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ON GENERALIZED SINE AND COSINE FUNCTIONS

Introduction

In the present paper we give an answer to R. Ger's question of finding addition formulas for functions f, g which satisfy the functional equation

$$(\mathrm{I}) \qquad (f(x))^n + (g(x))^n = 1;$$

f, g are supposed to be real functions on a given group $(X,+)$ and $n \in \mathbb{N}$ is fixed, $\mathbb{N}$ being the set of positive integers.

The idea of replacing the usual trugonometric identity $\sin^2 x + \cos^2 x = 1$ by (I) with $n \geq 2$ was considered in R. Tardiff's paper [2], where some geometric aspects of this question have been discussed.

The addition formulas we have obtained coincide with the well known representations of $\cos(x+y)$ and $\sin(x+y)$ in the case of $(X,+) = (\mathbb{R},+)$ and $n = 2$, where $\mathbb{R}$ is the set of real numbers.

We denote the set of all integers by $\mathbb{Z}$; $\mathbb{C}$ will stand for the set of complex numbers.

1. Let X be an arbitrary non-empty set. We shall make use of the following result.

LEMMA 1.1. *Two real functions f, g defined on X satisfy functional equation* (I) *if and only if there exists a function $t : X \to \mathbb{R}$ such that*

$$(1.0) \qquad f(x) = \frac{\cos t(x)}{\sqrt[n]{\cos^n t(x) + \sin^n t(x)}}, \qquad g(x) = \frac{\sin t(x)}{\sqrt[n]{\cos^n t(x) + \sin^n t(x)}}$$

for all $x \in X$.

Proof. The sufficiency is obvious. We shall prove the necessity. Let $m : X \to \mathbb{C}$ be a function defined by the formula

$$(1.1) \qquad m(x) := f(x) + ig(x), \quad x \in X.$$

Since functions f, g satisfy functional equation (I), they do not vanish simultaneously and therefore $|m(x)| \neq 0$, $x \in X$. Let $t : X \to \mathbb{R}$ be any function such that $t(x) \in \arg m(x)$, $x \in X$. Then

$$(1.2) \qquad f(x) = |m(x)| \cos t(x), \quad g(x) = |m(x)| \sin t(x), \qquad x \in X.$$

Moreover

$$(1.3) \quad 1 = (f(x))^n + (g(x))^n = |m(x)|^n (\cos^n t(x) + \sin^n t(x)), \qquad x \in X,$$

implying

$$|m(x)| = \frac{1}{\sqrt[n]{\cos^n t(x) + \sin^n t(x)}}, \qquad x \in X.$$

Hence and from conditions (1.2) we obtain the required assertion.

COROLLARY 1.1. *If $n \in \mathbb{N}$ is odd, functions $f, g : X \to \mathbb{R}$ satisfy functional equation* (I) *and $t(x) \in \arg m(x)$, $x \in X$, then*

$$t(x) \in \bigcup_{k \in \mathbb{Z}} \left(-\frac{\pi}{4} + 2k\pi, \frac{3}{4}\pi + 2k\pi \right), \qquad x \in X.$$

Proof. Condition (1.3) implies $\cos^n t(x) + \sin^n t(x) > 0$, $x \in X$, whence our assertion follows.

Now, suppose that X is a subset of a group $(G, +)$. We look for formulas expressing $f(x+y)$ and $g(x+y)$ in terms of $f(x)$, $f(y)$, $g(x)$, $g(y)$, in the case of $x, y, x+y \in X$.

THEOREM 1.1. *Suppose that functions $f, g : X \to \mathbb{R}$ satisfy functional equation* (I) *and function $t : X \to \mathbb{R}$ is such that the relations* (1.0) *hold for all $x \in X$. If t is invertible on X, then, for every $x, y \in X$ such that $x + y \in X$ and $a := t^{-1}(t(x+y) - t(x)) \in X$, the following equalities are satisfied*

$$f(x+y) = \frac{f(x)f(a) - g(x)g(a)}{\sqrt[n]{(f(x)f(a) - g(x)g(a))^n + (f(x)g(a) + f(a)g(x))^n}},$$

$$g(x+y) = \frac{f(x)g(a) + f(a)g(x)}{\sqrt[n]{(f(x)f(a) - g(x)g(a))^n + (f(x)g(a) + f(a)g(x))^n}}.$$

Moreover, if t is an additive and invertible function, then $a = y$.

Proof. Note that

$$(1.4) \quad f(x)f(a) - g(x)g(a) = \frac{\cos t(x) \cos(t(x+y) - t(x)) - \sin t(x) \sin(t(x+y) - t(x))}{\sqrt[n]{\cos^n t(x) + \sin^n t(x)}\, \sqrt[n]{\cos^n t(a) + \sin^n t(a)}}$$

$$= \frac{\operatorname{cost}(x+y)}{\sqrt[n]{\cos^n t(x) + \sin^n t(x)}\,\sqrt[n]{\cos^n t(a) + \sin^n t(a)}}$$

and, similarly,

$$(1.5)\qquad f(x)g(a) + f(a)g(x) = \frac{\operatorname{sint}(x+y)}{\sqrt[n]{\cos^n t(x) + \sin^n t(x)}\,\sqrt[n]{\cos^n t(a) + \sin^n t(a)}}.$$

Hence

$$(1.6)\qquad (f(x)f(a) - g(x)g(a))^n + (f(x)g(a) + f(a)g(x))^n = \frac{\cos^n t(x+y) + \sin^n t(x+y)}{(\cos^n t(x) + \sin^n t(x))(\cos^n t(a) + \sin^n t(a))}.$$

Since $x + y \in X$, we have $\cos^n t(x+y) + \sin^n t(x+y) \neq 0$ and, by (1.0), $f(x+y) = \frac{\operatorname{cost}(x+y)}{\sqrt[n]{\cos^n t(x+y) + \sin^n t(x+y)}}$. Now, from conditions (1.4) and (1.6) we infer that

$$\frac{f(x)f(a) - g(x)g(a)}{\sqrt[n]{(f(x)f(a) - g(x)g(a))^n + (f(x)g(a) + f(a)g(x))^n}} = \frac{\operatorname{cost}(x+y)}{\sqrt[n]{\cos^n t(x+y) + \sin^n t(x+y)}} = f(x+y).$$

Analogously, one can prove the other equality occuring in assertion of the theorem.

Note that any group $(X, +)$ which is isomorphic to a subgroup of the additive group $(\mathbb{R}, +)$ yields an example of a group admitting an injective homomorphism t spoken of in Theorem 1.1.

In the sequel we are going to solve the system of two functional equations occurring in Theorem 1.1, in the case of $a = y$ and $n \in \mathbb{N}$ even. The case of $n \in \mathbb{N}$ odd requires different methods and will be a subject of another paper.

2. Let $(X, +)$ be a group and let an even $n \in \mathbb{N}$ be fixed. Suppose that f, g are two real functions on X.

LEMMA 2.1. *If f and g do not vanish simultaneously, then*

$$(f(x)f(y) - g(x)g(y))^n + (f(x)g(y) + f(y)g(x))^n > 0, \qquad x, y \in X.$$

Proof. The assumption implies $f(x) + ig(x) \neq 0$ for every $x \in X$. Hence

$$0 \neq (f(x) + ig(x))(f(y) + ig(y)) = (f(x)f(y) - g(x)g(y)) + i(f(x)g(y) + f(y)g(x)), \qquad x, y \in X.$$

Therefore, $f(x)f(y) - g(x)g(y) \neq 0$ or $f(x)g(y) + f(y)g(x) \neq 0$ for all $x, y \in X$, and we obtain the assertion, because of the eveness of n.

If functions $f, g : X \to \mathbb{R}$ do not vanish simultaneously, then we can define a function $w : X^2 \to \mathbb{R}$ by the formula

$$w(x,y) := \sqrt[n]{(f(x)f(y) - g(x)g(y))^n + (f(x)g(y) + f(y)g(x))^n} \tag{2.1}$$

for every $(x,y) \in X^2$. From Lemma 2.1 it follows that $w(x,y) > 0$ for all $(x,y) \in X^2$.

In the sequel $(T, \cdot)$ will stand for multiplicative group of the unit circle.

THEOREM 2.1. *Functions $f, g : X \to \mathbb{R}$ do not vanish simultaneously and satisfy for all $x, y \in X$ the following system of functional equations*

$$f(x+y) = \frac{f(x)f(y) - g(x)g(y)}{w(x,y)}, \qquad g(x+y) = \frac{f(x)g(y) + f(y)g(x)}{w(x,y)} \tag{II}$$

if and only if there exists a character $h : X \to T$ with $u :=$ Reh, *$v :=$* Imh *such that*

$$\begin{aligned} f(x) &= \frac{u(x)}{\sqrt[n]{(u(x))^n + (v(x))^n}}, \\ g(x) &= \frac{v(x)}{\sqrt[n]{(u(x))^n + (v(x))^n}}, \qquad x \in X. \end{aligned} \tag{2.2}$$

Moreover, if functions $f, g : X \to \mathbb{R}$ satisfy system (II) *or conditions* (2.2), *then $(f(x))^n + (g(x))^n = 1$, $x \in X$.*

Proof. The latter part of our assertion is obvious. We shall first prove the necessity. Let $m : X \to T$ be defined by (1.1) and let $h : X \to T$ be such a function that

$$h(x) := \frac{m(x)}{|m(x)|}, \qquad x \in X. \tag{2.3}$$

We shall prove that h is a character.

By (1.1) and (II), we get

$$\begin{aligned} m(x)m(y) &= (f(x)f(y) - g(x)g(y)) + i(f(x)g(y) + f(y)g(x)) \\ &= (f(x+y) + ig(x+y))w(x,y) = m(x+y)w(x,y), \qquad x, y \in X. \end{aligned}$$

Therefore

$$\frac{m(x+y)}{m(x)m(y)} = \frac{1}{w(x,y)} > 0, \qquad x, y \in X.$$

Consequently, we have the equality

$$\frac{m(x+y)}{m(x)m(y)} = \left|\frac{m(x+y)}{m(x)m(y)}\right|, \qquad x \in X,$$

implying, by (2.3), the following one

$$h(x+y) = \frac{m(x+y)}{|m(x+y)|} = \frac{m(x)}{|m(x)|}\cdot\frac{m(y)}{|m(y)|} = h(x)h(y), \qquad x, y \in X.$$

By (1.1), (2.3), we get

$$u = \frac{f}{|m|}, \qquad v = \frac{g}{|m|}. \tag{2.4}$$

Since $1 = (f(x))^n + (g(x))^n = ((u(x))^n + (v(x))^n)|m(x)|^n$, one has

$$|m(x)| = \frac{1}{\sqrt[n]{(u(x))^n + (v(x))^n}}, \qquad x \in X, \tag{2.5}$$

and hence, by (2.4), we get (2.2).

Sufficiency. If functions f, g satisfy conditions (2.2) with $h = u+iv$ being character of X, then f, g do not vanish simultaneously. Moreover, we have the relations

$$u(x+y) = u(x)u(y) - v(x)v(y), v(x+y) = u(x)v(y) + u(y)v(x), \qquad x, y \in X,$$

implying, by (2.1), (2.2),

$$\frac{f(x)f(y) - g(x)g(y)}{w(x,y)} = \frac{u(x+y)}{\sqrt[n]{(u(x+y))^n + (v(x+y))^n}} = f(x+y), \qquad x, y \in X.$$

Analogously, one can prove the other equality in system (II).

COROLLARY 2.1. *If $a : X \to \mathbb{R}$ is an additive function and $f, g : X \to \mathbb{R}$ are such that for every $x \in X$ we have*

$$f(x) = \frac{\cos a(x)}{\sqrt[n]{\cos^n a(x) + \sin^n a(x)}}, \qquad g(x) = \frac{\sin a(x)}{\sqrt[n]{\cos^n a(x) + \sin^n a(x)}},$$

then f, g do not vanish simultaneously and satisfy the system (II) *and the equation* (I).

THEOREM 2.2. *If functions $f, g : X \to \mathbb{R}$ do not vanish simultaneously and satisfy the system* (II), *then f is even, g is odd and $f(0) = 1$, $g(0) = 0$.*

Proof. By Theorem 2.1, there exists character $h = u + iv$ on X such that (2.2) hold. Since $1 = h(0) = u(0) + iv(0)$, we have $u(0) = 1$, $v(0) = 0$ and, consequently, $f(0) = 1$, $g(0) = 0$. Moreover u is even and v is odd. Therefore f is even and g is odd, as well.

3. In this section we shall show some connections between solutions of the system (II) and solutions of well-known d'Alembert's and Wilson's trigonometric functional equations.

Let $(X,+)$ be a group and let an even $n \in \mathbb{N}$ be fixed. Put $k := \frac{n}{2}$. In the sequel, for any real function f on X with $|f|$ bounded by 1, the symbol $p(f,\cdot)$ will denote the real function defined on X by the formula

$$p(f,x) := \frac{1}{\sqrt{(f(x))^2 + \sqrt[k]{1-(f(x))^n}}}, \qquad x \in X.$$

THEOREM 3.1. *If functions $f, g : X \to \mathbb{R}$ do not vanish simultaneously and satisfy the system* (II), *then the function $fp(f,\cdot)$ satisfies d'Alembert's cosine functional equation and the function $gp(g,\cdot)$ satisfies Wilson's sine functional equation.*

Proof. By Theorem 2.1, there exists a character $h = u + iv$ on X such that (2.2) hold. Moreover f, g satisfy (I) and therefore $|f|$, $|g|$ are bounded by 1.

Let m be defined by (1.1). Since $(u(x))^2 + (v(x))^2 = 1$ for every $x \in X$, we have the relation

$$|m(x)| = \sqrt{(f(x))^2 + g(x))^2} = \frac{1}{\sqrt[n]{(u(x))^n + (v(x))^n}}, \qquad x \in X,$$

implying (2.4) and, by the equation (I),

$$p(f,x) = \frac{1}{\sqrt{(f(x))^2 + (g(x))^2}} = \frac{1}{|m(x)|} = p(g,x), \qquad x \in X. \tag{3.1}$$

From conditions (2.4) and (3.1) we get $fp(f,\cdot) = u$, $gp(g,\cdot) = v$. Therefore, $fp(f,\cdot)$ and $gp(g,\cdot)$ satisfy the cosine and sine functional equation, respectively.

THEOREM 3.2. *Suppose that $(X,+)$ is an Abelian group. Let $f : X \to \mathbb{R}$ be a function with $|f|$ bounded by 1. Then the function $fp(f,\cdot)$ satisfies the cosine functional equation if and only if $f = 0$ or there exists a character $h = u + iv$ on X such that the first relation of* (2.2) *holds.*

Proof. Note that $|fp(f,\cdot)|$ is bounded by 1, as well. It is known (see [1], p. 3, proof of Theorem 9) that a real function is a bounded solution of the cosine functional equation on X if and only if it vanishes on X or it is a real part of a character of the group $(X,+)$. Hence, if $fp(f,\cdot)$ satisfies the cosine functional equation, then $fp(f,\cdot) = 0$ and therefore $f = 0$ or there exists a character $h = u + iv$ on X such that $fp(f, \cdot) = u$. Then

$$u(x) = \frac{f(x)}{\sqrt{(f(x))^2 + \sqrt[k]{1 - (f(x))^n}}}, \quad x \in X. \tag{3.2}$$

Now, an easy calculation shows that

$$(f(x))^n = \frac{(u(x))^n}{(u(x))^n + (v(x))^n}, \quad x \in X.$$

From (3.2) it follows that $\operatorname{sgn} f(x) = \operatorname{sgn} u(x)$ for every $x \in X$ and we get the required assertion.

Conversely, if $f = 0$, then $fp(f, \cdot) = 0$. However, if f admits the form (2.2), then $fp(f, \cdot) = u$. Therefore $fp(f, \cdot)$ satisfies the cosine functional equation.

THEOREM 3.3. *Suppose that $(X, +)$ is an uniquely 2-divisible Abelian group and $g : X \to \mathbb{R}$ is a function with $|g|$ bounded by 1. Then the function $gp(g, \cdot)$ satisfies the sine functional equation if and only if there exists a character $h = u + iv$ on X and a real constant c such that $|cv(x)| \le 1$ and*

$$g(x) = \frac{cv(x)}{\sqrt[n]{(1 - (cv(x))^2)^k + (cv(x))^n}}, \quad x \in X. \tag{3.3}$$

Moreover, if $c = 1$, then the second relation of (2.2) holds.

Proof. Observe that $|gp(g, \cdot)|$ is bounded by 1, as well.

Moreover, it is known (see [1], p. 3, proof of Theorem 11) that a real function is a bounded solution of the sine functional equation on X if and only if it is proportional to the imaginary part of a character of the group $(X, +)$. Hence, if $gp(g, \cdot)$ is a solution of the sine functional equation, then there exists a character $h = u + iv$ on X and a real constant c such that $gp(g, \cdot) = cv$. Then

$$cv(x) = \frac{g(x)}{\sqrt{(g(x))^2 + \sqrt[k]{1 - (g(x))^n}}}, \quad x \in X.$$

An easy calculation shows that

$$(cv(x))^2 \sqrt[k]{1 - (g(x))^n} = (g(x))^2 (1 - (cv(x))^2), \quad x \in X. \tag{3.4}$$

Thus $1 - (cv(x))^2 \ge 0$, $x \in X$, and therefore $|cv(x)| \le 1$, $x \in X$. Now, from (3.4) we have

$$(cv(x))^n = (g(x))^n ((1 - (cv(x))^2)^k + (cv(x))^n), \quad x \in X.$$

Since $(1-(cv(x))^2)^k+(cv(x))^n>0$, $x\in X$, and $\operatorname{sgn} g(x)=\operatorname{sgn} cv(x)$, $x\in X$, we get finally

$$g(x)=\frac{cv(x)}{\sqrt[n]{(1-(cv(x))^2)^k+(cv(x))^n}},\qquad x\in X.$$

Clearly, if $c=1$ then g admits the form (2.2). Conversely, if g admits the form (3.3), then $gp(g,\cdot)=cv$ and therefore $gp(f,\cdot)$ satisfies the sine functional equation.

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