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ON THE CONVERGENCE OF THE ISHIKAWA ITERATES TO A COMMON FIXED POINT FOR A PAIR OF MAPPINGS

In [5], [6] and [7] it has shown that for a mapping T satisfying certain conditions, if the sequence of Mann iterates converges, then it converges to a fixed point of T. In [4], the author has shown that, for a pair of mappings S, T satisfying some contractive conditions, if the sequence of Mann iterates associated with S or T is convergent, then its limit point is a common fixed point of S and T.

In the present paper we extend the results of [4] for a pair of mappings S, T in a normed space, using Ishikawa iterates process.

Let S, T be self-mappings on a Banach space X. We shall consider the Ishikawa iterates $\{x_n\}$ [3] associated with S as:

$$(1)\quad x_{n+1} = (1-\alpha_n)x_n + \alpha_n S y_n, \qquad y_n = (1-\beta_n)x_n + \beta_n S x_n, \qquad n > 0,$$

where $\{\alpha_n\}$, $\{\beta_n\}$ satisfy

$$0 \le \alpha_n \le \beta_n \le 1 \text{ for all } n, \lim_{n\to\infty} \beta_n = 0 \text{ and } \sum_{n=1}^{\infty} \alpha_n\beta_n = \infty.$$

We shall use the conditions

(i) $0 \le \alpha_n, \beta_n \le 1$, (ii) $\lim_{n\to\infty} \beta_n = 0$,

(iii) $\lim_{n\to\infty} \alpha_n = h$, $0 < h < 1$.

If $\beta_n = 0$, the Ishikawa iterates process reduces to the Mann iterates process.

We present our main result in the form of the following theorem.

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THEOREM. *Let K be a closed bounded and convex subset of a normed space N, and S, T be two self-mappings on K satisfying one of the following conditions*:

(I) $\|Sx-Ty\| \leq a\|x-y\|+b[\|x-Sx\|+\|y-Ty\|]+c[\|x-Ty\|+\|y-Sx\|]$, *for all x, y in K and* $1-b-c>0$, $a>0$, $b\geq 0$, $c\geq 0$.

(II) $\|Sx-Ty\| \leq q\max\{\|x-y\|, \|x-Sx\|+\|y-Ty\|, \|x-Ty\|+\|y-Sx\|\}$, *for all x, y in K and* $0<q<1$.

(III) $\|Sx-Ty\| \leq q\max\{\|x-y\|, \|x-Sx\|, \|y-Ty\|, \|x-Ty\|, \|y-Sx\|\}$, *for all x, y in K and* $0<q<1$.

If for some x_0 in K, the sequence $\{x_n\}$ of the Ishikawa iterates associated with either S or T is convergent, then its limit is a common fixed point of S and T. Moreover, if (III) holds, then this common fixed point is unique.

Proof. Suppose first that $Su=u$ for a point u in K. Then putting $x=y=u$ into any of the inequalities (I)–(III), we easily see that $Tu=u$. Similarly $Tu=u$ implies $Su=u$. Now, let $\{x_n\}$ be a sequence of the Ishikawa iterates associated with S such that $\lim_{n\to\infty} x_n = u$. From (1) $x_{n+1}-x_n=\alpha_n(Sy_n-x_n)$. Since $\lim_{n\to\infty} x_n=u$, we have $\|x_{n+1}-x_n\|\to 0$. The condition (iii) implies $\|Sy_n-x_n\|\to 0$. It follows that $\|u-Sy_n\|\to 0$.

If S, T satisfy (I), then

$$\|Sy_n-Tu\| \leq a\|y_n-u\|+b[\|y_n-Sy_n\|+\|u-Tu\|] + c[\|y_n-Tu\|+\|u-Sy_n\|]. \tag{2}$$

From (1), we have

$$\begin{cases} \|y_n-u\| \leq (1-\beta_n)\|x_n-u\|+\beta_n\|Sx_n-u\|, \\ \|y_n-Sy_n\| \leq (1-\beta_n)\|x_n-Sy_n\|+\beta_n\|Sx_n-Sy_n\|, \\ \|y_n-Tu\| \leq (1-\beta_n)\|x_n-Tu\|+\beta_n\|Sx_n-Tu\|; \end{cases} \tag{3}$$

introducing (3) in (2), we get

$$\begin{aligned} \|Sy_n-Tu\| \leq\ & a[(1-\beta_n)\|x_n-u\|+\beta_n\|Sx_n-u\|] \\ & + b[(1-\beta_n)\|x_n-Sy_n\|+\beta_n\|Sx_n-Sy_n\|+\|u-Tu\|] \\ & + c[(1-\beta_n)\|x_n-Tu\|+\beta_n\|Sx_n-Tu\|+\|u-Sy_n\|]. \end{aligned} \tag{4}$$

If S, T satisfy (II), then

$$\begin{aligned} &\|Sy_n-Tu\| \\ &\leq q\max\{\|y_n-u\|,\ \|y_n-Sy_n\|+\|u-Tu\|, \|y_n-Tu\|+\|u-Sy_n\|\}; \end{aligned} \tag{5}$$

introducing (3) in (5), we get

$$(6)\ \|Sy_n - Tu\| \le q\max\{(1-\beta_n)\|x_n - u\| + \beta_n\|Sx_n - u\|,\ (1-\beta_n)\|x_n - Sy_n\| + \beta_n\|Sx_n - Sy_n\|,\ (1-\beta_n)\|x_n - Tu\| + \beta_n\|Sx_n - Tu\| + \|u - Sy_n\|\}.$$

If S, T satisfy (III), then obviously satisfy (II) as well.

Taking in (4), (6) the limit as $n \to \infty$, we have

$$\|u - Tu\| \le \lambda\|u - Tu\|, \quad \text{where} \quad \lambda = \max\{b + c, q\} < 1.$$

In view of our remark on the beginning of the proof, u is a fixed point of S, as well.

In order to show the uniqueness of u in the case (III), suppose that v ($v \ne u$) is another common fixed point of S and T; then, using (III), we have

$$\begin{aligned}\|u - v\| &\le \|Su - Tv\| \\ &\le q\max\{\|u - Su\|, \|u - Sv\|, \|v - Tv\|, \|u - Tv\|, \|v - Su\|\} \\ &\le q\|u - v\|,\end{aligned}$$

whence $v = u$ follows.

Remark 1. Note that if (I) holds and in addition $a + 2b + 2c < 1$, then, by Theorem 1 in [1], each of S, T has a unique fixed point and these points coincide. Thus in this case the above theorem adds fact that this common fixed point is the limit of any convergent sequence of Ishikawa iterates associated with either S or T.

Remark 2. The uniqueness of a common fixed point does not hold in the case of the conditions (I), (II) as is shown by the example of $S = T = \text{Id}$, where Id denotes the identity mapping.

References

[1] Chi Song Wong, *Common fixed points of two mappings*, Pacific J. Math., 48 (1973), 299–312.

[2] L. A. Khan, *On the convergence of Mann iterates to a common fixed point of two mappings*, J. Pure Appl. Sci., 5 (1986), 57–58.

[3] S. A. Naimpally and K. L. Singh, *Extension of some fixed point theorem of Rhoades*, J. Math. Anal. Appl., 96 (1983), 437–446.

[4] R. A. Rashwan, *On the convergence of Mann iterates to a common fixed point for a pair of mappings*, Demonstratio Math., 23 (1990), 709–712.

[5] B. E. Rhoades, *Fixed point theorems using infinite matrices*, Trans. Amer. Soc., 196 (1974), 161–176.

[6] B. E. Rhoades, *Extensions of some fixed point theorems of Ciric, Maiti and Pal*, Math. Seminar. Notes, Kobe University, (1978), 41–46.
[7] K. L. Singh, *Fixed point iterations using infinite matrices*, Proc. of an International Conference on Applied Nonlinear Analysis. Academic Press, (1979), 689–703.

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