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ON THE SOLUTION OF THE FIRST FOURIER PROBLEM FOR THE SYSTEM OF DIFFUSION EQUATIONS

1. Introduction

In this paper we are studying the first Fourier problem (F) for the parabolic (in Petrovskii's sense) system

$$v_t(t,x) = \mathbf{A}(\Delta v)(t,x) + \varphi(t,x) \quad (t,x) \in]0,T[\times \Omega,$$

with $\mathbf{A}$ given real $k \times k$ matrix, $\varphi : [0,T] \times \overline{\Omega} \ni (t,x) \to \varphi(t,x) \in \mathbb{R}^k$ given function and $v : [0,T] \times \overline{\Omega} \ni (t,x) \to v(t,x) \in \mathbb{R}^k$, unknown function where $\Omega = \{(x_1,x_2) : x_1^2 + x_2^2 \le a^2\}$.

The initial condition

$$v(0,x) = g(x), \quad x \in \overline{\Omega}, \tag{1.2}$$

where $g : \overline{\Omega} \ni x \to g(x) \in \mathbb{R}^k$ is a given function and boundary condition

$$v(t,x) = 0 \quad \text{on } [0,T] \times \partial\Omega \tag{1.3}$$

are considered. This corresponds physically to diffusion of several gases and evolution of their concentrations.

The solution of this problem is represented as a sum of two integrals being counterparts of the Poisson–Weierstrass integral and potential of plane domain. Kerneks of these integrals are represented by the matrix-function G introduced in this paper and playing crucial role in a representation of a solution v of given Fourier problem (F).

Similar problems were solved by Majchrowski [3] and by Majchrowski and Rogulski [2], but for $\Omega = [0,1]$ only.

2. Assumptions

We make the following assumptions for the functions φ, g and for the matrix $\mathbf{A}$:

(2.1) $\varphi(t,x) = 0$ for $(t,x) \in [0,T] \times \partial\Omega$ and the function $\widetilde{\varphi} : [0,T] \times [0,a] \times [0,2\pi] \to \mathbb{R}^k$, defined by $\widetilde{\varphi}(t,\rho,\gamma) := \varphi(t,\rho\cos\gamma,\rho\sin\gamma)$ is of the class C^1, of the class C^2 in ρ, of the class C^4 in γ, the derivatives $\frac{\partial^{2+l}}{\partial\gamma^l\partial\rho^2}\widetilde{\varphi}$ exist, are continuous, bounded and vanish on $\partial\Omega$ for $l \in \{1,2\}$, and besides $\frac{\partial^s}{\partial\gamma^s}\widetilde{\varphi}(t,\rho,0) = \frac{\partial^s}{\partial\gamma^s}\widetilde{\varphi}(t,\rho,2\pi)$ for $s \in \{0,1,2,3\}$,

(2.2) $g(x) = 0$ for $x \in \partial\Omega$; the function $\widetilde{g} : [0,a] \times [0,2\pi] \to \mathbb{R}^k$, defined by $\widetilde{g}(\rho,\gamma) := g(\rho\cos\gamma,\rho\sin\gamma)$ is of the class C^2 and of the class C^4 in γ, the derivatives $\frac{\partial^{2+l}}{\partial\gamma^l\partial\rho^2}\widetilde{g}$ exist, are continuous, bounded and vanish on $\partial\Omega$ for $l \in \{1,2\}$, and besides $\frac{\partial^s}{\partial\gamma^s}\widetilde{g}(\rho,0) = \frac{\partial^s}{\partial\gamma^s}\widetilde{g}(\rho,2\pi)$ for $s \in \{0,1,2,3\}$,

(2.3) all eigenvalues of $\mathbf{A}$ have positive real parts.

3. Matrix-function $G(t,r,\rho,\vartheta)$

We denote by $\mu_{n,m}$ the m-th positive real zero of equation $J_n(z) = 0$, where $n \in \{0\} \cup \mathbb{N}$, $m \in \mathbb{N}$, and J_n is the Bessel function of degree n.

LEMMA 3.1. *Under the assumption* (2.3) *for every* $T \in]0,\infty[$ *there exist positive constants* C, β, k_0, E *such that for every* $m \in \mathbb{N} \cap [E,\infty]$ *there exists* $l \in \mathbb{N}$ *such that for every* $n \in \mathbb{N}$ *and for every* $t \in]0,T[$

$$\left\| \exp\left[-(\mu_{n,m})^2 \frac{t}{a^2}\mathbf{A}\right] \right\| \le C \exp\left[-(n+l)^2 \frac{\pi^2}{4} t\beta \right] (n+1)^{2k_0}.$$

Proof. In virtue of [2] (p. 1077), there exists a canonical decomposition $\frac{1}{a^2}\mathbf{A} = S + N$ such that $SN = NS$, where S is a semisimple matrix and $N^{k_0+1} = 0$ for $k_0 \le k-1$. Let $S = BCB^{-1}$, where C is the matrix in the Jordana form for the matrix S, whereas B is the matrix of likeness. Then

$$\begin{aligned}\left\| \exp\left[-(\mu_{n,m})^2 \frac{t}{a^2}\mathbf{A}\right] \right\| &\le \| \exp[-(\mu_{n,m})^2 tS] \exp[-(\mu_{n,m})^2 tN] \| \le \\ &\le \| \exp[-(\mu_{n,m})^2 tS] \| \, \| \exp[-(\mu_{n,m})^2 tN] \| = \\ &= \| B \exp[-(\mu_{n,m})^2 tC] B^{-1} \| \, \| \exp[-(\mu_{n,m})^2 tN] \| \le \\ &\le \|B\| \cdot \|B^{-1}\| \exp[-(\mu_{n,m})^2 t\beta] \| \exp[-(\mu_{n,m})^2 tN] \|,\end{aligned}$$

where $\beta = \min(\mathrm{Re}\,\lambda, \lambda$ is an eigenvalue of the matrix $\frac{1}{a^2}\mathbf{A})$. If we take into consideration the inequality

$$\|\exp[-(\mu_{n,m})^2 tN]\| = \left\| \sum_{j=0}^{k_0} \frac{[-(\mu_{n,m})^2 t]^j}{j!} N^j \right\| \leq \tag{3.1}$$

$$\leq [\max\{1, (\mu_{n,m})^2\}]^{k_0} \sum_{j=0}^{k_0} \frac{T^j}{j!} \|N\|^j,$$

then we have

$$\left\| \exp -(\mu_{n,m})^2 \frac{t}{a^2} \mathbf{A} \right\| \leq$$

$$\leq \|B\| \cdot \|B^{-1}\| \exp[-(\mu_{n,m})^2 t\beta](\mu_{n,m})^{2k_0} \sum_{j=0}^{k_0} \frac{T^j}{j!} \|N\|^j.$$

The last inequality follows from the inequalities $\mu_{n,m} \geq 1$ and $\mu_{n,m} > n$ for each $(n,m) \in \mathbb{N} \times \mathbb{N}$ and from the fact that there exists $\bar{k} \in \mathbb{N} \cup \{0\}$ such that $\mu_{0,m} \in]\bar{k}\pi + \frac{3}{4}\pi, \bar{k}\pi + \frac{7}{8}\pi[$ for each $m \in \mathbb{N}$ (see [4] p. 485 and p. 490). From Hankel's asymptotic formula for $x \to \infty$ (see [4] p. 488)

$$J_n(x) = \sqrt{\frac{2}{\pi x}} \left[\cos\left(x - \frac{n\pi}{2}\right) + O\left(\frac{1}{x}\right) \right]$$

it follows that $\cos(\mu_{n,m} - \frac{n\pi}{2}) = O(\frac{1}{\mu_{n,m}})$. Consequently, there exists a constant $\delta > 0$ such that

$$-\delta + \frac{\pi}{2}(n+1) + l\pi \leq \mu_{n,m} \leq \delta + \frac{\pi}{2}(n+1) + l\pi$$

holds for sufficiently large $m \in \mathbb{N}$ and for some $l \in \mathbb{N}$ because of the inequality $0 < \mu_{n,m} < \mu_{n,m+1}$ for $(n,m) \in (\mathbb{N} \cup \{0\}) \times \mathbb{N}$. Hence, for sufficiently large $m \in \mathbb{N}$, there exists $l \in \mathbb{N}$ such that for each $n \in \mathbb{N} \cup \{0\}$ we have

$$\frac{1}{2}(n+l)\pi \leq \mu_{n,m} \leq (n+l)\pi. \tag{3.2}$$

Finally

$$\left\| \exp\left[-(\mu_{n,m})^2 \frac{t}{a^2} \mathbf{A} \right] \right\| \leq C \exp\left[-(n+l)^2 \cdot \frac{\pi^2}{4} r\beta \right] (n+l)^{2k_0}.$$

This ends the proof of Lemma 3.1.

LEMMA 3.2. *Under the assumption* (2.3) *there exists a constant* $b \in \mathbb{R}$ *such that for each* $t > 0$ *the inequality* $\|\exp[-(\mu_{n,m})^2 \frac{t}{a^2}\mathbf{A}]\| \leq b$ *holds.*

Proof. Observe that (see [2])

$$e^{-k^2 z}k^{2j} \le \frac{j^j}{z^j}e^{-j} \quad \text{for all } j,k,z>0.$$

Then from (3.1) we obtain the inequality

$$\|\exp[-(\mu_{n,m})^2 t(S+N)]\| \le$$
$$\le \|B\|\cdot\|B^{-1}\|\sum_{j=0}^{k_0}\exp\Big[-(n+l)^2\frac{\pi^2}{16}t\beta\Big]\frac{(n+l)^{2j}\pi^{2j}t^j}{j!}\|N\|^j \le$$
$$\le \|B\|\cdot\|B^{-1}\|\Big(1+\sum_{j=1}^{k_0}\frac{(4j\|N\|)^j}{\beta^j j!}e^{-j}\Big)$$

for sufficiently large m, for instance $m>m_0$. The methods applied above allow us also to show for $m\le m_0$ that all the functions of the form

$$\mathbb{R}_+ \ni t \to \Big\|\exp\Big[-\Big(\frac{\mu_{n,m}}{a}\Big)^2 t\mathbf{A}\Big]\Big\| \in \mathbb{R}$$

are bounded. The proof of Lemma 3.2 is complete.

Let us introduce now a matrix-valued function

$$G :]0,\infty[\, \times [0,a]\times[0,a]\times\mathbb{R}\to\mathbb{R}^{k^2}$$

given by the formula

$$G(t,r,\rho,\vartheta) = \tag{3.3}$$
$$= \sum_{n=0}^{\infty}\sum_{n=1}^{\infty}\frac{J_n(\mu_{n,m}\frac{r}{a})J_n(\mu_{n,m}\frac{\rho}{a})}{\varepsilon_n[J_{n+1}(\mu_{n,m})]^2}\cos n\vartheta\cdot\exp\Big[-(\mu_{n,m})^2\frac{t\mathbf{A}}{a^2}\Big],$$

with

$$\varepsilon_n = \begin{cases} 2 & \text{for } n=0,\\ 1 & \text{for } n>0,\end{cases}$$

which plays the same role for the system (1.1) as the function θ_3 for a single parabolic equation in Cannon's paper [1] or the matrix-function M introduced by Majchrowski and Rogulski ([2]).

Now we shall consider the properties of the matrix-function G.

THEOREM 3.1. *Under the assumption* (2.3) *the matrix-valued function G defined by the formula* (3.3) *is of the class C^∞ and all its partial derivatives can be calculated by term-by-term differentiation of the series* (3.3).

Proof. From the fact that $J_n(z)$ and $I_{n+l}(z)$, where $l \in \mathbb{N}$, have no common roots (see [4] p. 484), from the inequality $|J_n(x)| \leq 1$ for $n \in \mathbb{N} \cup \{0\}$ and $x \in \mathbb{R}$ (see [4] p. 31), and from Hankel's asymptotic formula we obtain the inequality

$$\left| \frac{J_n(\mu_{n,m}\frac{r}{a})J_n(\mu_{n,m}\frac{\rho}{a})}{\varepsilon_n[J_{n+1}(\mu_{n,m})]^2} \right| \leq C$$

for all $n \in \mathbb{N} \cup \{0\}$, $m \in \mathbb{N}$ and $r, \rho \in [0, a]$, where C is a constant independent of r, ρ, m, n. It follows from Lemma 3.1 that the series is uniformly convergent on the arbitrary subset

$$P_\delta = \{(t, r, \rho, \vartheta) : \delta < t < T; r, \rho \in [O, a];\ \vartheta \in \mathbb{R}\}$$

of the set $]0, \infty[\times [0, a] \times [0, a] \times \mathbb{R}$.

Taking also into consideration Lemma 3.2 we complete the proof of theorem.

4. Auxiliary theorems

LEMMA 4.1. *If*

1° $\sum_{n=0}^{\infty} \sum_{m=1}^{\infty} c_{n,m}$ *is convergent,*

2° *the functions* $f_{n,m} : [0, T] \to \mathbb{R}$, *where* $(n, m) \in J = (\mathbb{N} \cup \{0\}) \times \mathbb{N}$, *have the properties*

(a) $$\bigwedge_{(n,m)\in J} \bigwedge_{\delta\in]0,T[} \bigvee_{q\in]0,1[} \bigvee_{l(m)\in \mathbb{N}} \bigwedge_{t\in[\delta,T]} \left(\lim_{m\to\infty} l(m) = \infty \wedge 0 \leq f_{n,m}(t) \leq q^{n+l(m)} \right),$$

(b) $$\bigwedge_{(n,m)\in J} \lim_{t\to 0^+} f_{n,m}(t) = 1 = f_{n,m}(0),$$

(c) $$\bigwedge_{t\in[0,T]} \bigwedge_{(n,m)\in J} ((f_{n+1,m}(t) \leq f_{n,m}(t)) \wedge (f_{n,m+1}(t) \leq f_{n,m}(t))),$$

then

1) $\bigwedge_{\delta\in[0,T]} \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} (c_{n,m} f_{n,m}(\cdot))$ *is uniformlý convergent on* $[\delta, T]$,

2) *there exists*

$$\lim_{t\to 0^+} \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} c_{n,m} f_{n,m}(t) = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} c_{n,m}.$$

Proof. At first we shall prove that for every $\varepsilon > 0$ there exists $p_0 \in \mathbb{N}$, such that for all $k > p_0$ and $r > p_0$ and all $p, q \in \mathbb{N}$ such that $p \geq k$ and $q \geq r$ the following inequality holds

$$\Big|\sum_{n=0}^{p}\sum_{m=1}^{q} c_{n,m} f_{n,m}(t) - \sum_{n=0}^{k-1}\sum_{m=1}^{r-1} c_{n,m} f_{n,m}(t)\Big| < \varepsilon$$

which we can rewrite as

$$\Big|\sum_{n=0}^{p}\sum_{m=r}^{q} c_{n,m} f_{n,m}(t) + \sum_{n=k}^{p}\sum_{m=1}^{r-1} c_{n,m} f_{n,m}(t)\Big| < \varepsilon.$$

Let us denote

$$S_1(t) := \sum_{n=0}^{p}\sum_{m=r}^{q} c_{n,m} f_{n,m}(t), \qquad S_2(t) := \sum_{n=k}^{p}\sum_{m=1}^{r-1} c_{n,m} f_{n,m}(t),$$

$$\sigma_w^{(n)} = \sum_{s=1}^{w} c_{n,s}, \qquad \Delta_{w,r} := \sum_{s=0}^{w} M_r^{(s)}, \qquad \delta_{w,r} := \sum_{s=0}^{w} m_r^{(s)},$$

where

$$M_r^{(s)} := \begin{cases} \max(\sigma_1^{(s)}, \ldots, \sigma_{r-1}^{(s)}, -\sigma_{r-1}^{(s)}, \sigma_r^{(s)}, \ldots, \sigma_q^{(s)}) & \text{for } 1 < r \leq q, \\ \max(\sigma_1^{(s)}, \ldots, \sigma_q^{(s)}) & \text{for } r = 1, \text{ and for } s \in \mathbb{N}, \end{cases}$$

$$m_r^{(s)} := \begin{cases} \min(\sigma_q^{(s)}, \ldots, \sigma_{r-1}^{(s)}, -\sigma_{r-1}^{(s)}, \sigma_r^{(s)}, \ldots, \sigma_q^{(s)}) & \text{for } 1 < r \leq q, \\ \min(\sigma_1^{(s)}, \ldots, \sigma_q^{(s)}) & \text{for } r = 1, \text{ and for } s \in \mathbb{N} \cup \{0\}, \end{cases}$$

$$M_k := \begin{cases} \max(\Delta_{0,r}, \ldots, \Delta_{k-1,r}, -\Delta_{k-1,r}, \Delta_{k,r}, \ldots, \Delta_{p,r}) & \text{for } 0 < k \leq p, \\ \max(\Delta_{0,r}, \ldots, \Delta_{p,r}) & \text{for } k = 0, \end{cases}$$

$$m_k := \begin{cases} \min(\delta_{0,r}, \ldots, \delta_{k-1,r}, -\delta_{k-1,r}, \delta_{k,r}, \ldots, \delta_{p,r}) & \text{for } 0 < k \leq p, \\ \min(\delta_{0,r}, \ldots, \delta_{p,r}) & \text{for } k = 0. \end{cases}$$

Next, using the Abel transformation, we get the inequality

$$\begin{aligned} \sum_{m=r}^{q} c_{n,m} f_{n,m}(t) &= \sum_{s=0}^{q-r} f_{n,r+s}(t)(\sigma_{r+s}^{(n)} - \sigma_{r+s-1}^{(n)}) = \\ &= -\sigma_{r-1}^{(n)} f_{n,r}(t) + \sum_{s=0}^{q-1-r} \sigma_{r+s}^{(n)}(f_{n,r+s}(t) - f_{n,r+s+1}(t)) + \\ &\quad + \sigma_q^{(n)} f_{n,q}(t) \leq 2M_r^{(n)} f_{n,r}(t) \end{aligned}$$

and analogously

$$\sum_{m=r}^{q} c_{n,m} f_{n,m}(t) \geq 2m_r^{(n)} f_{n,r}(t)$$

which imply

$$S_1(t) \leq \sum_{n=0}^{p} (2m_r^{(n)} f_{n,r}(t)) = 2\Big\{ \sum_{s=1}^{p} (\Delta_{s,r} - \Delta_{s-1,r}) f_{s,r}(t) + \Delta_{0,r} f_{0,r}(t) \Big\} =$$

$$= 2\Big\{ \sum_{s=1}^{p} \Delta_{s-1,r}(f_{s-1,r}(t) - f_{s,r}(t)) + \Delta_{p,r} f_{p,r}(t) \Big\}.$$

Finally, we have

$$|S_1(t)| \leq 2D_k f_{0,r}(t) \leq 2D_k q_0^{l(r)},$$

where $D_k = \max(|M_k|, |m_k|)$. The last inequality holds independently of $t \in [\delta, T]$ and for all $\delta \in]0, T]$.

Using the Abel transformation, we have

$$\sum_{m=1}^{r-1} c_{n,m} f_{n,m}(t) = \sigma_1^{(n)} f_{n,1}(t) + \sum_{s=2}^{r-1} (\sigma_s^{(n)} - \sigma_{s-1}^{(n)}) f_{n,s}(t) =$$

$$= \sum_{s=1}^{r-2} \sigma_s^{(n)} (f_{n,s}(t) - f_{n,s+1}(t) + \sigma_{r-1}^{(n)} f_{n,r-1}(t)$$

and, by the assumptions, we obtain the inequalities

$$\sum_{m=1}^{r-1} c_{n,m} f_{n,m}(t) \leq M_1^{(n)} f_{n,1}(t) \leq M_r^{(n)} f_{n,1}(t),$$

$$\sum_{m=1}^{r-1} c_{n,m} f_{n,m}(t) \geq m_1^{(n)} f_{n,1}(t) \geq m_r^{(n)} f_{n,1}(t).$$

Next, the Abel transformation gives

$$\sum_{n=k}^{p} M_r^{(n)} f_{n,1}(t) = -\Delta_{k-1,r} f_{k,1}(t) +$$

$$+ \sum_{n=k}^{p-1} \Delta_{n,r}(f_{n,1}(t) - f_{n+1,1}(t)) + \Delta_{p,r} f_{p,1}(t),$$

and from the assumptions we have $2m_k f_{k,1}(t) \leq S_2(t) \leq 2M_k f_{k,1}(t)$ or $|S_2(t)| \leq 2D_k f_{k,1}(t)$ and then

$$|S_1(t) + S_2(t)| \leq 2D_k(q_0^{l(r)} + q_0^k q_0^{l(1)}) \leq \varepsilon$$

for $k > k_0$, $r > r_0$, where k_0 and r_0 ate sufficiently large non-negative integers, $q_0 \in]0,1[$ and there exists $p_0 = \max(k_0, r_0)$.

This ends the proof of part 1). Part 2) follows immediately from the equality

$$\lim_{t\to 0^+} \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} c_{n,m} f_{n,m}(t) = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} c_{n,m} f_{n,m}(0). \blacksquare$$

LEMMA 4.2. *The functions of the form* $f_{n,m}(t) = \exp[-(\mu_{n,m})^2 \alpha t]$, *where* $\alpha > 0$, $t \in [0,T]$, $(n,m) \in J$ *fulfil assumptions of Lemma* 4.1.

Proof. The inequality (a) of Lemma 4.1 follows immediately from (3.2). Continuity of the functions $f_{n,m}$ is evident. From [4] (p. 479) it follows that the positive zeros of $J_n(x)$ are interlaced with those of $J_{n+1}(x)$, i.e. $0 < \ldots < \mu_{n,m} < \mu_{n+1,m} < \mu_{n,m+1} < \mu_{n+1,m+1}$ what implies that for every $t \in [0,T]$ the functions $f_{n,m}$ satisfy 2°(c).

LEMMA 4.3. *If*

1° S *is a semisimple real matrix* $k \times k$ *with eigenvalues* λ *such that* $\operatorname{Re}\lambda > 0$,

2° $\sum_{n=0}^{\infty}\sum_{m=1}^{\infty} c_{n,m}$ *is convergent*,

3° β *is a positive constant*,

then $\bigwedge_{t>0} \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} c_{n,m} \exp(-(\mu_{n,m})^2 \beta t S)$ *is convergent and there exists the limit*

$$\lim_{t\to 0^+} \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} c_{n,m} \exp[-(\mu_{n,m})^2 \beta t S] = \Big(\sum_{n=0}^{\infty}\sum_{m=1}^{\infty} c_{n,m}\Big) I \tag{4.1}$$

where I *is the unit matrix* $k \times k$.

Proof. In view of the relation $\|\exp(-cS)\| \le D\exp(-c\alpha)$, with $\alpha = \min\{\operatorname{Re}\lambda : \det(S - \lambda I) = 0\}$ and c, D positive constants (see [2]), we have

$$\|\exp[-(\mu_{n,m})^2 \beta t S]\| \le \|M\|\,\|M^{-1}\|\exp[-(\mu_{n,m})^2 \alpha\beta t]\|,$$

where M is the matrix of likeness. From the assumptions and from Lemmas 4.1, 4.2 the convergence of considered series follows. From the first part of Lemma 4.3 we obtain (4.1), because the series is uniformly convergent on $[\delta, T]$ for all $\delta \in]0,T[$.

Let us introduce now some denotations which will be used in the next theorem:

$F : [0,T] \times [0,1] \times [0,2\pi] \ni (\eta,\rho,\gamma) \to F(\eta,\rho,\gamma) \in \mathbb{R}$,

$$a_{n,m}(\eta,\gamma) := \frac{2\int_0^1 \rho F(\eta,\rho,\gamma) J_n(\mu_{n,m}\rho)\,d\rho}{(J_{n+1}(\mu_{n,m}))^2},$$

$$h_{n,m}(\eta,r,\beta) := J_n(\mu_{n,m}r)[\cos n\beta \overset{1}{h}_{n,m}(\eta) + \sin n\beta \overset{2}{h}_{n,m}(\eta)],$$

where

$$\overset{1}{h}_{n,m}(\eta) := \frac{2}{(J_{n+1}(\mu_{n,m}))^2} \int\limits_0^{2\pi} \Big[\int\limits_0^1 \rho \cos n\gamma J_n(\mu_{n,m}\rho) F(\eta,\rho,\gamma)\,d\rho \Big]\,d\gamma,$$

$$\overset{2}{h}_{n,m}(\eta) := \frac{2}{(J_{n+1}(\mu_{n,m}))^2} \int\limits_0^{2\pi} \Big[\int\limits_0^1 \rho \sin n\gamma J_n(\mu_{n,m}\rho) F(\eta,\rho,\gamma)\,d\rho \Big]\,d\gamma,$$

$$c_{n,m}(\eta,\gamma,\beta) := \mu_{n,m}^2 h_{n,m}(\eta,r,\beta),$$

$$(r,\beta) \in [0,1] \times [0,2\pi] \quad \text{and} \quad (n,m) \in J.$$

THEOREM 4.1. *If the function F fulfils the same assumptions* (2.1) *as $\tilde{\varphi}$ with $a = 1$ and $k = 1$, then the series $\sum_{n=0}^{\infty}\sum_{m=1}^{\infty} c_{n,m}(\eta,r,\beta)$, $\sum_{n=0}^{\infty}\sum_{m=1}^{\infty} h_{n,m}(\eta,r,\beta)$ are uniformly convergent in η on every interval $[\delta, T-\delta_1]$, where $0 < \delta < T - \delta_1 < T$, for all $(r,\beta) \in [0,1] \times [0,2\pi]$.*

Proof. If we denote $a_m := a_{n,m}(\eta,\gamma)$ for fixed η, γ and $n \in \{0\} \cup \mathbb{N}$, then

$$\sum_{m=1}^{\infty} a_m J_n(\mu_{n,m} r)$$

is the Fourier–Bessel series of the function F with respect to ρ. By assumptions, for $l = 0$ and $l = 1$ the derivatives $\frac{\partial^l}{\partial r^l} F(\eta,r,\beta)$ have limited total fluctuation in $[\varepsilon, 1-\varepsilon]$ for all $\varepsilon \in \,]0,\frac{1}{2}[$ and there exist the integrals $\int_0^1 \sqrt{\rho} \frac{\partial^l}{\partial \rho^l} F(\eta,\rho,\beta)\,d\rho$, $\int_0^1 \rho^{n+\frac{1}{2}} \frac{\partial}{\partial \rho}(\rho^{-n} \frac{\partial^l}{\partial \rho^l} F(\eta,\rho,\beta))\,d\rho$, $n \in \mathbb{N} \cup \{0\}$, and the limits $\lim_{r\to 0^+} \frac{\partial^l}{\partial r^l} F(\eta,r,\beta) = 0$, $\lim_{r\to 1^-} \frac{\partial^l}{\partial r^l} F(\eta,r,\beta) = 0$. Hence the series $\sum_{m=1}^{\infty} a_m \mu_{n,m} J_n'(\mu_{n,m})$ (see [4] p. 605) is convergent to $\frac{\partial}{\partial r} F(\eta,r,\beta)$ for fixed η, β and n. From the recurrence formulae

$$zJ_n'(z) + nJ_n(z) = zJ_{n-1}(z), \qquad zJ_n'(z) - nJ_n(z) = -zJ_{n+1}(z),$$

for $n \in \{0\} \cup \mathbb{N}$, it follows immediately that

$$J_n''(z) = \left(\frac{n}{z}J_n(z) - J_{n+1}(z)\right)' =$$
$$= -J_{n+1}'(z) - \frac{n}{z^2}J_n(z) + \frac{n}{z}J_n'(z) =$$
$$= \frac{n+1}{z}J_{n+1}(z) - J_n(z) - \frac{n}{z^2}J_n(z) + \frac{n}{z}\left(\frac{n}{z}J_n(z) - J_{n+1}(z)\right) =$$
$$= \frac{1}{z}J_{n+1}(z) + \frac{n^2-n}{z^2}J_n(z) - J_n(z).$$

By assumptions on F, the series

$$\sum_{m+1}^{\infty} \frac{n^2-r}{r^2\mu_{n,m}^2}a_{n,m}^2 J_n(\mu_{n,m}r) = \frac{n^2-n}{r^2}\sum_{m=1}^{\infty} a_m J_n(\mu_{n,m}r)$$

is the Fourier–Bessel expansion of the function F. Analogously, the series $\frac{1}{r}\sum_{m=1}^{\infty} a_m\mu_{n,m}J_{n+1}(\mu_{n,m}r)$ is the Dini expansion of the form $\sum_{m=1}^{\infty} b_m J_{n+1}(\lambda_{n+1,m}\rho)$, where

$$b_m = 2\lambda_{n+1,m}^2 \int_0^1 t\left(\frac{\partial}{\partial t}F(\eta,t,\beta) - \frac{n}{t}F(\eta,t,\beta)\right)J_{n+1}(\lambda_{n+1,m}t)\,dt\times$$
$$\times \{(\lambda_{n+1,m}^2 - (n+1)^2)J_{n+1}^2(\lambda_{n+1,m}) + \lambda_{n+1,m}^2(J_{n+1}'(\lambda_{n+1,m}))^2\}^{-1},$$

where $\lambda_{n+1,m}$ denotes m-th positive real zero of the function $\{zJ_{n+1}'(z) + (n+1)J_{n+1}(z)\}$. Taking advantage of the recurrence formulae for the Bessel function, we see that $\lambda_{n+1,m} = \mu_{n,m}$ and, in virtue of assumptions, $b_m = a_m\mu_{n,m}$. The assumptions are sufficient to the uniformly convergence of the Fourier–Bessel series and of the Dini series (see [4] p. 593 and p. 601) with respect to the variable r on all intevrals $[\varepsilon, 1-\varepsilon]$, where $0 < \varepsilon < 1$.

Taking into consideration the recurrence formulae since furthermore the series $\sum_{m=1}^{\infty} a_m\mu_{n,m}J_n'(\mu_{n,m}r)$ is convergent to the function $\frac{\partial}{\partial r}F(\eta,r,\beta)$, (see [4] p. 605) and the series $\sum_{m=1}^{\infty} a_m\mu_{n,m}^2 J_n''(\mu_{n,m}r)$ is uniformly convergent to $\frac{\partial^2}{\partial r^2}F(\eta,r,\beta)$, with respect to r, the series $\sum_{m=1}^{\infty} c_{n,m}(\eta,r,\beta)$ is convergent in η for $n \in \{0\} \cup \mathbb{N}$.

Next, applying [4] (p. 583 and p. 598), we can represent the partial sum of a Fourier–Bessel series and of a Dini series as a sum of residues of one function of complex variable having poles at the points $\mu_{n,m}$ in the case of Fourier–Bessel series and at the points $\mu_{n,m}$ and $\lambda_{n,m}$ in the case of Dini series. Therefore, the function which is the sum of the series $\sum_{m=1}^{\infty} c_{n,m}(\eta,r,\beta)$ is continuous with respect to the parameters η and β for all $n \in \{0\} \cup \mathbb{N}$. It follows from the definition of residue and from the compactness of the contour on which we calculate integrals. This means that the functions whose variables are parameters of integrals are continuous.

Let us denote $S_n(\eta, r, \beta) := \sum_{m=1}^{\infty} c_{n,m}(\eta, r, \beta)$. Making use of the mathematical induction with respect to $p \in \mathbb{N}$ and integrating by parts, one can prove that

$$\int_0^{2\pi} \cos n\gamma F(\eta,\rho,\gamma)\,d\gamma = (-1)^p \frac{1}{n^{2p}} \int_0^{2\pi} \cos n\gamma \frac{\partial^{2p}}{\partial\gamma^{2p}} F(\eta,\rho,\gamma)\,d\gamma, \tag{4.2}$$

by assumption

$$\frac{\partial^{2s-1}}{\partial\gamma^{2s-1}} G(\eta,\rho,0) = \frac{\partial^{2s-1}}{\partial\gamma^{2s-1}} F(\eta,\rho,2\pi)$$

for $s \in \mathbb{N} \cap [1,p]$, $(\eta,\rho) \in [0,T] \times [0,1]$, and that

$$\int_0^{2\pi} \sin n\gamma F(\eta,\rho,\gamma)\,d\gamma = (-1)^p \frac{1}{n^{2p}} \int_0^{2\pi} \sin n\gamma \frac{\partial^{2p}}{\partial\gamma^{2p}} F(\eta,\rho,\gamma)\,d\gamma, \tag{4.3}$$

by assumption

$$\frac{\partial^{2(s-1)}}{\partial\gamma^{2(s-1)}} G(\eta,\rho,0) = \frac{\partial^{2(s-1)}}{\partial\gamma^{2(s-1)}} F(\eta,\rho,2\pi)$$

for $s \in \mathbb{N} \cap [1,p]$, $(\eta,\rho) \in [0,T] \times [0,1]$. Applying the equalities (4.2) and (4.3) for $p = 2$, we obtain the uniformly convergence of the series $\sum_{n=0}^{\infty} S_n(\eta, r, \beta)$, by assumptions on F. Observe that the series $\sum_{m=1}^{\infty} h_{n,m}(\eta, r, \beta)$ is a Fourier–Bessel series which is uniformly convergent on $[\delta, 1-\delta_1]$, with respect to the variable r. So, by proceeding as before, we prove the convergence of the series $\sum_{n=0}^{\infty}(\sum_{m=1}^{\infty} h_{n,m}(\eta, r, \beta))$.

5. The solution of the problem (F)

In order to construct a solution of the problem (F) given by (1.1)–(1.3) we shall prove two existence theorems.

Denote

$$\begin{aligned} v_1(t,x) &= v_1(t, r\cos\beta, r\sin\beta) = \tilde{v}_1(t,r,\beta) = \\ &= \frac{2}{\pi a^2} \int_0^a \Big[\int_0^{2\pi} G(t,r,\rho,\beta-\gamma)\tilde{g}(\rho,\gamma)\,d\gamma \Big] \rho\,d\rho, \end{aligned} \tag{5.1}$$

where $\tilde{g}(\rho,\gamma) = g(\rho\cos\gamma, \rho\sin\gamma) = g(y)$, $y \in \Omega$, $(r\cos\beta, r\sin\beta) = x \in \Omega$, $t \in\,]0,T[$, the function g occurs in (1.2) and G is given by (3.3).

THEOREM 5.1. *If the function g fulfils the assumptions* (2.2), *then the function* v_1 *given by the formula* (5.1) *is a solution of the problem* (F) *with* $\varphi = 0$.

Proof. Making use of the properties of the function G and of the equality

$$G(t,r,\rho,\beta-\gamma) = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} \frac{J_n(\mu_{n,m}\frac{r}{a})J_n(\mu_{n,m}\frac{\rho}{a})}{\varepsilon_n[J_{n+1}(\mu_{n,m})]^2}\times$$
$$\times(\cos n\beta\cos n\gamma + \sin n\beta \sin n\gamma)\exp\left[-(\mu_{n,m})^2\frac{t}{a^2}\mathbf{A}\right],$$

(see (3.3) and defining $A_{n,m}$, $C_{n,m}$ — the $\mathbb{R}^k$-valued coefficients of the Fourier–Bessel expansion of the function $\widetilde{g}$ by formulas

$$A_{n,m} = \frac{2}{\varepsilon_n\pi a^2[J_{n+1}(\mu_{n,m}]^2}\times\int_0^a \rho J_n\left(\mu_{n,m}\frac{\rho}{a}\right)\left[\int_0^{2\pi}\cos n\gamma\widetilde{g}(\rho,\gamma)\,d\gamma\right]d\rho,$$

$$C_{n,m} = \frac{2}{\varepsilon_n\pi a^2[J_{n+1}(\mu_{n,m})]^2}\times\int_0^a \rho J_n\left(\mu_{n,m}\frac{\rho}{a}\right)\left[\int_0^{2\pi}\sin n\gamma\widetilde{g}(\rho,\gamma)\,d\gamma\right]d\rho,$$

where $(n,m)\in J$, we can represent the function $\widetilde{v}_1$ in the form

$$\widetilde{v}_1(t,r,\beta) = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\exp\left[-(\mu_{n,m})^2\frac{t}{a^2}\mathbf{A}\right]J_n\left(\mu_{n,m}\frac{r}{a}\right)\times$$
$$\times\{\cos n\beta A_{n,m} + \sin n\beta C_{n,m}\}.$$

Next, we are going to prove that the function $\widetilde{v}_1$ fulfils the equation

$$\frac{\partial\widetilde{v}_1}{\partial t}(t,r,\beta) = \mathbf{A}\widetilde{\Delta}\widetilde{v}_1(t,r,\beta). \tag{5.3}$$

Calculating derivatives of $\widetilde{v}_1$, we have

$$\frac{\partial\widetilde{v}_1}{\partial t} = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\left(-(\mu_{n,m})^2\frac{1}{a^2}\mathbf{A}\right)\exp\left[-(\mu_{n,m})^2\frac{t}{a^2}\mathbf{A}\right]\times$$
$$\times\left\{J_n\left(\mu_{n,m}\frac{r}{a}\right)\cos n\beta A_{n,m} + J_n\left(\mu_{n,m}\frac{r}{a}\right)\sin n\beta C_{n,m}\right\},$$

$$\widetilde{\Delta}\widetilde{v}_1(t,r,\beta) = \frac{\partial^2\widetilde{v}_1}{\partial r^2} + \frac{1}{r}\frac{\partial\widetilde{v}_1}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v_1}{\partial\beta^2} =$$

$$= \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} \exp\left[-(\mu_{n,m})^2 \frac{t}{a^2}\mathbf{A}\right]\left\{\left[\frac{d^2}{dr^2}J_n\left(\mu_{n,m}\frac{r}{a}\right)+\right.\right.$$

$$\left. + \frac{1}{r}\frac{d}{dr}J_n\left(\mu_{n,m}\frac{r}{a}\right) - \frac{n^2}{r^2}J_n\left(\mu_{n,m}\frac{r}{a}\right)\right]\cos n\beta A_{n,m} +$$

$$+ \left[\frac{d^2}{dr^2}J_n\left(\mu_{n,m}\frac{r}{a}\right) + \frac{1}{r}\frac{d}{dr}J_n\left(\mu_{n,m}\frac{r}{a}\right)+\right.$$

$$\left.\left. - \frac{n^2}{r^2}J_n\left(\mu_{n,m}\frac{r}{a}\right)\right]\sin n\beta C_{n,m}\right\} =$$

$$= -\sum_{n=0}^{\infty}\sum_{m=1}^{\infty} \exp\left[-(\mu_{n,m})^2 \frac{t}{a^2}\mathbf{A}\right]\times$$

$$\times\left\{\left(\frac{\mu_{n,m}}{a}\right)^2 J_n\left(\mu_{n,m}\frac{r}{a}\right)(\cos n\beta A_{n,m} + \sin n\beta C_{n,m})\right\}.$$

Since $W = J_n(\mu_{n,m}\frac{r}{a})$ is a solution of the Bessel equation

$$\frac{d^2W}{dr^2} + \frac{1}{r}\frac{dW}{dr} + (k_0^2 - \frac{n^2}{r^2})W = 0.$$

Thus the function $\widetilde{v}_1$ is a solution of the equation (5.3).

Now we can observe that

$$\widetilde{v}_1(0,r,\beta) = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\mathbf{I}\left\{J_n\left(\mu_{n,m}\frac{r}{a}\right)(\cos n\beta A_{n,m} + \sin n\beta C_{n,m})\right\} = \widetilde{g}(r,\beta),$$

where $\mathbf{I}$ is the unit matrix $k \times k$. It follows from the definitions of $\widetilde{v}_1$ and $\mu_{n,m}$ that $\lim_{r\to a^-}\widetilde{v}_1(t,r,\beta) = 0 \in \mathbf{R}^k$. The proof of Theorem 5.1 is now complete.

Next, let us denote

(5.4) $v_2(t,x) = v_2(t, r\cos\beta, r\sin\beta) = \widetilde{v}_2(t,r,\beta) =$

$$= \frac{2}{\pi a^2}\int_0^t \left[\int_0^{2\pi}\left[\int_0^a \rho G(t-\tau, r, \rho, \beta-\gamma)\widetilde{\varphi}(\eta,\rho,\gamma)\,d\rho\right]d\gamma\right]d\eta,$$

where $\widetilde{\varphi}(t,r,\beta) = \varphi(t, r\cos\beta, r\sin\beta) = \varphi(t,x)$, $(r\cos\beta, r\sin\beta) = x \in \Omega$, $t \in]0,T[$.

THEOREM 5.2. *If the function φ fulfils the assumptions* (2.1), *then the function v_2 given by the formula* (5.4) *is a solution of the problem* (F) *with* $g = 0$.

Proof. Analogously as the function $\widetilde{g}$ in the proof of Theorem 5.1 we can represent the function $\widetilde{\varphi}$ for the fixed t in the form of the series

$$\widetilde{\varphi}(t,r,\beta) = \sum_{m=1}^{\infty} J_0\left(\mu_{0,m}\frac{r}{a}\right)A_{0,m}(t) + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} J_n\left(\mu_{n,m}\frac{r}{a}\right)\times$$
$$\times\{\cos n\beta A_{n,m}(t) + \sin n\beta C_{n,m}(t)\}$$

where $A_{n,m}(t)$, $C_{n,m}(t)$ are the following $\mathbb{R}^k$-valued coefficients of the Fourier–Bessel expansion of the $\mathbb{R}^k$-valued function $\widetilde{\varphi}$ (for fixed t)

$$A_{n,m}(t) = \frac{2}{\varepsilon_n\pi a^2[J_{n+1}(\mu_{n,m})]^2}\times$$
$$\times\int_0^a \rho J_n\left(\mu_{n,m}\frac{\rho}{a}\right)\left[\int_0^{2\pi}\widetilde{\varphi}(t,\rho,\gamma)\cos n\gamma\,d\gamma\right]d\rho, \quad (n,m)\in J,$$

$$C_{n,m}(t) = \frac{2}{\varepsilon_n\pi a^2[J_{n+1}(\mu_{n,m})]^2}\times$$
$$\times\int_0^a \rho J_n\left(\mu_{n,m}\frac{\rho}{a}\right)\left[\int_0^{2\pi}\widetilde{\varphi}(t,\rho,\gamma)\sin n\gamma\,d\gamma\right]d\rho, \quad (n,m)\in J.$$

Making use of the properties of G, we can represent the function $\widetilde{v}_2$ in the form

$$\widetilde{v}_2(t,r,\beta) =$$
$$= \sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\left\{\cos n\beta J_n\left(\mu_{n,m}\frac{r}{a}\right)\int_0^t \exp\left[-\left(\mu_{n,m}\frac{1}{a}\right)^2(t-\eta)\mathbf{A}\right]A_{n,m}(t)\,d\eta+\right.$$
$$\left.+\sin n\beta J_n\left(\mu_{n,m}\frac{r}{a}\right)\int_0^t \exp\left[-\left(\mu_{n,m}\frac{1}{a}\right)^2(t-\eta)\mathbf{A}\right]C_{n,m}(t)\,d\eta\right\}.$$

Calculating derivatives of $\widetilde{v}_2$ we obtain

$$\frac{\partial}{\partial t}\widetilde{v}_2(t,r,\beta) = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\left\{\cos n\beta J_n\left(\mu_{n,m}\frac{r}{a}\right)A_{n,m}(t)+\right.$$
$$\left.+\sin n\beta J_n\left(\mu_{n,m}\frac{r}{a}\right)C_{n,m}(t)\right\}+$$

$$+\sum_{n=0}^{\infty}\sum_{m=1}^{\infty} J_n\left(\mu_{n,m}\frac{r}{a}\right)\left(-\left(\mu_{n,m}\frac{1}{a}\right)^2\mathbf{A}\right)\times$$

$$\times\left[\cos n\beta\int_0^t \exp\left[-\left(\mu_{n,m}\frac{1}{a}\right)^2(t-\eta)\mathbf{A}\right]A_{n,m}(t)\,d\eta+\right.$$

$$\left.+\sin n\beta\int_0^t \exp\left[-\left(\mu_{n,m}\frac{1}{a}\right)^2(t-\eta)\mathbf{A}\right]C_{n,m}(t)\,d\eta\right],$$

$$\widetilde{\Delta}\widetilde{v}_2(t,r,\beta)=$$

$$=\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\left[\frac{d^2}{dr^2}J_n\left(\mu_{n,m}\frac{r}{a}\right)+\frac{1}{r}\frac{d}{dr}J_n\left(\mu_{n,m}\frac{r}{a}\right)-\left(\frac{n}{r}\right)^2J_n\left(\mu_{n,m}\frac{r}{a}\right)\right]\times$$

$$\times\left[\cos n\beta\int_0^t \exp\left[-\left(\mu_{n,m}\frac{1}{a}\right)^2(t-\eta)\mathbf{A}\right]A_{n,m}(t)d\eta+\right.$$

$$\left.+\sin n\beta\int_0^t \exp\left[-\left(\mu_{n,m}\frac{1}{a}\right)^2(t-\eta)\mathbf{A}\right]C_{n,m}(t)\,d\eta\right]=$$

$$=-\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\left[\left\{\left(\mu_{n,m}\frac{1}{a}\right)^2\cos n\beta J_n\left(\mu_{n,m}\frac{r}{a}\right)\times\right.\right.$$

$$\left.\times\int_0^t \exp\left[-\left(\mu_{n,m}\frac{1}{a}\right)^2(t-\eta)\mathbf{A}\right]A_{n,m}(t)\,d\eta\right\}+$$

$$+\left\{\left(\frac{\mu_{n,m}}{a}\right)^2\sin n\beta J_n\left(\mu_{n,m}\frac{r}{a}\right)\times\right.$$

$$\left.\left.\times\int_0^t \exp\left[-\left(\mu_{n,m}\frac{1}{a}\right)^2(t-\eta)\mathbf{A}\right]C_{n,m}(t)\,d\eta\right\}\right],$$

since the function $J_n(\mu_{n,m}\frac{r}{a})$ is a solution of the Bessel equation, analogously as in the proof of Theorem 5.1. Thus, we obtain

$$\frac{\partial}{\partial t}\widetilde{v}_2(t,r,\beta)=\mathbf{A}\widetilde{\Delta}\widetilde{v}_2(t,r,\beta)+\widetilde{\varphi}(t,r,\beta).$$

From (5.4) it follows that $\widetilde{v}_2(0,r,\beta)=0\in\mathbf{R}^k$ and $\lim_{r\to a^-}\widetilde{v}_2(t,r,\beta)=0\in\mathbf{R}^k$.

COROLLARY 5.1. *Suppose that the functions g and φ fulfil the assumptions* (2.2) *and* (2.1), *respectively. Then the function $v = v_1 + v_2$ is a solution of the problem* (F).

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