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ON HARDY TYPE INTEGRAL INEQUALITIES
FOR FUNCTIONS OF TWO VARIABLES

1. Introduction

In the course of attempts to simplify the proof of Hilbert's double series theorem [3] (p. 226), Hardy [2] first proved in 1920 the most famous integral inequality which is now known in the literature as Hardy's inequality [3] (p. 240). Many papers (see [1]–[6]) deal with the alternative proofs and various generalizations of Hardy's integral inequality. In particular, in [4] are proved some interesting variants of Hardy's integral inequality given in [3] (p. 240). The purpose of this paper is to establish some new integral inequalities of Hardy's type involving functions of two independent variables which claim their origin to a certain variant of Hardy's inequality given in [4] (Theorem 2) and to our recent results given in [6]. Our proofs are simple and based on the applications of the well known Fubini's theorem.

2. Statement of results

Throughout the paper without further mention, we assume that all the integrals exist on the respective domains of their definitions.

Our main result is given in the following theorem.

THEOREM 1. *Let f be a nonnegative integrable function on $\Delta = (0, a) \times (0, b)$, where a, b are positive constants, and r_1, r_2 be positive and absolutely continuous functions on $(0, \infty)$ such that*

$$(1) \quad 1 + \frac{px}{p-1} \frac{r'_1(x)}{r_1(x)} \geq \frac{1}{\alpha},$$

$$(2) \quad 1 + \frac{py}{p-1} \frac{r'_2(y)}{r_2(y)} \geq \frac{1}{\beta}$$

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for almost all $x, y \in (0, \infty)$ and for some constants $\alpha > 0, \beta > 0, p > 1$. If F is defined by

$$(3) \quad F(x, y) = \frac{1}{r_1(x)r_2(y)} \int_{x/2}^x \int_{y/2}^y \frac{r_1(s)r_2(t)f(s, t)}{st} dt ds, \quad (x, y) \in \Delta,$$

then

$$(4) \quad \int_0^a \int_0^b \left\{ \frac{F(x, y)}{xy} \right\}^p dy dx \leq (\alpha\beta)^p \left(\frac{p}{p-1} \right)^{2p} \int_0^a \int_0^b \left\{ \frac{1}{xyr_1(x)r_2(y)} \times \right. \\ \times |r_2(y)(r_1(x)f(x, y) - r_2(x/2)f(x/2, y)) - r_2(y/2)(r_1(x)f(x, y/2) \right. \\ \left. - r_1(x/2)f(x/2, y/2))| \right\}^p dy dx.$$

We next establish the following slightly different versions of Theorem 1 which, we believe, are of independent interest.

THEOREM 2. Let g be a nonnegative integrable function on Δ and the functions r_1, r_2 be as defined in Theorem 1. If G is defined by

$$(5) \quad G(x, y) = \frac{1}{r_1(x)r_2(y)} \int_{x/2}^x \int_0^y \frac{r_1(s)r_2(t)g(s, t)}{st} dt ds, \quad (x, y) \in \Delta,$$

then

$$(6) \quad \int_0^a \int_0^b \left\{ \frac{G(x, y)}{xy} \right\}^p dy dx \\ \leq (\alpha\beta)^p \left(\frac{p}{p-1} \right)^{2p} \int_0^a \int_0^b \left\{ \frac{1}{xyr_1(x)} |r_1(x)g(x, y) \right. \\ \left. - r_1(x/2)g(x/2, y)| \right\}^p dy dx,$$

for $p > 1$.

THEOREM 3. Let h be a nonnegative integrable function on Δ and the functions r_1, r_2 be as defined in Theorem 1. If H is defined by

$$(7) \quad H(x, y) = \frac{1}{r_1(x)r_2(y)} \int_0^x \int_{y/2}^y \frac{r_1(s)r_2(t)h(s, t)}{st} dt ds, \quad (x, y) \in \Delta,$$

then

$$(8) \quad \int_0^a \int_0^b \left\{ \frac{H(x, y)}{xy} \right\}^p dy dx \\ \leq (\alpha\beta)^p \left(\frac{p}{p-1} \right)^{2p} \int_0^a \int_0^b \left\{ \frac{1}{xyr_2(y)} |r_2(y)h(x, y) - r_2(y/2)h(x, y/2)| \right\}^p dy dx,$$

for $p > 1$.

Remark 1. The inequalities (4), (6), (8) extend our result in [6] (Theorem 1 with $p = m$ therein) to the case of two independent variables. In the special case of $r_1(x) = r_2(y) = 1$, inequalities (4), (6) and (8) reduce to

$$(9) \quad \int_0^a \int_0^b \left\{ \frac{F(x, y)}{xy} \right\}^p dy dx \\ \leq \left(\frac{p}{p-1} \right)^{2p} \int_0^a \int_0^b \left\{ \frac{1}{xy} |f(x, y) - f(x/2, y) - f(x, y/2) + f(x/2, y/2)| \right\}^p dy dx,$$

$$(10) \quad \int_0^a \int_0^b \left\{ \frac{G(x, y)}{xy} \right\}^p dy dx \\ \leq \left(\frac{p}{p-1} \right)^{2p} \int_0^a \int_0^b \left\{ \frac{|g(x, y) - g(x/2, y)|}{xy} \right\}^p dy dx,$$

and

$$(11) \quad \int_0^a \int_0^b \left\{ \frac{H(x, y)}{xy} \right\}^p dy dx \\ \leq \left(\frac{p}{p-1} \right)^{2p} \int_0^a \int_0^b \left\{ \frac{|h(x, y) - h(x, y/2)|}{xy} \right\}^p dy dx,$$

respectively. The inequalities (9)–(11) can be considered as the two independent variables generalizations of the slight variant of Hardy's inequality given in [4] (Theorem 2).

3. Proof of Theorem 1

For $f = 0$ the inequality (4) is trivially true. We assume that f is positive and denote by I the integral on the left-hand side of (4). By using Fubini's theorem [1] (p. 18), we observe that

$$(12) \quad I = \int_0^a (xr_1(x))^{-p} I_1(x) dx,$$

where

$$(13) \quad I_1(x) = \int_0^b y^{-p} \left\{ \frac{1}{r_2(y)} \int_{y/2}^y \frac{r_2(t)}{t} \left(\int_{x/2}^x \frac{r_1(s)f(s, t)}{s} ds \right) dt \right\}^p dy.$$

By keeping x fixed in (13) and integrating by parts, we have the relation

$$\begin{aligned}
(14) \quad I_1(x) = & -\frac{b^{-p+1}}{p-1} \left\{ \frac{1}{r_2(b)} \int_{b/2}^b \frac{r_2(t)}{t} \left[\int_{x/2}^x \frac{r_1(s)f(s,t)}{s} ds \right] dt \right\}^p \\
& + \frac{p}{p-1} \int_0^b y^{-p+1} \left\{ \frac{1}{r_2(y)} \int_{y/2}^y \frac{r_2(t)}{t} \left[\int_{x/2}^x \frac{r_1(s)f(s,t)}{s} ds \right] dt \right\}^{p-1} \\
& \times \frac{1}{r_2^2(y)} \left[r_2(y) \left(\frac{r_2(y)}{y} \int_{x/2}^x \frac{r_1(s)f(s,y)}{s} ds - \frac{1}{2} \frac{r_2(y/2)}{y/2} \int_{x/2}^x \frac{r_1(s)f(s,y/2)}{s} ds \right) \right. \\
& \left. - r_2'(y) \int_{y/2}^y \frac{r_2(t)}{t} \left(\int_{x/2}^x \frac{r_1(s)f(s,t)}{s} ds \right) dt \right] dy,
\end{aligned}$$

implying

$$\begin{aligned}
(15) \quad & \int_0^b \left(1 + \frac{py}{p-1} \frac{r_2'(y)}{r_2(y)} \right) y^{-p} \left\{ \frac{1}{r_2(y)} \int_{y/2}^y \frac{r_2(t)}{t} \left[\int_{x/2}^x \frac{r_1(s)f(s,t)}{s} ds \right] dt \right\}^p dy \\
& \leq \frac{p}{p-1} \int_0^b \left\{ y^{-p+1} \left[\frac{1}{r_2(y)} \int_{y/2}^y \frac{r_2(t)}{t} \left(\int_{x/2}^x \frac{r_1(s)f(s,t)}{s} ds \right) dt \right] \right\}^{p-1} \\
& \times \frac{1}{y r_2(y)} \left| r_2(y) \int_{x/2}^x \frac{r_1(s)f(s,y)}{s} ds - r_2(y/2) \int_{x/2}^x \frac{r_1(s)f(s,y/2)}{s} ds \right| dy.
\end{aligned}$$

From (1), (15), using Hölder's inequality with indices $\frac{p}{p-1}$, p , we get

$$\begin{aligned}
(16) \quad I_1(x) \leq & \frac{\alpha p}{p-1} I_1^{(p-1)/p}(x) \left[\int_0^b \left\{ \frac{1}{y r_2(y)} \left| r_2(y) \int_{x/2}^x \frac{r_1(s)f(s,y)}{s} ds \right. \right. \right. \\
& \left. \left. \left. - r_2(y/2) \int_{x/2}^x \frac{r_1(s)f(s,y/2)}{s} ds \right| \right\}^p dy \right]^{1/p}.
\end{aligned}$$

Dividing both sides of (16) by $I_1^{(p-1)/p}(x)$ and raising the result to the p -th power, we get

$$\begin{aligned}
(17) \quad I_1(x) \leq & \left(\frac{\alpha p}{p-1} \right)^p \int_0^b y^{-p} \left\{ \frac{1}{r_2(y)} \left| r_2(y) \int_{x/2}^x \frac{r_1(s)f(s,y)}{s} ds \right. \right. \\
& \left. \left. - r_2(y/2) \int_{x/2}^x \frac{r_1(s)f(s,y/2)}{s} ds \right| \right\}^p dy.
\end{aligned}$$

Substituting (17) in (12) and using Fubini's theorem, we obtain

$$(18) \quad I \leq \left(\frac{\alpha p}{p-1} \right)^p \int_0^b y^{-p} I_2(y) dy,$$

where

$$(19) \quad I_2(y) = \int_0^a x^{-p} \left\{ \frac{|m(x, y)|}{r_1(x)r_2(y)} \right\}^p dx,$$

$$(20) \quad m(x, y) = r_2(y) \int_{x/2}^x \frac{r_1(s)f(s, y)}{s} ds - r_2(y/2) \int_{x/2}^x \frac{r_1(s)f(s, y/2)}{s} ds.$$

Now, keeping y fixed in (19), using the relation $|m(x, y)| = m(x, y) \times \operatorname{sgn} m(x, y)$ and integrating by parts, we have the equality

$$(21) \quad I_2(y) = \frac{a^{-p+1}}{-p+1} \left\{ \frac{|m(a, y)|}{r_1(a)r_2(y)} \right\}^p + \frac{p}{p-1} \int_0^a x^{-p+1} \left\{ \frac{|m(x, y)|}{r_1(x)r_2(y)} \right\}^{p-1} \\ \times \frac{\operatorname{sgn} m(x, y)}{x r_1(x) r_2(y)} [r_2(y)(r_1(x)f(x, y) - r_1(x/2)f(x/2, y)) \\ - r_2(y/2)(r_1(x)f(x, y/2) - r_1(x/2)f(x/2, y/2))] dx \\ - \frac{p}{p-1} \int_0^a x^{-p+1} \left\{ \frac{|m(x, y)|}{r_1(x)r_2(y)} \right\}^{p-1} \frac{r'_1(x)}{r_1(x)} \times \frac{m(x, y) \operatorname{sgn} m(x, y)}{r_1(x)r_2(y)} dx$$

which implies

$$(22) \quad \int_0^a \left(1 + \frac{px}{p-1} \frac{r'_1(x)}{r_1(x)} \right) x^{-p} \left\{ \frac{|m(x, y)|}{r_1(x)r_2(y)} \right\}^p dx \\ \leq \frac{p}{p-1} \int_0^a \left[x^{-p+1} \left\{ \frac{|m(x, y)|}{r_1(x)r_2(y)} \right\}^{p-1} \right] \\ \times \left[\frac{1}{x r_1(x) r_2(y)} [r_2(y)(r_1(x)f(x, y) - r_1(x/2)f(x/2, y)) \\ - r_2(y/2)(r_1(x)f(x, y/2) - r_1(x/2)f(x/2, y/2))] \right] dx.$$

From (2), (22), using Hölder's inequality with indices $p/(p-1)$, p , we get

$$(23) \quad I_2(y) \leq \frac{\beta p}{p-1} I_2^{(p-1)/p}(y) \left[\int_0^a x^{-p} \left\{ \frac{1}{r_1(x)r_2(y)} \right. \right. \\ \times |r_2(y)(r_1(x)f(x, y) - r_1(x/2)f(x/2, y)) \\ \left. \left. - r_2(y/2)(r_1(x)f(x, y/2) - r_1(x/2)f(x/2, y/2))| \right\}^p dx \right]^{1/p}.$$

Dividing both sides of (23) by $I_2^{(p-1)/p}(y)$ and raising the result to the p -th power, we get

$$(24) \quad I_2(y) \leq \left(\frac{\beta p}{p-1} \right)^p \int_0^a x^{-p} \left\{ \frac{1}{r_1(x)r_2(y)} \right. \\ \times |r_2(y)(r_1(x)f(x,y) - r_1(x/2)f(x/2,y)) \\ \left. - r_2(y/2)(r_1(x)f(x,y/2) - r_1(x/2)f(x/2,y/2))| \right\}^p dx.$$

Substituting (24) in (18) and using Fubini's theorem, we get (4), which completes the proof of Theorem 1.

We omit the proofs of Theorems 2, 3, being similar to that of Theorem 1 given above.

Remark 2. We note that the method employed in the proofs of Theorems 1–3 can be very easily extended to the case of two independent variables in [6] (Theorems 2–5). We do not discuss here the details of these results, as similar to those given in Theorems 1–3.

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