

Applying Hölder's Inequality to the integral on the right hand side of the above inequality, we get

$$\mu \leq \sum_{i=1}^{n} \sum_{j=1}^{m} \Big(\int_{t_0}^{b} \|A_{ij}(t)\|^p\, dt\Big)^{1/p} \Big(\int_{t_0}^{b} \|x^{(n-i)}(t - \Delta_j(t))\|^q\, dt\Big)^{1/q}.$$

Using Lemma 4 (applied to $x \in C^{n-1}(\langle t_0, b\rangle, R^v)$) to evaluate the last term of the above inequality and then magnifying the right hand side of this inequality by writing a instead of t_0, we obtain

$$\mu \leq \mu \sum_{i=1}^{n}\sum_{j=1}^{m} \frac{v^{1-1/q}}{(i-1)!} \times \left\{ \frac{(b-a)^{i-1+1/q}}{((i-1)q+1)^{1/q}} + \right.$$
$$\left. + \sum_{k=1}^{i-1} \left[\binom{i-1}{i-1-k} \frac{(b-a)^{(i-1-k)2q+1}}{(i-1-k)2q+1} \int_a^b \Delta_j^{k2q}(t)\, dt \right]^{\frac{1}{2q}} \right\} \times$$
$$\times \left[\int_a^b \|A_{ij}(t)\|^p\, dt \right]^{1/p}.$$

Dividing this inequality by $\mu > 0$ we obtain an inequality which is contrary to (8).

Remark 1. Theorem 1 and Theorem 2 are valid also in the case $\Delta_j(t) = 0$, $t \in \langle a, b\rangle; j = 1, \ldots, m$. In this special case they are reduced almost to the respective criteria for disconjugacy of linear vector differential equations without delay proved in [8]: in these criteria in (5) resp. (8) the sign $\leq$ occurs. This is caused by the fact, that in $B'(E_n^v, t_0)$ there are nontrivial solutions with a zero point of multiplicity n.

Remark 2. In the case $v = 1$ Theorem 1 and Theorem 2 provide criteria for strict disconjugacy of scalar linear differential equations which are supplementary to that of [4].

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