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NOTES ON SOME OPERATOR EQUATIONS IN THE ORLICZ SPACES OF VECTOR MULTIFUNCTIONS

1. Introduction

In this paper we apply the results of [1] and [3] to the case of the spaces $M^1_{Y,\varphi}$, $M^0_{Y,\varphi}$ of multifunctions. All definitions and theorems connected with the idea of Musielak–Orlicz space can be found in [2].

Let I be a bounded interval. Let (I,Σ,μ) be the Lebesgue measure space. Let X be a real separable Hilbert space with the norm $\|\circ\|_X$. We denote by $L^\varphi(I,X)$ the Orlicz space of all strongly measurable functions $x: I\to X$ for which the number $\|x\|_\varphi = \inf\{r>0 : \int_I \varphi(\frac{\|x(t)\|_X}{r})d\mu \le 1\}$, is finite, where φ is a N-function i.e. $\varphi: R\to R_+, \varphi(u)$ is an even convex function, equal zero iff $u=0$, $\lim_{u\to 0}\frac{\varphi(u)}{u}=0$, $\lim_{u\to\infty}\frac{\varphi(u)}{u}=\infty$. It is known that $\langle L^\varphi(I,X), \|\circ\|_\varphi\rangle$ is a Banach space. We shall denote by $E^\varphi(I,X)$ the closure in $L^\varphi(I,X)$ of the set of all bounded functions.

We will apply the following theorem from [3]:

THEOREM 1. *Assume that $D=[0,d]$ and M,N are complementary N-functions. We shall consider the operator A defined by*

$$A(x)(t) = \int_0^t f(t,s,x(s))\,ds \quad \textit{for } t\in D,\ x\in L^\varphi(D,X).$$

Let:

C1. *$(t,s,x)\to f(t,s,x)$ is a function from $D^2\times X$ into X which is continuous in x for a.e. $t,s\in D$, and strongly measurable in (t,s) for every $x\in X$,*

C2. $\|f(t,s,x)\|_X \le K(t,s)(b(s)+S(\|x\|_X))$ *for $t,s\in D$, $x\in X$, where*

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$b \in L^N(D,R)$, S is a nonnegative nondecreasing continuous functions defined on $[0,\infty)$ and $K(t,s) \geq 0$ for $t,s \in D$,

C3. *N satisfies the condition Δ', i.e. there exist $\lambda, u_0 \geq 0$ such that $N(uv) \leq \lambda N(u)N(v)$ for $u,v \geq u_0$, $K \in E^M(D^2,R)$, φ is an N-function and there exist $\alpha,\gamma,u_0 \geq 0$ such that $N(\alpha S(u)) \leq \gamma\varphi(u) \leq \gamma M(u)$ for $u \geq u_0$,*

C4. *N satisfies the condition Δ_3, i.e. there exist $\lambda, u_0 \geq 0$ such that $N(\lambda u) \geq uN(u)$ for $u \geq u_0$, $K \in L^M(D^2,R)$, there exist $\lambda, u_0 > 0$ such that $S(u) \leq \lambda\frac{M(u)}{u}$ for $u \geq u_0$, φ is an N-function satisfying the condition Δ' and such that $\int\int_{D^2} \varphi(M(K(t,s)))\,ds\,dt < \infty$,*

C5. *φ is an N-function and the function N satisfies the condition Δ_2, i.e. there exist $\lambda, u_0 \geq 0$ such that $N(2u) \leq \lambda N(u)$ for $u \geq u_0$, there exist $\gamma > 0$ such that $S(u) \leq \gamma N^{-1}(\varphi(u))$ for $u \geq 0$, $K(t,\circ) \in E^M(D,R)$ for a.e. $t \in D$ and the function $t \to \|K(t,\circ)\|_M$ belongs to $E^\varphi(D,R)$.*

Let $(t,s,u) \to h(t,s,u)$ be a nonnegative functions defined for $0 \leq s \leq t \leq d$, $u \geq 0$, satisfying the following conditions:

h is continuous and nondecreasing in u,

for any nonnegative $u \in L^1(D,R)$ there exists the integral $\int_0^t h(t,s,u(s))ds$ for a.e. $t \in D$,

for any a, $0 < a \leq d$, $g = 0$ is the only nonnegative integrable function on $[0,a]$ which satisfies $g(t) \leq \int_0^t h(t,s,g(s))ds$ a.e. on $[0,a]$.

Assume that the conditions C1.,C2., *and any one of* C3.–C5. *are fulfiled. Assume in additioon that*

for any $r > 0$

$$\lim_{\tau\to 0} \sup_{\|x\|_\varphi \leq r} \int_0^d \|A(x)(t+\tau) - A(x)(t)\|_X dt = 0,$$

$\beta(f(t,s,Z) \leq h(t,s,\beta(Z))$ for a.e. $t,s \in D$ and every bounded subset Z of X, where $\beta(Z)$ is a measure of noncompactnes of Z (see [3], p. 239). Then for any $p \in E^\varphi(D,X)$ there exist an interval $J = [0,a]$ and a function $u \in L^\varphi(J,X)$ which satisfies the equation

$$x(t) = p(t) + A(x)(t)$$

for a.e. $t \in J$. In addition, under the assumption C4. *$J = D$.*

Remark 1. If $Z \subset R$ is bounded, then $\beta(Z) = 0$.

2. Main Definitions

Let Y be a real separable Hilbert space. Let Θ denotes the zero element in Y. Let

$$M(I,Y) = \{x : I \to Y : x \text{ is strongly measurable}\},$$

$$M(I,R) = \{q : I \to R : q \text{ is measurable}\},$$
$$M_Y(I) = \{F : I \to 2^Y : F(s) \text{ is nonempty for every } s \in I, \text{ closed and bounded for a.e. } s \in I\}.$$

We denote for all $a \in Y$, $r \geq 0$,

$$B(a,r) = \{x \in Y : \|x - a\|_Y \leq r\},$$

Let:

$$M_Y^0(I) = \{F \in M_Y(I) : F(s) = B(\Theta, r_F(s)) \text{ for every } s \in I,\ r_F(\circ) \in M(I,R)\},$$
$$M_Y^1(I) = \{F \in M_Y(I) : F(s) = B(a_F(s), r_F(s)) \text{ for every } s \in I,\ a_F(\circ) \in M(I,Y),\quad r_F(\circ) \in M(I,R)\}.$$

If $F, G \in M_Y^1(I)$ and $F(t) = G(t)$ for a.e. $t \in I$, then $F = G$ in $M_Y^1(I)$.

In the set $M_Y^1(I)$ we introduce the operations $\odot : R \times M_Y^1(I) \to M_Y^1(I)$, $\oplus : M_Y^1(I) \times M_Y^1(I) \to M_Y^1(I)$ as follows: let $F_1, F_2 \in M_Y^1(I)$, $a \in R$, $F_1(s) = B(a_{F_1}(s), r_{F_1}(s))$, $F_2(s) = B(a_{F_2}(s), r_{F_2}(s))$, for every $s \in I$, if $F = F_1 \oplus F_2$ then

$$F(s) = B(a_{F_1}(s) + a_{F_2}(s), r_{F_1}(s) + r_{F_2}(s))$$

for every $s \in I$, if $G = a \odot F_1$, then

$$G(s) = B(a a_{F_1}(s), a r_{F_1}(s))$$

for every $s \in I$. It is easy to see that $F, G \in M_Y^1(I)$.

Let now

$$M_{Y,\varphi}^0(I) = \{F \in M_Y^0(I) : r_F(\circ) \in L^\varphi(I,R)\},$$
$$M_{Y,\varphi}^1(I) = \{F \in M_Y^1(I) : a_F(\circ) \in L^\varphi(I,Y),\quad r_F(\circ) \in L^\varphi(I,R)\},$$
$$EM_{Y,\varphi}^1(I) = \{F \in M_Y^1(I) : a_F(\circ) \in E^\varphi(I,Y),\quad r_F(\circ) \in E^\varphi(I,R)\}.$$

The spaces $M_{Y,\varphi}^1(I)$ and $M_{Y,\varphi}^0(I)$ will be called the Orlicz spaces of vector multifunctions (see [1]).

3. On the operator H

Let $H : I \times Y \to Y$ and let

$$\mathbf{H}(F)(t) = \{H(t,x) : x \in F(t)\} \text{ for every } t \in I,\ F \in M_Y(I).$$

In [1] is proved the following

LEMMA 1. *Let the function H fulfil the following conditions:*

a) *$H(s,x)$ is a strongly measurable function as a function of s for every $x \in Y$,*

b) *there exists* $L > 0$ *such that* $\|H(s,x) - H(s,y)\|_Y \le L\|x-y\|_Y$ *for all* $s \in I$, $x, y \in Y$,
c) $H(s,\Theta) = \Theta$ *for every* $s \in I$,
d) *if* $\|x\|_Y < \|y\|_Y$, *then* $\|H(s,x)\|_Y < \|H(s,y)\|_Y$ *and if* $\|x\|_Y = \|y\|_Y$, *then* $\|H(s,x)\|_Y = \|H(s,y)\|_Y$ *for every* $s \in I$,
e) *for every* $t \in I$ *and every* $y \in Y$ *there is* $x \in Y$ *such that* $y = H(t,x)$.
Then $\mathbf{H} : M^0_{Y,\varphi}(I) \to M^0_{Y,\varphi}(I)$.

Remark 2. Let $\mathcal{C}(F)(t) = \mathbf{H}(F + (-a_F))(t)$ for every $t \in I$, where $F(t) = B(a_F(t), r_F(t))$ for every $t \in I$. If the assumptions of Lemma 1 hold, then

$$\mathcal{C} : M^1_{Y,\varphi}(I) \to M^0_{Y,\varphi}(I).$$

4. On the operator T

Let $D = [0,d]$, $0 < d < \infty$. Let $K : D \times D \to R_+$ be measurable. Let $q : D \to R$ be measurable. We introduce the operator A^1 by the formula:

$$A^1(q)(t) = \int_0^t K(t,s)q(s)\,ds$$

for every $t \in D$.

Let $x : D \to Y$ be strongly measurable. We introduce the operator A^2 by the formula:

$$A^2(x)(t) = \begin{cases} \int_0^t K(t,s)x(s)\,ds, & \text{if } \int_0^t K(t,s)\|x(s)\|_Y\,ds < \infty \\ \Theta & \text{if } \int_0^t K(t,s)\|x(s)\|_Y\,ds = \infty \end{cases}$$

for every $t \in D$.

Let $\mathcal{B}(F) = \{A^2(x) : x \in M(D,Y) \cap F\}$ for every $F \in M_Y(D)$.

Remark 3. If $A^1 : L^\varphi(D,R) \to L^\varphi(D,R)$, then

$$\mathcal{B} : M^0_{Y,\varphi}(D) \to M^0_{Y,\varphi}(D).$$

Proof. Let $F \in M^0_{Y,\varphi}(D)$. We have for $t \in D$

$$\sup_{x \in M(D,Y)\cap F} \Big\| \int_0^t K(t,s)x(s)\,ds \Big\|_Y \le \sup_{x \in M(D,Y)\cap F} \Big\{ \int_0^t K(t,s)\|x(s)\|_Y\,ds \Big\}$$
$$= \int_0^t K(t,s)r_F(s)\,ds.$$

On the other hand for $x(s) = xr_F(s)/\|x\|_Y$ for every $s \in D$, where $x \in Y$

and $x \neq \Theta$, we have for $t \in D$

$$\Big\| \int_0^t K(t,s)x(s)\,ds \Big\|_Y = \Big\| \frac{x}{\|x\|_Y} \int_0^t K(t,s)r_F(s)\,ds \Big\|_Y = \int_0^t K(t,s)r_F(s)\,ds.$$

Let

$$0 < \int_0^t K(t,s)r_F(s)\,ds < \infty$$

and let $y \in B(\Theta, \int_0^t K(t,s)r_F(s)\,ds)$. Let

$$x_t(s) = y r_F(s) / \int_0^t K(t,s)r_F(s)\,ds$$

for every $s \in D$. We have

$$\int_0^t K(t,s)x_t(s)\,ds = y \quad \text{and} \quad x_t \in M(D,Y) \cap F \text{ because}$$

$$\|x_t(s)\|_Y = \Big\| y r_F(s) \Big/ \int_0^t K(t,s)r_F(s)\,ds \Big\|_Y \leq r_F(s) \text{ for every } s \in D.$$

So $\mathcal{B}(F)(t) = B(\Theta, r_{\mathcal{B}(F)}(t))$ for every $t \in D$, where

$$r_{\mathcal{B}(F)}(t) = \begin{cases} \int_0^t K(t,s)r_F(s)\,ds, & \text{if } A^1_v(r_F)(t) < \infty \\ 0, & \text{if } A^1_v(r_F)(t) = \infty \end{cases}$$

for every $t \in D$. It is easy to see that $r_{\mathcal{B}(F)} \in L^\varphi(D,R)$.

Let $F \in M^1_{Y,\varphi}(D)$ and let $F(s) = B(a_F(s), r_F(s))$ for every $s \in D$. We introduce the operator T by the formula:

$$T(F)(s) = \begin{cases} B(A^2(a_F)(s), A^1(r_F)(s)), & \text{if } A^1(r_F)(s) < \infty \\ \{A^2(a_F)(s)\}, & \text{if } A^1(r_F)(s) = \infty \end{cases}$$

for every $s \in D$.

Remark 4. If $A^1 : L^\varphi(D,R) \to L^\varphi(D,\overline{R})$, where $\overline{R} = [-\infty,+\infty]$, then $T : M^1_{Y,\varphi}(D) \to M^1_{Y,\varphi}(D)$.

Proof. Let $F \in M^1_{Y,\varphi}(D)$, $F(s) = B(a_F(s), r_F(s))$, for every $s \in D$. It is easy to see that

$$B(A^2(a_F)(s), A^1(r_F)(s)) = B(A^2(a_F)(s), 0) \oplus B(\Theta, A^1(r_F)(s))$$

for every $s \in D$ and

$$A^2 : L^\varphi(D,Y) \to L^\varphi(D,Y),$$

so $T(F) \in M^1_{Y,\varphi}(D)$.

COROLLARY 1. *If the assumptions of Lemma* 1 *and Remarks* 2, 4 *hold, then*

$$T(\mathcal{C}) : M^1_{Y,\varphi}(D) \to M^0_{Y,\varphi}(D).$$

Remark 5. $\mathcal{B}(F) = T(F)$ for every $F \in M^0_{Y,\varphi}(D)$.

5. General Theorem

THEOREM 2. *Let the assumptions of Theorem* 1 *without the asumptions* C3. *and* C5. *hold. Let* $H(s,x) = H(x)$ *for every* $s \in I$ *satisfy the assumptions of Lemma* 1 *with* $I = D$. *Let there exists a function* $R_1 : R_+ \to R_+$ *such that:* $\|H(x)\|_Y = R_1(\|x\|_Y)$ *for every* $x \in Y$, R_1 *is a increasing continuous function on* $[0,\infty)$ *and there exists* $L_1 > 0$ *such that* $R_1(u) \le L_1 u$ *for every* $u \in [0,\infty)$. *If the assumption* C4. *of Theorem* 1 *holds with the function* $S = R_1$ *and* $\int_0^d \|K(t,\circ)\|_M \, dt < \infty$, *then for every* $P \in M^1_{Y,\varphi}(D)$ *there exists* $F \in M^1_{Y,\varphi}(D)$ *which satisfies the equation*

$$F(t) = P(t) \oplus A(a_F)(t) \oplus \mathcal{B}(\mathcal{C}(F))(t)$$

for a.e. $t \in D$.

Proof. From the assumptions $A^2 : L^\varphi(D,Y) \to L^\varphi(D,Y)$. So $\mathcal{B}(\mathcal{C}) : M^1_{Y,\varphi}(D) \to M^0_{Y,\varphi}(D)$. So we must solve two equations

$$a_F(t) = a_P(t) + A(a_F)(t),$$

$$r_F(t) = r_P(t) + \int_0^t K(t,s) R_1(r_F(s))\, ds$$

for a.e. $t \in D$.

It is easy to see that there exists $a' \in L^\varphi(D,Y)$ such that

$$a'(t) = a_P(t) + A(a')(t)$$

for a.e. $t \in D$.

It is easy to see that $\beta(K(t,s)R_1(Z)) = 0$ for every $t,s \in D$ and every bounded subset Z of R and also

$$\lim_{\tau\to 0} \sup_{\|x\|_\varphi \le r} \int_0^d \left| \int_0^{t+\tau} K(t,s)R_1(x(s))\, ds - \int_0^t K(t,s)R_1(x(s))\, ds \right| dt = 0$$

for any $r > 0$.

So there is $r' \in L^\varphi(D,R)$ such that $r' \ge 0$ for $t \in D$ and

$$r'(t) = r_P(t) + \int_0^t K(t,s)R_1(r'(s))\, ds$$

for a.e. $t \in D$. Let $\Phi(t) = B(a'(t), r'(t))$ for a.e. $t \in D$. It is easy to see that $\Phi \in M^1_{Y,\varphi}(D)$ and

$$\Phi(t) = P(t) \oplus A(a_\Phi)(t) \oplus \mathcal{B}(\mathcal{C}(\Phi))(t)$$

for a.e. $t \in D$.

Remark 6. The function

$$H(x) = \begin{cases} \frac{x}{\|x\|_Y} \ln(1 + \|x\|_Y), & \text{for } x \neq \Theta \\ \Theta, & \text{for } x = \Theta \end{cases}$$

satisfies the assumptions of Theorem 2.

Remark 7. Let φ fulfil the condition Δ_2. If in Theorem 2 we replece the assumption C4. by the assumption C5. and we assume that $\int_0^d \|K(t, \circ)\|_\psi \, dt < \infty$ instead of $\int_0^d \|K(t, \circ)\|_M \, dt < \infty$, where ψ is complementary to φ, then for any $P \in M^1_{Y,\varphi}(D)$ there exist an interval $J = [0, a]$ and $F \in M^1_{Y,\varphi}(J)$ such that

$$F(t) = P(t) \oplus A(a_F)(t) \oplus \mathcal{B}(\mathcal{C}(F))(t)$$

for a.e. $t \in J$.

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