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Agnieszka Plucińska, Edmund Pluciński

ON STOCHASTIC DIFFERENCE EQUATIONS ASSOCIATED WITH QUASI-DIFFUSION PROCESSES

We consider a sequence of stochastic difference equations. We define the notion "the consistency" of this sequence. This consistent sequence is an analogue of consistent sequence of solutions of Kołmogorow type parabolic equations. We give relations between coefficients of these parabolic equations and coefficients of equations considered in the present paper. We find solutions when the coefficients are linear. Every difference equation of the considered sequence has a form similar as in ARiMA models.

1. Introduction and formulation of the results

Let $(\Omega, \mathfrak{F}, P)$ be a complete probability space, $\mathfrak{F} = (\mathfrak{F}_t,\ t \in [0,T])$ an increasing family of sub σ-field of $\mathfrak{F}$, $W = (W_t, \mathfrak{F}_t)$ a Wiener process.

Let $X = (X_t)$, $t \in [0,T]$, be a real-valued continuous stochastic process $(\mathfrak{F}_t)$-adapted such that $E(X_t) = 0$, $E(X_t^2) < \infty$.

For $0 \le t_1 \le t_2 \le \ldots \le T$ we put $\mathbf{t}_n = (t_1, t_2, \ldots, t_n)$, $\mathbf{X}_n = (X_{t_1}, \ldots \ldots, X_{t_n})$ and $\mathbf{x}_n = (x_1, \ldots, x_n) \in R^n$. We suppose that for every n, every $\mathbf{t}_n$ the random variables $X_{t_1}, \ldots, X_{t_n}$ are linearly independent.

Let $a_n(\mathbf{t}_{n+1}, \mathbf{x}_n)$ and $b_n(\mathbf{t}_{n+1}, \mathbf{x}_n)$ be continuous functions.

For a given probability space $(\Omega, \mathfrak{F}, P)$ and a given Wiener process W, we consider a stochastic process X satisfying for every n, every fixed sequence $t_1 < t_2 < \ldots < t_n$ and optional t_{n+1} stochastic difference equations

$$(1.1)\quad X_{t_{n+1}} = X_{t_n} + \int_{t_n}^{t_{n+1}} a_n(\mathbf{t}_n, s, \mathbf{X}_n)\,ds + \int_{t_n}^{t_{n+1}} b_n(\mathbf{t}_n, s, \mathbf{X}_n)\,dW_s = \\ = F_n(\mathbf{t}_{n+1}, \mathbf{X}_n, W), \qquad n \ge 1.$$

We shall say that the solutions of system (1.1) are a consistent family (CF) of solutions if for every n the random functionals F_n satisfy the relations

$$(1.2)\qquad F_n(\mathbf{t}_{n+1}, \mathbf{X}_{n-1}, F_{n-1}(\mathbf{t}_n, \mathbf{X}_{n-1}, W), W) \stackrel{d}{=} \stackrel{d}{=} F_{n-1}(\mathbf{t}_{n-1}, t_{n+1}, \mathbf{X}_{n-1}, W).$$

By properties of Ito integrals, almost surely,

$$(1.3)\qquad \begin{cases} E(X_{t_{n+1}} \mid \mathfrak{F}_{t_n}) = X_{t_n} + \int\limits_{t_n}^{t_{n+1}} a_n(\mathbf{t}_n, s, \mathbf{X}_n)\, ds, \\ E(X_{t_{n+1}} - E(X_{t_{n+1}} \mid \mathfrak{F}_{t_n}))^2 \mid \mathfrak{F}_{t_n}) = \int\limits_{t_n}^{t_{n+1}} b_n^2(\mathbf{t}_{n+1}, s, \mathbf{X}_n)\, ds. \end{cases}$$

If X is a Markov process the functions a_n, b_n of $2n+1$ arguments turn into the functions of 3 arguments

$$a_n(\mathbf{t}_n, s, \mathbf{X}_n) = a_1(t_n, s, X_{t_n}), \qquad b_n(\mathbf{t}_n, s, \mathbf{X}_n) = b_1(t_n, s, X_{t_n}).$$

Therefore the consistency conditions (1.2) reduce to the one condition:

$$(1.2')\qquad F_1(t_2, t_3, F_1(t_1, t_2, X_{t_1}, W), W) \stackrel{d}{=} F_1(t_1, t_3, X_{t_1}, W).$$

Formula (1.2′) is similar to a property of semigroups operators for Markov processes. In a non-markovian case for the considered process X we have the system of stochastic difference equations (1.1) satisfying (1.2).

The main aim of this paper is to prove the following Proposition.

PROPOSITION 1. *If $a_n(\mathbf{t}_{n+1}, \mathbf{x}_n) \in C^0$ are linear functions of $\mathbf{x}_n$, $b_n(\mathbf{t}_{n+1}, \mathbf{x}_n) = b_n(\mathbf{t}_{n+1})$ are continuous positive functions, then solutions of (1.1) are (CF) gaussian solutions. For every $n > 1$*

$$(1.4)\qquad \operatorname{Var}\Big(X_{t_{n+1}} - X_{t_n} - \int\limits_{t_n}^{t_{n+1}} a_n(\mathbf{t}_n, s, \mathbf{X}_n)\, ds\Big) \le \le \operatorname{Var}\Big(X_{t_{n+1}} - X_{t_n} - \int\limits_{t_n}^{t_{n+1}} a_{n-1}(t_2, \ldots, t_n, s, X_{t_2}, \ldots, X_{t_n})\, ds\Big).$$

For fixed n equation (1.1) permit to express the state $X_{t_{n+1}}$ at the point t_{n+1} by W and preceding states of the considered process, i.e. by the states at the points $t_1, \ldots, t_n$. In this sense we get forecasting at the point t_{n+1} depending on W and the preceding states.

By virtue of (1.4) we can say that such forecasting given by (1.1) for n past states is better than for $n-1$ past states. The greater number of conditions improve the forecasting.

The idea of the sequence of stochastic difference equations (1.1) satisfying (1.2) refers to various stochastic investigations. We mention some relations.

De Haan and Karandicar [2] have considered stochastic difference equation of the form

$$X_t = A_t^s X_s + B_t^s$$

with the random functionals satisfying almost surely

(i) $\quad A_t^s = A_u^s A_t^u, \quad B_t^s = A_t^u B_u^s + B_t^u \quad \text{for } 0 \le s \le u \le t$

and some further conditions (ii), (iii). For fixed n relation (1.1) is a stochastic difference equation with memory depending on the preceding states.

Relations (1.2) have similar character to (i). Relations (1.2) for linear coefficients a_n, b_n have a strict connection with (i). The essential difference consists on the dependence of functionals F_n on the precedings states.

On the other hand by (1.3) equations (1.1) can be written in the following form

$$(1.5) \qquad X_{t_{n+1}} - E(X_{t_{n+1}} \mid \mathfrak{F}_{t_n}) = \int_{t_n}^{t_{n+1}} b_n(\mathbf{t}_n, s, \mathbf{X}_n)\, dW_s, \qquad n \ge 1.$$

Every equation of system (1.5) can be treated as a version of Clark's formula with a special form of the functional under the integral sign. Clark's formula was considered for example by Karatzas, Ocone and Jinlu Li [3].

It seems also interesting to mention that under the assumptions of Proposition 1 relations (1.1) are a continuous analogue of ARIMA models (see e.g. Box, Jenkins [1], Priestley [7]).

The most essential connection of the sequence (1.1) is with the quasi-diffusion processes (the definition is quoted in pargraph 2)

The main idea of quasi-diffusion processes is to give a tool for finding conditional distributions. This tool is a sequence of Kolmogorov type parabolic equations. If we have the initial condition and we solve first n equations of this sequence we get n-dimensionals distributions. For a Markov process the solution of Kolmogorov equation determines all multi-dimensionsl distributions. For non-markovian process the n-dimesional distributions (for finite n) give only some partial information; when n increases we get more informations. Now we propose a method to express the state of the process at the moment t_{n+1} by the preceding states of the process at the moments $t_1, t_2, \ldots, t_n$ plus some functional of the Wiener process. This method is based on stochastic difference equations (1.1). The coefficients of equations (1.1) have the strict connection with the coefficients of Kolmogorov type parabolic equation (see chapter 2). When n increases we get better information in the sense of relation (1.4).

Therefore we extend the idea of quqsi-diffusion processes permitting to find conditional distributons (conditioning by preceding states) to the idea of expressing the state of the process by some preceding states plus some functionals. The idea of quasi-diffusion processes is based on infinitesimal moments (some limits of conditional moments). The idea of the present paper is based on conditional moments.

2. Quasi-diffusion processes

Solutions of (1.1) have a strict connection with quasi-diffusion processes. For quasi-diffusion processes conconsidered by Plucińska [4], [5] there exist limits

$$\lim_{\Delta_n \to 0+} \frac{1}{\Delta_n} E(X_{t_n+\Delta_n} - X_{t_n}) \mid \mathbf{X}_n = \mathbf{x}_n) = \widehat{a}_n(\mathbf{t}_n, \mathbf{x}_n), \tag{2.1}$$

$$\lim_{\Delta_n \to 0+} \frac{1}{\Delta_n} E(X_{t_{n+\Delta_n}} - X_{t_n})^2 \mid \mathbf{X}_n = \mathbf{x}_n) = \widehat{b}_n(\mathbf{t}_n, \mathbf{x}_n). \tag{2.2}$$

The conditional densities f_n of quasi-diffusion processes satisfy (under some additional regularity assumptions) the Kolmogorov type equations

$$\begin{aligned}\frac{\partial}{\partial t_n} f_n(\mathbf{t}_{n-1}, \mathbf{x}_{n-1}; t_n, x_n) + \frac{\partial}{\partial x_n}[\widehat{a}_n(\mathbf{t}_n, \mathbf{x}_n) f_n(\mathbf{t}_{n-1}, \mathbf{x}_{n-1}; t_n, x_n)] =\\ = \frac{1}{2}\frac{\partial^2}{\partial x_n^2}[\widehat{b}_n(\mathbf{t}_n, \mathbf{x}_n) f_n(\mathbf{t}_{n-1}, \mathbf{x}_{n-1}; t_n, x_n)], \qquad n > 1.\end{aligned} \tag{2.3}$$

It is obvious that for solutions of (1.1) there exist limits (2.1) and (2.2) and the following relations hold:

$$\begin{cases} \widehat{a}_n(\mathbf{t}_n, \mathbf{x}_n) = a_n(\mathbf{t}_n, t_n, \mathbf{x}_n) \\ \widehat{b}_n(\mathbf{t}_n, \mathbf{x}_n) = b_n^2(\mathbf{t}_n, t_n, \mathbf{x}_n). \end{cases} \tag{2.4}$$

For Markov processes, for $n = 2$, equation (2.3) with some initial conditions determines all finite dimensional distributions. In the non-markovian case we have the sequence of equations (2.3). These equations give a partial information about multi-dimensional distributions. If we consider these equations for $n \leq N$, then we can find the N-dimensional distributions.

Similarly, if we consider equations (1.1) for $n \leq N$ with given a_n, b_n, then by (2.4) and (2.3) we can find conditional densities f_n and next N-dimensional densities.

On the other hand, for given n, equation (1.1) provides a forecasting for $X_{t_{n+1}}$ as a functional of n past states $X_{t_1}, \ldots, X_{t_n}$ and a Wiener process W. A forecasting given by (1.1) for n past states is better than for $n-1$ past states in the sense of (1.4).

3. Auxiliary results

We shall use the following lemmas.

LEMMA 1. *Let $X = (X_t)$, $t \in [0,T]$ be a zero mean stochastic process with $E(X_t^4) < \infty$ and with a continuous covariance function $k_{ij} = E(X_{t_i} X_{t_j})$.*

If for every n, $t_1 < t_2 < \ldots < T$, the random variables

(3.1) $X_{t_1}, X_{t_2}, \ldots, X_{t_n}$ *are linearly independent,*

(3.2) $\mu_n = E(X_{t_n} \mid X_{t_1}, \ldots, X_{t_{n-1}})$ *is a linear function of* $X_{t_1}, \ldots, X_{t_{n-1}}$,

(3.3) $v_n^2 = \mathrm{Var}(X_{t_n} \mid X_{t_1}, \ldots, X_{t_{n-1}}) = E[(X_{t_n} - \mu_n)^2 \mid X_{t_1}, \ldots, X_{t_{n-1}}]$ *is a deterministic function*

(3.4) $$E[(X_{t_n} - \mu_n)^4 \mid X_{t_1}, \ldots, X_{t_{n-1}}] \le o(t_n - t_{n-1})$$

then X is a gaussian process. Moreover

(3.5) $$\mu_n = \sum_{i=1}^{n-1} c_{in}(\mathbf{t}_n) X_{t_i},$$

(3.6) $$v_n^2 = \frac{K^{(n)}}{K_{nn}^{(n)}},$$

where

(3.7) $$c_{in} = -\frac{K_{in}^{(n)}}{K^{(n-1)}},$$

$k_{ij} = E(X_{t_i} X_{t_j})$, $K_{in}^{(n)}$ is the cofactor of k_{in} in the matrix

$$[k_{ij}]_{i,j=1}^{n}, \ K^{(n-1)} = \det[k_{i,j}]_{i,j=1}^{n-1} \quad (i.e.\ K^{(n-1)} = K_{nn}^{(n)}).$$

Evidently for every i the coefficients c_{in} are functions of n arguments: $t_1, \ldots$ $\ldots, t_n$.

We omit the proof of Lemma 1 because it is analogous to the proof of Theorem 1 given by Plucińska [6]. In that paper the conditions of type (3.2), (3.3) are assumed for all the point $(t_1, \ldots, t_n)$ (not necesserity ordered). In the present paper conditions (3.2), (3.3) must be satisfied only for $t_1 < t_2 < \ldots < t_n$. But in the present paper we have the additional condition (3.4).

LEMMA 2. *If conditions* (3.1), (3.2) *and* (3.3) *hold then*

(3.8) $$c_{i,n+1}(\mathbf{t}_{n+1}) + c_{n,n+1}(\mathbf{t}_{n+1}) c_{i,n}(\mathbf{t}_n) = c_{i,n}(\mathbf{t}_{n-1}, t_{n+1}),$$

(3.9) $$c_{n,n+1}^2(\mathbf{t}_{n+1}) \mathrm{Var}(X_{t_n} \mid X_{t_1}, \ldots, X_{t_{n-1}}) + \\ + \mathrm{Var}(X_{t_{n+1}} \mid X_{t_1}, \ldots, X_{t_n}) = \mathrm{Var}(X_{t_{n+1}} \mid X_{t_1}, \ldots, X_{t_{n-1}}).$$

Proof of Lemma 2. We shall use the following formula for determinants of symmetric matrices

$$(3.10)\qquad K^{(n+1)}K_{ij}^{(n)} = K_{ij}^{(n+1)}K_{n+1,n+1}^{(n+1)} - K_{i,n+1}^{(n+1)}K_{j,n+1}^{(n+1)}.$$

By virtue of (3.7) and (3.10) we have

$$(3.11)\qquad K^{(n+1)}c_{i,n+1}(\mathbf{t}_{n+1}) = = -\frac{K_{i,n+1}^{(n+1)}}{K^{(n)}K^{(n-1)}}[K_{nn}^{(n+1)}K_{n+1,n+1}^{(n+1)} - (K_{n,n+1}^{(n+1)})^2],$$

$$(3.12)\qquad c_{n,n+1}(\mathbf{t}_{n+1})c_{i,n}(\mathbf{t}_n) = \frac{K_{n,n+1}^{(n+1)}}{K^{(n)}}\frac{K_{i,n,n+1,n+1}^{(n+1)}}{K_{n,n,n+1,n+1}^{(n+1)}} = = \frac{K_{n,n+1}^{(n+1)}}{K^{(n-1)}K^{(n)}K^{(n+1)}}[K_{in}^{(n+1)}K_{n+1,n+1}^{(n+1)} - K_{i,n+1}^{(n+1)}K_{n,n+1}^{(n+1)}],$$

$$(3.13)\qquad c_{in}(\mathbf{t}_{n-1}, t_{n+1}) = -\frac{K_{i,n,n,n+1}^{(n+1)}}{K_{n,n,n+1,n+1}^{(n+1)}} = = \frac{-K_{i,n+1}^{(n+1)}K_{nn}^{(n+1)} + K_{in}^{(n+1)}K_{n,n+1}^{(n+1)}}{K^{(n-1)}K^{(n+1)}}.$$

Formula (3.8) follows immediately from (3.11), (3.12) and (3.13).

Taking into account (3.6), (3.7) and (3.8) we have

$$(3.14)\qquad c_{n,n+1}^2(\mathbf{t}_{n+1})\operatorname{Var}(X_{t_n} \mid X_{t_1}, \ldots, X_{t_{n-1}}) = = \left[\frac{K_{n,n+1}^{(n+1)}}{K^{(n+1)}}\right]^2 \frac{K^{(n)}}{K^{(n-1)}} = \frac{[K_{n,n+1}^{(n+1)}]^2}{K^{(n)}K^{(n-1)}},$$

$$(3.15)\qquad \operatorname{Var}(X_{t_{n+1}} \mid X_{t_1}, \ldots, X_{t_n}) = \frac{K^{(n+1)}}{K^{(n)}} = = \frac{K^{(n+1)}[K_{nn}^{(n+1)}K_{n+1,n+1}^{(n+1)} - (K_{n,n+1}^{(n+1)})^2]}{K^{(n)}K^{(n-1)}K^{(n+1)}},$$

$$(3.16)\qquad \operatorname{Var}(X_{t_{n+1}} \mid X_{t_1}, \ldots, X_{t_{n-1}}) = \frac{K_{nn}^{(n+1)}}{K^{(n-1)}}.$$

Formula (3.9) follows immediately from (3.14), (3.15) and (3.16). Thus Lemma 2 is proved.

4. Proof of Proposition 1

First we are going to show that conditions (1.2) hold. It follows from the properties of the Itô integral and the linearity of a_n that

$$E(X_{t_{n+1}} \mid \mathfrak{F}_{t_n}) = X_{t_n} + \int_{t_n}^{t_{n+1}} a_n(\mathbf{t}_n, s, \mathbf{X}_n)\, ds = \sum_{i=1}^{n} \alpha_{i,n+1} X_{t_i}, \tag{4.1}$$

in other words the conditional expectation is a linear function of the states with some coefficients $\alpha_{i,n+1}$. Then by Lemma 1

$$\alpha_{i,n+1} = c_{i,n+1}. \tag{4.2}$$

By the properties of functions b_n and Itô integrals we have

$$E\{[X_{t_{n+1}} - E(X_{t_{n+1}} \mid \mathfrak{F}_{t_n})]^2 \mid \mathfrak{F}_{t_n}\} = \int_{t_n}^{t_{n+1}} b_n^2(\mathbf{t}_n, s)\, ds = \tag{4.3}$$

$$= E[X_{t_{n+1}} - E(X_{t_{n+1}} \mid \mathfrak{F}_{t_n})]^2 > 0.$$

It follows from (4.1), (4.3) that $X_{t_1}, \ldots, X_{t_{n+1}}$ are linearly independent i.e.

$$K^{(n+1)} > 0, \quad n = 1, 2, \ldots. \tag{4.4}$$

Therefore assumptions (3.1), (3.2) and (3.3) are satisfied and in virtue of Lemma 1 we have

$$c_{i,n+1}(\mathbf{t}_{n+1}) = -\frac{K^{(n+1)}_{i,n+1}}{K^{(n)}}, \tag{4.5}$$

$$b_n^2(\mathbf{t}_{n+1}) = \frac{\partial}{\partial t_{n+1}} v_{n+1}^2 = \frac{\partial}{\partial t_{n+1}} \frac{K^{(n+1)}}{K^{(n)}}. \tag{4.6}$$

By virtue of formula (3.8) and (4.5)

$$F_n[\mathbf{t}_{n+1}, \mathbf{X}_{n-1}, F_{n-1}(\mathbf{t}_n, \mathbf{X}_{n-1}, W)W] = \sum_{i=1}^{n-1} c_{i,n+1}(\mathbf{t}_{n+1})\mathbf{X}_{t_i} +$$

$$+ c_{n,n+1}(\mathbf{t}_{n+1})\Big[\sum_{i=1}^{n-1} c_{i,n}(\mathbf{t}_n) X_{t_i} + \int_{t_{n-1}}^{t_n} b_{n-1}(\mathbf{t}_{n-1}, s)\, dW_s\Big] +$$

$$+ \int_{t_n}^{t_{n+1}} b_n(\mathbf{t}_n, s)\, dW_s = \sum_{i=1}^{n-1} c_{in}(\mathbf{t}_{n-1}, t_{n+1}) X_{t_i} +$$

$$+ c_{n,n+1}(\mathbf{t}_{n+1}) \int_{t_{n-1}}^{t_n} b_{n-1}(\mathbf{t}_{n-1})\, dW_s + \int_{t_n}^{t_{n+1}} b_n(\mathbf{t}_n, s)\, dW_s = I_1 + I_2$$

For fixed $t_1, \ldots, t_{n+1}$ the sum

$$I_2 = c_{n,n+1}(\mathbf{t}_{n+1}) \int_{t_{n-1}}^{t_n} b_{n-1}(\mathbf{t}_{n-1}, s)\, dW_s + \int_{t_n}^{t_{n+1}} b_n(\mathbf{t}_n, s)\, dW_s$$

is the sum of two independent gaussian random variables. Then this sum has a gaussian distribution with the mean value equal to zero and by formula (3.9) the variance is equal to

$$(4.7)\quad E(I_2^2) = c_{n,n+1}^2(\mathbf{t}_{n+1}) \int_{t_{n-1}}^{t_n} b_{n-1}^2(\mathbf{t}_{n-1}, s)\, ds + \int_{t_n}^{t_{n+1}} b_n^2(\mathbf{t}_n, s)\, ds =$$
$$= c_{n,n+1}^2(\mathbf{t}_{n+1}) \operatorname{Var}(X_{t_n} \mid X_{t_1}, \dots, X_{t_{n-1}}) +$$
$$+ \operatorname{Var}(X_{t_{n+1}} \mid X_{t_1}, \dots, X_{t_n}) = \operatorname{Var}(X_{t_{n+1}} \mid X_{t_1}, \dots, X_{t_{n-1}}) =$$
$$= \int_{t_{n-1}}^{t_n} b_{n-1}^2(\mathbf{t}_{n-1}, s)\, ds = E\Big(\int_{t_{n-1}}^{t_{n+1}} b_{n-1}(\mathbf{t}_{n-1}, s)\, dW_s \Big)^2.$$

On the other hand

$$F_{n-1}(\mathbf{t}_{n-1}, t_{n+1}, \mathbf{X}_{n-1}, W) = \sum_{i=1}^{n-1} c_{in}(\mathbf{t}_{n-1}, t_{n+1}) X_{t_i} +$$
$$+ \int_{t_{n-1}}^{t_{n+1}} b_{n-1}(\mathbf{t}_{n-1}, s)\, dW_s = I_1 + I_3.$$

It is evident that for fixed $t_1, \dots, t_{n+1}$ the integral I_3 has a mean zero gaussian distribution and by (4.7) $I_2 \stackrel{\mathrm{d}}{=} I_3$. It follows from the indepedence of I_1, I_2 and the indepedence of I_1, I_3 that $I_1 + I_2 \stackrel{\mathrm{d}}{=} I_1 + I_3$. Thus (1.2) is proved.

Now we are going to show, by Lemma 1, that the solutions of (1.1) are gaussian. Conditions (3.1)–(3.3) follows from (4.1), (4.3) and (4.4). For $\delta = 2$ by virtue of properties of stochastic integrals and the continuity of functions b_n we have

$$E\{(X_{t_{n+1}} - E(X_{t_{n+1}} \mid \mathfrak{F}_{t_{n+1}}))^4 \mid \mathfrak{F}_{t_n}\} = E\Big(\int_{t_n}^{t_{n+1}} b_n(\mathbf{t}_n, s)\, d_s \Big)^4 \le$$
$$\le 36(t_{n+1} - t_n) \int_{t_n}^{t_{n+1}} b_n^4(\mathbf{t}_n s)\, ds = o(t_{n+1} - t_n).$$

Thus (3.4) holds. Therefore by virtue of Lemma 1 solutions of (1.1) are gaussian.

Now we are going to show (1.4). Taking into account (3.10), for $i = j = 1$, we have

$$(4.8)\qquad K^{(n+1)} K_{11}^{(n)} = K_{11}^{(n+1)} K^{(n)} - (K_{1,n+1}^{(n+1)})^2.$$

It follows from (3.6) and (4.8) that

$$\operatorname{Var}\Big(X_{t_{n+1}} - X_{t_n} - \int\limits_{t_n}^{t_{n+1}} a_n(\mathbf{t}_n, s, \mathbf{x}_n)\Big)\, ds = \frac{K^{(n+1)}}{K^{(n)}} =$$

$$= \frac{K_{11}^{(n+1)}}{K_{11}^{(n)}} - \frac{(K_{1,n+1}^{(n+1)})^2}{K_{11}^{(n)} K^{(n)}} \le \frac{K_{11}^{(n+1)}}{K_{11}^{(n)}} =$$

$$= \operatorname{Var}\Big(X_{t_{n+1}} - X_{t_n} - \int\limits_{t_n}^{t_{n+1}} a_{n-1}(t_2, \ldots, t_n, s, X_{t_2}, \ldots, X_{t_n})\, ds\Big).$$

Formula (1.4) is thus shown. Therefore Proposition 1 is proved.

5. Examples

EXAMPLE 1. Let assumptions of Proposition 1 be satisfied and $k(t_1, t_2) = E(X_{t_1} X_{t_2}) = \exp[-(t_2 - t_1)^2]$. Then the coefficients are given by formulas

$$c_{12}(t_1, t_2) = \exp[-(t_2 - t_1)^2],$$

$$c_{13}(\mathbf{t}_3) = \frac{\exp[-(t_3 - t_2)^2] - \exp[-(t_2 - t_1)^2 - (t_3 - t_2)^2]}{1 - \exp[-2(t_2 - t_1)^2]}$$

$$c_{23}(\mathbf{t}_3) = \frac{\exp[-(t_3 - t_2)^2] - \exp[-(t_2 - t_1)^2 - (t_3 - t_1)^2]}{1 - \exp[-2(t_2 - t_1)^2]}$$

$$b_1^2(\mathbf{t}_2) = \frac{\partial}{\partial t_2}\{1 - \exp[-2(t_2 - t_1)^2]\}$$

$$b_2^2(\mathbf{t}_3) = \frac{\partial}{\partial t_3}\{(1 + 2\exp[-(t_2 - t_1)^2 - (t_3 - t_2)^2 - (t_3 - t_1)^2] -$$
$$- \exp[-2(t_3 - t_1)] - \exp[-2(t_3 - t_2)^2] - \exp[-2(t_2 - t_1)^2]) \times$$
$$\times (1 - \exp[-2(t_2 - t_1)^2])^{-1}\}.$$

Formula (1.2) for $n = 3$ has, in virtue of (1.5), the following form

$$X_{t_3} = F_2(t_1, t_2, t_3, X_{t_1}, F_1(t_1, t_2, W), W) =$$

$$= c_{13}(\mathbf{t}_3) X_{t_1} + c_{23}(\mathbf{t}_3)\Big[c_{12}(\mathbf{t}_2) X_{t_1} + \int\limits_{t_1}^{t_2} b_1(t_1, s)\, dW_s\Big] +$$

$$+ \int\limits_{t_2}^{t_3} b_2(t_1, t_2, s)\, dW_s \overset{\mathrm{d}}{=} c_{12}(t_1, t_3) X_{t_1} + \int\limits_{t_1}^{t_3} b_1(t, s)\, dW_s =$$

$$= F_1(t_1, t_3, W).$$

Evidently we have

$$\frac{K^{(3)}}{K^{(2)}} \leq \frac{K_{11}^{(3)}}{K_{11}^{(2)}}.$$

Therefore formula (1.4) is satisfied.

EXAMPLE 2. An example of a stochastic process satisfying (1.1) is Ornstein–Uhlenbeck [8] process

$$X_t = e^{-\rho t} X_0 + \int_0^t e^{-\rho(t-u)}\, dW_u.$$

References

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INSTITUTE OF MATHEMATICS
WARSAW UNIVERSITY OF TECHNOLOGY
Plac Politechniki 1
00-661 WARSZAWA, POLAND

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