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ON TRANSLATE OF BERNSTEIN TYPE RATIONAL POLYNOMIALS

1. Introduction

Pàpanicolau [2] studied some approximation results on bounded continuous functions f by a class of linear operators $(L_{n,t}f)$ defined as

$$(1.1)\qquad (L_{n,t}f)(x) = \sum_{k=0}^{n} \binom{k+n-1}{k} \frac{t^k}{(1+t)^{n+k}} f\left(x+\frac{k}{n}\right), \quad \text{for } f \in C_B[0,\infty).$$

Now, following the operators (1.1), we define the following Bernstein type rational polynomials $(P_{n,t}f)$ as

$$(1.2)\qquad (P_{n,t}f)(x) = \sum_{k=0}^{n} A_{n,k,t} f\left(x+\frac{k}{n^\alpha}\right),$$

where

$$(1.3)\qquad A_{n,k,t} = \binom{n}{k}\left(\frac{n^{\alpha-1}t)^k}{(1+n^{\alpha-1}t)^n}\right.$$

$$f \in C_B[0,\infty) \quad \text{and} \quad a \in (0,1],$$

and study some approximation results on the operators (1.2).

2. In this section we prove some basic results which are useful in proving the main results.

LEMMA. *For $t \geq 0$ and $n \in \mathbb{N}$, the following identities hold*

$$(2.1)\qquad \sum_{k=0}^{n} A_{n,k,t} = t,$$

$$(2.2)\qquad \sum_{k=0}^{n} A_{n,k,t}\left(\frac{k}{n^{\alpha}}-t\right) = -\frac{n^{\alpha-1}t^{2}}{1+n^{\alpha-1}t},$$

$$(2.3)\qquad \sum_{k=0}^{n} A_{n,k,t}\left(\frac{k}{n^{\alpha}}-t\right)^{2} = \frac{(n^{2\alpha-2}t^{4}+n^{-\alpha}t)}{(t+n^{\alpha-1}t)^{2}} = B_{n,\alpha,t}\ (say).$$

Proof. On differentiating the expression (2.1) with respect to t and adjusting the terms, we get the required results (2.2) and (2.3). However the proof is similar to that of K. Balâzs [1].

3. In this section we prove our main results.

THEOREM 1. *For fixed $x \in [0,\infty)$ and all $T \geq 0$, we have*

$$(3.1)\qquad \sup_{0\leq t\leq T} |(P_{n,t}f)(x) - f(x+t)| \to 0 \text{ as } n \to \infty.$$

Proof. From the definition (1.2) we write

$$|(P_{n,t}f)(x) - f(x+t)| \leq \sum_{|\frac{k}{n^{\alpha}}-t|<\delta} A_{n,k,t}\left|f\left(x+\frac{k}{n^{\alpha}}\right) - f(x+t)\right| +$$

$$+ \sum_{|\frac{k}{n^{\alpha}}-t|\geq\delta} A_{n,k,t}\left|f\left(x+\frac{k}{n^{\alpha}}\right) - f(x+t)\right| = S_1 + S_2\ (say).$$

This first sum S_1 in the above expression is arbitrarily small if δ is chosen sufficiently small. The choice of δ depends only on powers of n.

Now, with $\delta > 0$ so chosen and fixed, and with $M = \sup_{x\geq 0}|f(x)|$, the following estimate

$$S_2 \leq \frac{2M}{\delta^2} \sum_{|\frac{k}{n^{\alpha}}-t|\geq\delta} A_{n,k,t}\left(\frac{k}{n^{\alpha}}-t\right)^2 \leq$$

$$\leq \frac{2M}{\delta^2}\sum_{k=0}^{n} A_{n,k,t}\left(\frac{k}{n^{\alpha}}-t\right)^2 = \frac{2M}{\delta^2}B_{n,\alpha,t}$$

(using the result (2.3)), approaches to 0 as $n \to \infty$.

Hence the theorem is proved.

THEOREM 2. *Let $f \in C^{(1)}[0,\lambda]$, $\lambda > 0$, be such that $w(f';\delta)$ is the modulus of continuity of f'. Then for $n \geq 1$ and $\delta > 0$ one gets*

$$(3.2)\qquad |(P_{n,t}f)(x) - f(x+t)| \leq \frac{n^{2\alpha-1}t^2}{1+n^{\alpha-1}t}\|f'\| + w(f';\delta)\{\sqrt{B_{n,\alpha,t}} + B_{n,\alpha,t}\}.$$

Proof. Using the mean value theorem, we write

$$f\left(x+\frac{k}{n^\alpha}\right)-f(x+t)=\left(\frac{k}{n^\alpha}-t\right)f'(x+t)+\left(\frac{k}{n^\alpha}-t\right)\{f'(x+\eta)-f'(x+t)\},$$

where η lies between t and $\frac{k}{n^\alpha}$.

Now, applying (1.2) on above, we get

$$|(P_{n,t}f)(x)-f(x+t)|=\left|\sum_{k=1}^{n}A_{n,k,t}\left[f\left(x+\frac{k}{n^\alpha}\right)-f(x+t)\right]\right|,$$

and, using the inequality

$$|f'(x+\eta)-f'(x+t)|\le\{1+|\eta-t|\delta^{-1}\}w(f';\delta), \tag{3.3}$$

we see that

$$|(P_{n,t}f)(x)-f(x+t)|\le|f'(x+t)|\left|\sum_{k=1}^{n}A_{n,k,t}\left(\frac{k}{n^\alpha}-t\right)\right|+ \\ +w(f';\delta)\left\{\sum_{k=1}^{n}A_{n,k,t}\left|\frac{k}{n^\alpha}-t\right|+\sum_{k=1}^{n}A_{n,k,t}\left(\frac{k}{n^\alpha}-t\right)^2\right\}. \tag{3.4}$$

Now, using the results (2.1) to (2.3) in the inequality (3.4), we get the required result (3.2).

Hence the theorem is proved.

Using a slight different method, we can get the following result.

THEOREM 3. *Under the conditions of Theorem* 2, *one gets*

$$|(P_{n,t}f)(x)-f(x+t)|\le \\ \le\left(\frac{n^{2\alpha-1}t^2}{1+n^{\alpha-1}t}\right)\|f'\|+w(f';\delta)\left\{\frac{x^2}{\delta}+\left(1+\frac{x}{\delta}\right)\sqrt{B_{n,\alpha,t}}+\left(\frac{B_{n,\alpha,t}}{2\delta}\right)\right\}. \tag{3.5}$$

Proof. We know that

$$f\left(x+\frac{k}{n^\alpha}\right)-f(x+t)=\left(\frac{k}{n^\alpha}-t\right)f'(t)+\int_{x+t}^{x+\frac{k}{n^\alpha}}(f'(y)-f'(t))\,dy.$$

Now, applying (1.2) and using the inequality (3.3) on above, we get

$$|(P_{n,t}f)(x)-f(x+t)|\le|f'(t)|\left|\sum_{k=1}^{n}A_{n,k,t}\left(\frac{k}{n^\alpha}-t\right)\right|+ \\ +w(f';\delta)\sum_{k=1}^{n}A_{n,k,t}\cdots\left|\int_{x+t}^{x+\frac{k}{n^\alpha}}\left\{1+\frac{|y-t|}{\delta}\right\}dy\right|\le$$

$$\leq |f'(t)| \left| \sum_{k=1}^{n} A_{n,k,t} \left(\frac{k}{n^\alpha} - t \right) \right| +$$

$$+ w(f';\delta) \sum_{k=1}^{n} A_{n,k,t} \left\{ \frac{x^2}{\delta} + \left(1 + \frac{x}{\delta}\right) \left| \frac{k}{n^\alpha} - t \right| + \ldots + \frac{1}{2\delta} \left(\frac{k}{n^\alpha} - t \right)^2 \right\}.$$

Now using the results (2.1) to (2.3) in the above expression, we get the required result (3.5).

Hence the theorem is proved.

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References

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