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NATURAL TRANSFORMATIONS OF ANY HIGHER ORDER COVELOCITIES BUNDLE FUNCTOR

Recently, we have determined in [3] all natural transformations of the $(1,r)$-covelocities bundle functor T_1^{r*} into itself, which constitute the r-parameter family linearly generated by the p-power natural transformations A_p for $p = 1,\ldots,r$.

Moreover, we have obtained in [4] all natural transformations of the $(2,r)$-covelocities bundle functor T_2^{r*} into T_1^{r*}, which constitute the $(2r + \frac{r(r-1)}{2})$-parameter family linearly generated by the p_1,p_2-power mixed transformations $A^{(2)}_{p_1,p_2}$ for $p_1,p_2 = 0,1,\ldots,r$ with $p_1+p_2 = 1,\ldots,r$.

In this paper, we determine all natural transformations of the (k,r)-covelocities bundle functor T_k^{r*} into the $(1,s)$-covelocities bundle functor T_1^{s*} in the cases $r = s, r < s, r > s$ and any k,l. We deduce that all natural transformations of the functor T_k^{r*} into the functor T_1^{r*} form the $(\binom{k+r}{k}-1)$-parameter family linearly generated by the $p_1,\ldots,p_k$-power mixed transformations $A^{(r)}_{p_1,\ldots,p_k}$ for $p_1,\ldots,p_k = 0,1,\ldots,r$ with $p_1+\ldots+p_k = 1,\ldots,r$. Moreover, we deduce that all natural transformations of the functor T_k^{r*} into the functor T_l^{r*} constitute the $l(\binom{k+r}{k}-1)$-parameter family of the above mentioned form for all l components.

Also, we deduce that all natural transformations of the functor T_k^{r*} into $T_1^{(r+q)*}$ form the $(\binom{k+r}{k}-1)$-parameter family linearly generated by a generalized $p_1,\ldots,p_k$-power mixed transformations $A^{(r,r+q)}_{p_1,\ldots,p_k}$ for $p_1,\ldots,p_k = 0,1,\ldots,r+q$ with $p_1+\ldots+p_k = q+1,\ldots,q+r$.

At last, we deduce that all natural transformations of the functor T_k^{r*} into $T_l^{(r-q)*}$ form the $l(\binom{k+r-q}{k}-1)$-parameter family linearly generated for all l components by a generalized $p_1,\ldots,p_k$-power mixed transformations $A^{(r,r-q)}_{p_1,\ldots,p_k}$ for $p_1,\ldots,p_k = 0,1,\ldots,r-q$ with $p_1+\ldots+p_k = 1,\ldots,r-q$.

1. Let M be a smooth n-dimensional manifold. Let $T_k^{r*}M = J^r(M, R^k)_0$ be the space of all r-jets from a manifold M to R^k with target 0.

A vector bundle $\pi_M : T_k^{r*}M \to M$ with a source r-jet projection is called the (k,r)-covelocities bundle on M.

Every local diffeomorphism $\varphi : M \to N$ is extended into a vector bundle morphism $T_k^{r*}\varphi : T_k^{r*}M \to T_k^{r*}N$ defined by $j_x^r F \mapsto j_{\varphi(x)}^r(F \cdot \varphi^{-1})$, where φ^{-1} is constructed locally. Thus, the (k,r)-covelocities bundle functor T_k^{r*} is defined on a category $\mathcal{M}f_n$ of smooth n-dimensional manifolds with local diffeomorphisms as morphisms and with values in a category $\mathcal{VB}$ of vector bundles.

There is a canonical identification

$$(1,1) \qquad T_k^{r*}M = T_1^{r*}M \times \ldots \times T_1^{r*}M \quad (k - \text{times})$$

of the form $j_x^r F = (j_x^r F^1, \ldots, j_x^r F^k)$, where $F = (F^1, \ldots, F^k) : M \to R^k$.

We have defined in [3], the p-power transformations A_p of T_1^{r*} into itself of the form

$$(1,2) \qquad A_p : j_x^r F \mapsto j_x^r(F)^p$$

where $(F)^p$ denotes the p-th power of $F : M \to R$ for $p = 1, \ldots, r$.

We define the $p_1, \ldots, p_k$-power mixed transformation $A^{(r)}_{p_1,\ldots,p_k}$ of the functor T_k^{r*} into T_1^{r*} as a generalization of the p-power transformation A_p of the functor T_1^{r*} into itself.

Definition 1. A natural transformation $A^{(r)}_{p_1,\ldots,p_k}$ of the (k,r)-covelocities bundle functor T_k^{r*} into the $(1,r)$-covelocities bundle functor T_1^{r*} defined by

$$(1,3) \qquad A^{(r)}_{p_1,\ldots,p_k} : j_x^r F \mapsto j_x^r(F^1)^{p_1} \ldots (F^k)^{p_k}$$

is called $p_1, \ldots, p_k$-power mixed transformation for $p_1, \ldots, p_k = 0, 1, \ldots, r$ with $p_1 + \ldots + p_k = 1, \ldots, r$, where $F = (F^1, \ldots, F^k)$ and $(F^m)^{p_m}$ denotes the p_m-th power of F^m.

Definition 2. A natural transformation $A^{(r,s)}_{p_1,\ldots,p_k}$ of the functor T_k^{r*} into the functor T_1^{s*} defined by

$$(1,4) \qquad A^{(r,s)}_{p_1,\ldots,p_k} : j_x^r F \mapsto j_x^s(F^1)^{p_1} \ldots (F^k)^{p_k}$$

is called a generalized $p_1, \ldots, p_k$-power mixed transformation for $p_1, \ldots, p_k = 0, 1, \ldots, s$ with $p_1 + \ldots + p_k = 1, \ldots, s$.

We note that definition of $A^{(r,s)}_{p_1,\ldots,p_k}$ in the case $s = r + q$ is correct for $p_1 + \ldots + p_k = q + 1, \ldots, q + r$.

DEFINITION 3. A natural transformation $P^{(r,r-q)}$ of the functor T_k^{r*} into the functor $T_k^{(r-q)*}$ defined by

$$P^{(r,r-q)} : j_x^r F \mapsto j^{r-q} F \tag{1,5}$$

is called a projection.

Note that the generalized $p_1, \dots, p_k$-power mixed transformation $A^{(r,r-q)}_{p_1,\dots,p_k}$ for $p_1, \dots, p_k = 0, 1, \dots, r-q$ with $p_1 + \dots + p_k = 1, \dots, r-q$ is a composition

$$A^{(r,r-q)}_{p_1,\dots,p_k} = A^{(r-q)}_{p_1,\dots,p_k} \circ P^{(r,r-q)}. \tag{1,6}$$

If (x^i) are some local coordinates on M, then we have the induced fibre coordinates $(u_i^m, u^m_{i_1 i_2,\dots}, u^m_{i_1,\dots,i_r})$ with $m = 1, \dots, k$ on $T_k^{r*}M$ (symmetric in all subscripts) of the form:

$$\begin{aligned} u_i^m(j_x^r F) &= \left.\frac{\partial F^m}{\partial x^i}\right|_x \\ u^m_{i_1 i_2}(j_x^r F) &= \left.\frac{\partial^2 F^m}{\partial x^{i_1} \partial x^{i_2}}\right|_x \\ &\cdots\cdots\cdots\cdots\cdots \\ u^m_{i_1 \dots i_r}(j_x^r F) &= \left.\frac{\partial^r F^m}{\partial x^{i_1} \dots \partial x^{i_r}}\right|_x . \end{aligned} \tag{1,7}$$

2. In this part, we determine all natural transformations of the functor T_k^{r*} into T_1^{r*} and then T_l^{r*} by an induction with respect to r.

THEOREM 1. *All natural transformations $A : T_k^{r*} \to T_1^{r*}$ of the (k,r)-covelocities bundle functor into the $(1,r)$-covelocities bundle functor T_1^{r*} form the $(\binom{k+r}{k} - 1)$-parameter family of the form*

$$A = \sum_{\substack{p_1,\dots,p_k = 0,\dots,r \\ p_1+\dots+p_k = 1,\dots,r}} t_{p_1,\dots,p_k} A^{(r)}_{p_1,\dots,p_k} \tag{2,1}$$

with any real parameters $t_{p_1,\dots,p_k} \in R$.

Proof. The functor T_k^{r*} is defined on the category $\mathcal{M}f_n$ of n-dimensional smooth manifolds with local diffeomorphisms as morphisms and is of order r.

According to a general theory, [1], [2], [3], the natural transformations $A : T_k^{r*} \to T_1^{r*}$ are in bijection with G_n^r-equivariant maps of the standard fibres $f : (T_k^{r*} R^n)_0 \to (T_1^{r*} R^n)_0$.

Let tilda $\widetilde{a} = a^{-1}$ denote the inverse element in G_n^r and let $(i_1, \dots, i_r)$ denote the symmetrization of indices.

By (1,7) the action of an element $(a_j^i, a^i_{j_1 j_2}, \dots, a^i_{j_1 \dots j_r}) \in G_n^r$ on $((u_i^m, u^m_{i_1 i_2}, \dots, u^m_{i_1 \dots i_r})_{m=1,\dots,k}) \in (T_k^{r*} R^n)_0$ is of the form

$$\overline{u}_i^m = u_j^m \widetilde{a}_i^j \tag{2,2}$$

$$\overline{u}^m_{i_1i_2} = u^m_{j_1j_2}\widetilde{a}^{j_1}_{i_1}\widetilde{a}^{j_2}_{i_2} + u^m_{j_1}\widetilde{a}^{j_1}_{i_1i_2}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\begin{aligned}\overline{u}^m_{i_1\ldots i_r} &= u^m_{j_1\ldots j_r}\widetilde{a}^{j_1}_{i_1}\ldots\widetilde{a}^{j_r}_{i_r}\\ &+ u^m_{j_1\ldots j_{r-1}}\frac{r!}{(r-2)!2!}\widetilde{a}^{j_1}_{(i_1}\ldots\widetilde{a}^{j_{r-2}}_{i_{r-2}}\widetilde{a}^{j_{r-1}}_{i_{r-1}i_r)}\\ &+\ldots+u^m_{j_1j_2}\left[\frac{r!}{(r-1)!1!}\widetilde{a}^{j_1}_{(i_1}\widetilde{a}^{j_2}_{i_2\ldots i_r)}+\ldots\right]+u^m_{j_1}\widetilde{a}^{j_1}_{i_1\ldots i_r}.\end{aligned}$$

The action of an element $a \in G^r_n$ on $(w_i, w_{i_1i_2,\ldots,}w_{i_1\ldots i_r}) \in (T^{r*}_1R^n)_0$ is of the same form (2,2).

1°. First consider the case $r = 2$. Equivariancy of G^2_n-equivariant map $f = (f_i, f_{i_1i_2}) : (T^{2*}_kR^n)_0 \to (T^{2*}_1R^n)_0$ in the form

$$(2,3)\qquad \begin{aligned} w_i &= f_i((u^m_i, u^m_{i_1i_2})_{m=1,\ldots,k}),\\ w_{i_1i_2} &= f_{i_1i_2}((u^m_i, u^m_{i_1i_2})_{m=1,\ldots,k})\end{aligned}$$

with respect to homotheties in $G^2_n : \widetilde{a}^i_j = t\delta^i_j$, $\widetilde{a}^i_{j_1j_2} = 0$, give a homogeneity conditions

$$(2,4)\qquad \begin{aligned} tf_i((u^m_i, u^m_{i_1i_2})_{m=1,\ldots,k}) &= f_i((tu^m_i, t^2u^m_{i_1i_2})_{m=1,\ldots,k})\\ t^2f_{i_1i_2}((u^m_i, u^m_{i_1i_2})_{m=1,\ldots,k}) &= f_{i_1i_2}((tu^m_i, t^2u^m_{i_1i_2})_{m=1,\ldots,k}).\end{aligned}$$

By the homogeneous function theorem, [2], we deduce firstly that f_i is linear in u^m_i and is independent on $u^m_{i_1i_2}$ for $m = 1,\ldots,k$ and secondly that $f_{i_1i_2}$ is linear in $u^m_{i_1i_2}$ and is bilinear in $u^{m_1}_{i_1}$, $u^{m_2}_{i_2}$ for $m_1, m_2 = 1,\ldots,k$.

Using the invariant tensor theorem, [2], for G^1_n, we obtain f in the form:

$$(2,5)\qquad \begin{aligned} f_i &= \sum_{1\le m\le k}\lambda_m u^m_i,\\ f_{i_1i_2} &= \sum_{1\le m\le k}\mu_m u^m_{i_1i_2} + \sum_{1\le m_1\le m_2\le k}\lambda_{m_1m_2}u^{m_1}_{(i_1}u^{m_2}_{i_2)}\end{aligned}$$

with any real parameters λ_m, μ_m, $\lambda_{m_1m_2} \in R$ for $m, m_1, m_2 = 1,\ldots,k$.

The equivariancy of f in the form (2,5) with respect to the kernel of the projection $G^2_n \to G^1_n : \widetilde{a}^i_j = \delta^i_j$ and $\widetilde{a}^i_{jk}$ are arbitrary, gives relationship

$$(2,6)\qquad \lambda_m = \mu_m \quad \text{for } m = 1,\ldots,k.$$

We define new parameters $t_{p_1,\ldots,p_k}$ for $p_1,\ldots,p_k = 0,1,2$ and satisfying $p_1+\ldots+p_k = 1,2$, in the following way

$$(2,7)\qquad t_{p_1\ldots p_m\ldots p_k} = \lambda_m$$

iff $p_m = 1$ for $m = 1,\ldots,k$ and others $p_n = 0$ for $n \neq m$,

$$t_{p_1\ldots p_{m_1}\ldots p_{m_2}\ldots p_k} = \lambda_{m_1m_2}$$

iff $p_{m_1} = p_{m_2} = 1$ for $1 \leq m_1 < m_2 \leq k$, and others $p_n = 0$ for $n \neq m_1, m_2$,

$$t_{p_1 \dots p_m \dots p_k} = \lambda_{m,m} \quad \text{iff } p_m = 2 \text{ for } m = 1, \dots, k.$$

If we use the parameters (2,7) in the formulas (2,5) and if we use the combinatoric relation: $1 + k + \binom{k+1}{2} = \binom{k+2}{k}$, then we obtain the $(\binom{k+2}{k} - 1)$-parameter family in the form (2,1) for $r = 2$.

2°. Assume that theorem holds for $(r-1)$ and G_n^{r-1}-equivariant map $f = (f_i, f_{i_1 i_2}, \dots, f_{i_1 \dots i_{r-1}}) : (T_k^{(r-1)*} R^n)_0 \to (T_1^{(r-1)*} R^n)_0$ define the $(\binom{k+r-1}{k} - 1)$-parameter family of the form

$$(2,8) \qquad A = \sum_{\substack{p_1, \dots, p_k = 0, \dots, r-1 \\ p_1 + \dots + p_k = 1, \dots, r-1}} t_{p_1 \dots p_k} A^{(r-1)}_{p_1, \dots, p_k}$$

with any real parameters $t_{p_1 \dots p_k}$. We assume that G_n^r-equivariant map $\bar{f} : (T_k^{r*} R^n)_0 \to (T_1^{r*} R^n)_0$ is of the form $\bar{f} = (f_i, f_{i_1 i_2}, \dots, f_{i_1 \dots i_{r-1}}, f_{i_1 \dots i_r})$ provided that $f = (f_i, \dots, f_{i_1 \dots i_{r-1}})$.

The equivariancy of $\bar{f}$ with respect to the homotheties in $G_n^r : a_j^i = t\delta_j^i$, $a^i_{j_1 j_2} = 0, \dots, a^i_{j_1 \dots j_r} = 0$, gives for the r-th component $f_{i_1 \dots i_r}$ a homogeneity condition:

$$(2,9) \qquad t^r f_{i_1 \dots i_r}((u_i^m, u^m_{i_1 i_2}, \dots, u^m_{i_1 \dots i_r})_{m=1,\dots,k}) = f_{i_1 \dots i_r}((t u_i^m, t^2 u^m_{i_1 i_2}, \dots, t^r u^m_{i_1 \dots i_r})_{m=1,\dots,k}).$$

Using the homogeneous function theorem and the invariant tensor theorem, [2], we obtain that the r-th component $f_{i_1 \dots i_r}$ is of the general form

$$(2,10) \qquad f_{i_1 \dots i_r} = \sum_{1 \leq m \leq k} \mu_m u^m_{i_1 \dots i_r} + \sum_{1 \leq m_1 \leq m_2 \leq k} \mu^{(2,r-1)}_{m_1 m_2} u^{m_1}_{(i_1} u^{m_2}_{i_2 \dots i_r)} + \dots + \sum_{1 \leq m_1 \leq \dots \leq m_{r-1} \leq k} \mu^{(r-1,2)}_{m_1 \dots m_{r-1}} u^{m_1}_{(i_1} \dots u^{m_{r-2}}_{i_{r-2}} u^{m_{r-1}}_{i_{r-1} i_r)} + \sum_{1 \leq m_1 \leq \dots \leq m_r \leq k} \mu^{(r)}_{m_1 \dots m_r} u^{m_1}_{i_1} \dots u^{m_r}_{i_r}.$$

Equivariancy of $\bar{f}$ with respect to the kernel of the projection $G_n^r \to G_n^{r-1}$: $\widetilde{a}_j^i = \delta_j^i$, $\widetilde{a}_{j_1j_2}^i = 0, \ldots, \widetilde{a}_{j_1\ldots j_{r-1}}^i = 0$ and $a_{j_1\ldots j_r}^i$ are arbitrary, gives

$$(2,11) \qquad \mu_m = t_{p_1\ldots p_m\ldots p_k}$$

iff $p_m = 1$ for $m = 1, \ldots, k$ and others $p_n = 0$ for $n \neq m$.

Considering the equivariancy of $\bar{f}$ with respect to the kernel of the projection $G_n^{r-1} \to G_n^1$ in $G_n^r : \widetilde{a}_j^i = \delta_j^i$ and $\widetilde{a}_{j_1j_2}^i, \ldots, \widetilde{a}_{j_1\ldots j_{r-1}}^i$ are arbitrary and $\widetilde{a}_{j_1\ldots j_r}^i = 0$, we obtain the following relationship for parameters:

$$(2,12) \qquad \mu_{m_1m_2}^{(2,r-1)} = \frac{r!}{(r-1)!1!} t_{p_1\ldots p_{m_1}\ldots p_{m_2}\ldots p_k}$$

iff $p_{m_1}, p_{m_2} = 1, 2$, $p_{m_1} + p_{m_2} = 2$ for $1 \leq m_1 \leq m_2 \leq k$ and others $p_n = 0$ for $n \neq m_1, m_2$

....

$$\mu_{m_1\ldots m_{r-1}}^{(r-1,2)} = \frac{r!}{(r-2)!2!} t_{p_1\ldots p_{m_1}\ldots p_{m_{r-1}}\ldots p_k}$$

iff $p_{m_1}, \ldots, p_{m_{r-1}} = 1, \ldots, r-1$ satisfy $p_{m_1} + \ldots + p_{m_{r-1}} = r-1$ for $1 \leq m_1 \leq \ldots \leq m_{r-1} \leq k$ and others $p_n = 0$ for $n \neq m_1, \ldots, m_{r-1}$.

Moreover, we put for the remaining $\binom{k+r-1}{r}$-parameters

$$(2,13) \qquad \mu_{m_1\ldots m_r}^{(r)} = t_{p_1\ldots p_{m_1}\ldots p_{m_r}\ldots p_k}$$

iff $p_{m_1}, \ldots, p_{m_r} = 1, \ldots, r$ satisfy $p_{m_1} + \ldots + p_{m_r} = r$ for $1 \leq m_1 \leq \ldots \leq m_r \leq k$ and others $p_n = 0$ for $n \neq m_1, \ldots, m_r$.

Thus, we obtain the $(\binom{k+r}{k} - 1)$-parameter family of natural transformations in the form (2,1), where $(\binom{k+r}{k} - 1) = (\binom{k+r-1}{k} - 1) + \binom{k+r-1}{r}$. This proves our theorem.

By the canonical identification (1,1), $T_l^{r*}M = T_1^{r*}M \times \ldots \times T_1^{r*}M$ (l times), any natural transformation $A : T_k^{r*} \to T_l^{r*}$ correspond bijectively to G_n^r-equivariant map $f = ((f^m)_{m=1,\ldots,l}) = ((f_i^m, f_{i_1i_2}^m, \ldots, f_{i_1\ldots i_r}^m)_{m=1,\ldots,l})$. Considering the G_n^r-equivariancy of f, we obtain by Theorem 1 that each component f^m of f for $m = 1, \ldots, l$ define that $(\binom{k+r}{k} - 1)$-parameter family in the form (2,1).

Corollary 2. *All natural transformations $A : T_k^{r*} \to T_l^{r*}$ form the $l \cdot (\binom{k+r}{k} - 1)$-parameter family of the form (2,1) for all l components.*

3. We are going to determine all natural transformations $T_k^{r*} \to T_l^{s*}$ in the cases: $r < s$, $r > s$ and any k, l.

THEOREM 3. *All natural transformations* $A : T_k^{r*} \to T_1^{(r+q)*}$ *form the* $(\binom{k+r}{k} - 1)$*-parameter family of the form*

$$A = \sum_{\substack{p_1,\dots,p_k=0,\dots,q+r \\ p_1+\dots+p_k=q+1,\dots,q+r}} t_{p_1\dots p_k} A^{(r,r+q)}_{p_1,\dots,p_k} \tag{3,1}$$

with any real parameters $t_{p_1\dots p_k} \in R$.

Proof. We apply induction on $q = s - r$.

1°. First consider the case $q = 1$. According to general standard methods, [1], [2], the natural transformations $A : T_k^{r*} \to T_1^{(r+1)*}$ are in bijection with G_n^{r+1}-equivariant maps of the standard fibres $f : (T_k^{r*}R^n)_0 \to (T_1^{(r+1)*}R^n)_0$.

Let $(u_1^m, u_2^m, \dots, u_r^m) := (u_{i_1}^m, u_{i_1 i_2}^m, \dots, u_{i_1\dots i_r}^m)$ for $m = 1, \dots, k$, denotes the fibre coordinates on $T_k^{r*}M$.

Considering the equivariancy of $f = (f_1, \dots, f_r, f_{r+1})$ with respect to homotheties in $G_n^{r+1} : a_j^i = t\delta_j^i, \dots, a^i_{j_1\dots i_{r+1}} = 0$, we obtain the homogeneity condition:

$$\begin{aligned} (3,2)\qquad & t f_1(u_1^m, \dots, u_r^m)_{m=1,\dots,k}) = f_1((tu_1^m, \dots, t^r u_r^m)_{m=1,\dots,k}), \\ & \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ & t^r f_r((u_1^m, \dots, u_r^m)_{m=1,\dots,k}) = f_r((tu_1^m, \dots, t^r u_r^m)_{m=1,\dots,k}), \\ & t^{r+1} f_{r+1}((u_1^m, \dots, u_r^m)_{m=1,\dots,k}) = f_{r+1}(tu_1^m, \dots, t^r u_r^m)_{m=1,\dots,k}). \end{aligned}$$

Moreover, using the equivariancy of $\bar{f} = (f_1, \dots, f_r)$ with respect to the kernel of the projection $G_n^r \to G_n^1$ in G_n^{r+1}, we obtain by Theorem 1 the $(\binom{k+r}{k} - 1)$-parameter family of natural transformations

$$A = \sum_{\substack{p_1,\dots,p_k=0,\dots,r \\ p_1+\dots+p_k=1,\dots,r}} t_{p_1\dots p_k} A^{(r)}_{p_1,\dots,p_k}$$

with any real parameters $t_{p_1\dots p_k} \in R$.

By the homogeneous function theorem and by the invariant tensor theorem, [2], we deduce that the $(r+1)$-th component f_{r+1} is of the form

$$\begin{aligned} (3,3)\qquad f_{i_1\dots i_{r+1}} = & \sum_{1\le m_1\le\dots\le m_{r+1}\le k} \tau^{(r+1)}_{m_1\dots m_{r+1}} u^{m_1}_{(i_1} \dots u^{m_{r+1}}_{i_{r+1})} \\ & + \sum_{1\le m_1\le\dots\le m_r\le k} \tau^{(r,2)}_{m_1\dots m_r} u^{m_1}_{(i_1} \dots u^{m_{r-1}}_{i_{r-1}} u^{m_r}_{i_r i_{r+1})} \\ & + \dots + \sum_{1\le m_1\le m_2\le k} \tau^{(2,r)}_{m_1 m_2} u^{m_1}_{(i_1} u^{m_2}_{i_2\dots i_{r+1})} \end{aligned}$$

with any real parameters $\tau^{(r+1)}_{m_1\dots m_{r+1}}, \tau^{(r,2)}_{m_1,\dots,m_2}, \dots, \tau^{(2,r)}_{m_1 m_2} \in R$.

The equivariancy of f with respect to the kernel of the projections $G_n^{r+1} \to G_n^r$ and $G_n^{r+1} \to G_n^1$, gives the following relations:

$$(3,4) \qquad t_{p_1 \dots p_k} = 0 \text{ iff } p_1, \dots, p_k = 0, 1 \text{ satisfy } p_1 + \dots + p_k = 1,$$

$$(3,5) \qquad \tau^{(2,r)}_{m_1 m_2} = \frac{(r+1)!}{r!1!} t_{p_1 \dots p_{m_1} \dots p_{m_2} \dots p_k}$$

iff $p_{m_1}, p_{m_2} = 1, 2$ satisfy $p_{m_1} + p_{m_2} = 2$ for $1 \le m_1 \le m_2 \le k$ and others $p_m = 0$ for $m \neq m_1, m_2$

....

$$\tau^{(r,2)}_{m_1 \dots m_r} = \frac{(r+1)!}{(r-1)!2!} t_{p_1 \dots p_{m_1} \dots p_{m_r} \dots p_k}$$

iff $p_{m_1}, \dots, p_{m_r} = 1, \dots, r$ satisfy $p_{m_1} + \dots + p_{m_r} = r$ for $1 \le m_1 \le \dots \le m_r \le k$ and others $p_m = 0$ for $m \neq m_1, \dots, m_r$.

Moreover, we put

$$(3,6) \qquad \tau^{(r+1)}_{m_1 \dots m_{r+1}} = t_{p_1 \dots p_{m_1} \dots p_{m_{r+1}} \dots p_k}$$

iff$p_{m_1}, \dots, p_{m_{r+1}} = 1, 2, \dots, r+1$ satisfy $p_{m_1} + \dots + p_{m_{r+1}} = r+1$ for $1 \le m_1 \le \dots \le m_{r+1} \le k$ and others $p_m = 0$ for $m \neq m_1, \dots, m_{r+1}$.

Then, the G_n^{r+1}-equivariant map $f = (f_1, \dots, f_r, f_{r+1})$ define the $(\binom{k+r}{k} - 1)$-parameter family of the form

$$(3,7) \qquad A = \sum_{\substack{p_1, \dots, p_k = 0, \dots, r+1 \\ p_1 + \dots + p_k = 2, \dots, r+1}} t_{p_1 \dots p_k} A^{(r,r+1)}_{p_1, \dots, p_k}.$$

2°. Assume that the theorem holds for $(q-1)$ and the G_n^{r+q-1}-equivariant map $\bar{f} : (T_k^{r*} R^n)_0 \to (T^{(r+q-1)*} R^n)_0$ define the $(\binom{k+r}{k} - 1)$-parameter family of natural transformations $A : T_k^{r*} \to T_1^{(r+q-1)*}$ of the form

$$(3,8) \qquad A = \sum_{\substack{p_1, \dots, p_k = 0, \dots, r+q-1 \\ p_1 + \dots + p_k = q, \dots, q+r-1}} t_{p_1 \dots p_k} A^{(r,r+q-1)}_{p_1, \dots, p_k}$$

with any real parameters $t_{p_1 \dots p_k} \in R$.

Consider the G_n^{r+q}-equivariant map $f : (T_k^{r*} R^n)_0 \to (T_1^{(r+q)*} R^n)_0$ of the form $f = (\bar{f}, f_{r+q})$. The equivariancy of f with respect to homotheties in $G_n^{r+q} : \widetilde{a}^i_j = t\delta^i_j, \widetilde{a}^i_{j_1 j_2} = 0, \dots, \widetilde{a}^i_{j_1 \dots j_{r+q}} = 0$, gives for the $(r+q)$-th component f_{r+q} the homogeneity condition

$$(3,9) \quad t^{r+q} f_{r+q}((u_1^m, \dots, u_r^m)_{m=1,\dots,k}) = f_{r+q}((t u_1^m, \dots, t^r u_r^m)_{m=1,\dots,k}).$$

By the homogeneous function theorem and the invariant tensor theorem, [2], we deduce that f_{r+q} is of the general form:

$$(3,10)\quad f_{i_1\dots i_{r+q}} = \sum_{1\le m_1\le\dots\le m_{r+q}\le k} \nu^{(r+q)}_{m_1\dots m_{r+q}} u^{m_1}_{i_1}\dots u^{m_{r+q}}_{i_{r+q}}$$
$$+ \sum_{1\le m_1\le\dots\le m_{r+q-1}\le k} \nu^{(r+q-1,2)}_{m_1\dots m_{r+q-1}} u^{m_1}_{(u_1}\dots u^{m_{r+q-2}}_{i_{r+q-2}} u^{m_{r+q-1}}_{i_{r+q-1}i_{r+q})}$$
$$+\dots+ \sum_{1\le m_1\le\dots\le m_{q+1}\le k} \nu^{(q+1,r)}_{m_1\dots m_{q+1}} u^{m_1}_{(i_1}\dots u^{m_q}_{i_q} u^{m_{q+1}}_{i_{q+1}\dots i_{q+r})}$$

with any real parameters $\nu^{(r+q)}_{m_1\dots m_{r+q}}, \nu^{(r+q-1,2)}_{m_1\dots m_{r+q-1}}, \dots, \nu^{(q+1,r)}_{m_1\dots m_{q+1}}$.

The equivariancy of f with respect to the kernel of the projections $G^{r+q}_n \to G^r_n$ and $G^{r+q}_n \to G^1_n$, gives the relationships

$(3,11)$ $t_{p_1\dots p_k} = 0$ iff $p_1,\dots,p_k = 0,\dots,r+q-1$ satisfy $p_1+\dots+p_k = q$

$$\nu^{(q+1,r)}_{m_1\dots m_{q+1}} = \frac{(q+r)!}{r!q!} t_{p_1\dots p_{m_1}\dots p_{m_{q+1}}\dots p_k}$$

iff $p_{m_1},\dots,p_{m_{q+1}} = 1,\dots,r+q-1$ satisfy $p_{m_1}+\dots+p_{m_{q+1}} = q+1$ for $1\le m_1\le\dots\le m_{q+1}\le k$ and others $p_m = 0$ for $m\ne m_1,\dots,m_{q+1}$

....

$$\nu^{(r+q-1,2)}_{m_1\dots m_{q+r-1}} = \frac{(q+r)!}{(r+q-2)!2!} t_{p_1\dots p_{m_1}\dots p_{m_{r+q-1}}\dots p_k}$$

iff $p_{m_1},\dots,p_{m_{r+q-1}} = 1,\dots,r+q-1$ satisfy $p_{m_1}+\dots+p_{m_{r+q-1}} = r+q-1$ for $1\le m_1\le\dots\le m_{r+q-1}\le k$ and others $p_m = 0$ for $m\ne m_1,\dots,m_{r+q-1}$.

Moreover, we put

$$(3,12)\qquad \nu^{(r+q)}_{m_1\dots m_{r+q}} = t_{p_1\dots p_{m_1}\dots p_{m_{r+q}}\dots p_k}$$

iff $p_{m_1},\dots,p_{m_{r+q}} = 1,2,\dots,r+q$ satisfy $p_{m_1}+\dots+p_{m_{r+q}} = r+q$ for $1\le m_1\le\dots\le m_{r+q}\le k$ and others $p_m = 0$ for $m\ne m_1,\dots,m_{r+q}$.

This gives the family of natural transformation (3,1) and proves our theorem.

From this theorem we obtain immediately

COROLLARY 4. *All natural transformations* $A: T^{r*}_k \to T^{(r+q)*}_l$ *form the* $l(\binom{k+r}{k}-1)$*-parameter family of the general form* $(3,1)$ *for all* l *components.*

Finally, we have

THEOREM 5. *All natural transformations* $A: T^{r*}_k \to T^{(r-q)*}_1$ *form* $(\binom{k+r-q}{k}-1)$*-parameter family*

$$(3,13)\qquad A = \sum_{\substack{p_1,\dots,p_k=0,\dots,r-q\\ p_1+\dots+p_k=1,\dots,r-q}} t_{p_1\dots p_k} A^{(r,r-q)}_{p_1\dots p_k}$$

with any real parameters $t_{p_1\dots p_k}\in R$.

P r o o f. Applying the previous procedure, we obtain that any $A : T_k^{r*} \to T_1^{r-q)*}$ is the composition of the projection $P^{(r,r-q)} : T_k^{r*} \to T_k^{(r-q)*}$ and any $\overline{A} : T_k^{(r-q)*} \to T_1^{(r-q)*}$, $A = \overline{A} \circ P^{(r,r-q)}$. By result of Theorem 1 this proves our theorem.

COROLLARY 6. *All natural transformations* $A : T_k^{r*} \to T_l^{(r-q)*}$ *form the* $l(\binom{k+r-q}{k} - 1)$*-parameter family of the form* $(3,13)$ *for all* l *components.*

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