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GENERALIZED SOLUTIONS OF A GOURSAT-TYPE PROBLEM FOR THE POLYWAVE EQUATION IN $\mathbb{R}^n$-SPACE

Introduction

Generalized solutions of Goursat-type problems in $\mathbb{R}^2$-space were defined and studied in papers [1], [2]. Similar results concerning the $\mathbb{R}^3$- space were obtained in Chapter II of the unpublished paper [5] (which was based on [4]), and for another Goursat problem in [3]*). In this paper, which contains the results of [5], we examine generalized solutions of a Goursat-type problem in $\mathbb{R}^n$-space where n is an arbitrary positive integer not less than three. Our argument is based on papers [7] and [8].

1. The problem and assumptions

Let Ω be the parallelepiped

$$\Omega = \{x \in \mathbb{R}^n : 0 \leq x \leq A\}$$

($x = (x_s)$, where $s = 1, 2, \ldots, n$) and Y a Banach space with the norm $\|\cdot\|$.

In what follows N denotes the set of all positive integers.

For fixed $p \in N$, we consider the polywave (or polyvibrating) equation of Mangeron (cf [6])

$$L^p u(x) = F(x) \tag{1.1}$$

($x \in \Omega$), where $L = \prod_{\mu=1}^{n}$ with $D_\mu = \frac{\partial}{\partial x_\mu}$, $L^k = L(L^{k-1})$ for $k = 1, 2, \ldots, p$; $L^0 u = u$, and F is a given function.

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*) Concerning the classical solutions of Goursat-type problems in $\mathbb{R}^n$, where $n \geq 3$, see [4], [8] and the references therein

By a solution of equation (1.1) in Ω we mean a function $u : \Omega \to Y$ such that (cf [8])

$$D_{i_1} \dots D_{i_l} L^k u \in C^{p-k-1} \quad \text{for } k = 0, 1, \dots, p-1$$

$(1 \le i_1 < \dots < i_l \le n;\ l = 1, 2, \dots, n)$ satisfying (1.1) for $x \in \Omega$.

Let $x^{(i)} = (x_s^{(i)})$, where $x_s^{(i)} = x_s$ for $1 \le s \le i-1$ $(2 \le i \le n)$; $x_s^{(i)} = x_{s+1}$ for $i \le s \le n-1$ $(1 \le i \le n-1)$, denote by Ω_i the set of all points $x^{(i)}$ for $x \in \Omega$ (of course $\Omega_i = \mathop{\mathrm{X}}\limits_{\substack{s=1 \\ s \ne i}}^{n} [0, A_s]$), and consider a system of surfaces $S_1, \dots, S_n$ given by the equations

$$x_i = f_i(x^{(i)})$$

$(x^{(i)} \in \Omega_i)$, respectively, where $f_i : \Omega_i \to [0, A_i]$ for $i = 1, 2, \dots, n$.

We examine the Goursat-type problem $(\mathfrak{G})$ that consists in finding a solution of equation (1.1) in Ω, subject to the boundary conditions

$$L^r u(x) = N_{i,r}(x^{(i)}) \quad \text{for } x \in S_i \tag{1.2}$$

$(x^{(i)} \in \Omega_i;\ i = 1, 2, \dots, n;\ r = 0, 1, \dots, p-1)$, where $N_{i,r} : \Omega_i \to Y$ are given functions.

Each function having the said properties is called a classical solution (briefly c.s.) of the $(\mathfrak{G})$-problem.

Now, we are going to define generalized solutions (briefly g.s.) of the $(\mathfrak{G})$-problem (our definition originates from those in [1], [2]).

To this end let us consider a sequence $\{(\mathfrak{G}^m)\}$ (where $m \in N$; $m > m_0$ with m_0 being a sufficiently large positive integer) of Goursat problems which are formulated analogously to $(\mathfrak{G})$ with the replacement of F, $N_{i,r}$ and S_i by F^m, $N_{i,r}^m$ and S_i^m, respectively (S_i^m denotes a surface of equation $x_i = f_i^m(x^{(i)})$, where

$$F^m : \Omega \to Y,\ N_{i,r}^m : \Omega_i \to Y \text{ and } f_i^m : \Omega_i \to [0, A_i]$$

$(i = 1, 2, \dots, n;\ r = 0, 1, \dots, p-1)$ are given functions.

We admit the following definition

DEFINITION 1.1 A function $u : \Omega \to Y$ is called a g.s. of the $(\mathfrak{G})$-problem if there is a sequence $\{u^m\}$ of functions $u^m : \Omega \to Y$ $(m \in N;\ m > m_0)$ such that

1° Each of the functions u^m is a c.s. of the corresponding Goursat problem $(\mathfrak{G}^m)$ in which the given functions satisfy the relations

$$F^m \rightrightarrows F;\ f_i^m \rightrightarrows f_i;\ N_{i,r}^m \rightrightarrows N_{i,r} \quad \text{when } m \to \infty \tag{1.3}$$

($i = 1,2,\ldots,n$; $r = 0,1,2,\ldots,p-1$ and $\rightrightarrows$ denotes the uniform convergence), and

2° The following relation

$$u^m \rightrightarrows u \quad \text{when } m \to \infty \tag{1.4}$$

holds good.

We make the following assumptions:

I. The functions $f_i : \Omega_i \to [0, A_i]$ ($i = 1,2,\ldots,n$) are Hölder-continuous (exponent $h_f \in (0,1]$), the surfaces S_i ($i = 1,2,\ldots,n$) do not intersect one another at the points of Ω placed outside the axes of coordinates and the following inequality is satisfied

$$f_i(x^{(i)}) \le K_1[\min_{1\le s\le n-1} x_s^{(i)}]^{n-1} \tag{1.5}$$

($i = 1,2,\ldots,n$), where K_1 is a positive constant such that

$$\vartheta := K_1 A^{n-2} < 1 \tag{1.6}$$

with $A = \max_{1\le i\le n} A_i$.

II. The functions $N_{i,r} : \Omega_i \to Y$ ($i = 1,2,\ldots,n$; $r = 0,1,\ldots,p-1$) are Hölder-continuous (exponent $h_N \in (0,1]$) and satisfy the inequality

$$\|N_{i,r}(x^{(i)})\| \le K_2[\min_{1\le s\le n-1} x_s^{(i)}]^{c_r} \tag{1.7}$$

($i = 1,2,\ldots,n$; $r = 0,1,\ldots,p-1$), where K_2 is a positive constant and $c_r = n+p-r-1$.

III. The function F is continuous.

2. Auxiliary theorems

Set $\vec{k}(n) = (k_v)$, where $v = 1,2,\ldots,n$; $v \ne i$; $x^{(i)}_{\vec{k}^i(n),m} = x^{(i)}$ with $x_s^{(i)} = A_s \frac{k_s}{m}$ for $s = 1,2,\ldots,n$, $s \ne i$;

$$w_{\vec{k}^i(n)}(x^{(i)}) = \prod_{v=i}^{n} \binom{m}{k_v} x_v^{k_v}(A_v - x_v)^{m-k_v}; \qquad B_i = \prod_{v=1}^{n} A_v \tag{2.1}$$

($v \ne i$), and consider the Bernstein polynomials

$$f_i^m(x^{(i)}) = B_i^{-m} \sum_{k_v=n-1}^{m} f_i(x^{(i)}_{\vec{k}^i_{(n)},m}) w_{\vec{k}^i_{(n)}}(x^{(i)}) \tag{2.2}$$

($v = 1,2,\ldots,n$; $v \ne i$), where $i = 1,2,\ldots,n$; $m \in N$; $m \ge n-1$.

LEMMA 2.1. *The following relations*

$$0 \le f_i^m(x^{(i)}) \le A_i; \quad f_i^m \in C^\infty(\Omega_i); \tag{2.3}$$

$$f_i^m \rightrightarrows f_i \quad \textit{when } m \to \infty; \tag{2.4}$$

$$D^l f_i^m(x^{(i)}) = 0 \quad \textit{when} \prod_{s=1}^{n-1} x_s^{(i)} = 0; \quad 0 \le |l| \le n-2 \tag{2.5}$$

hold good, where

$$D^l = \prod_{v=1}^{n} D_v^{l_v}; \quad |l| = \sum_{v=1}^{n} l_v \quad (v \ne i). \tag{2.5'}$$

Proof. Relations (2.3) and (2.5) follow immediately from (2.1) and (2.2). In order to prove (2.4), let us observe that by (2.2) we can write

$$|f_i^m(x^{(i)}) - f_i(x^{(i)})| = \Big| f_i^m(x^{(i)}) - f_i(x^{(i)}) B_i^{-m} \sum_{k_v=0}^{m} w_{\vec{k}^i(n)}(x^{(i)}) \Big| \le \tag{2.6}$$

$$\le B_i^{-m} \Big\{ \sum_{k_v=0}^{m} |f_i(x^{(i)}) - f_i(x^{(i)}_{\vec{k}^i(n),m})| w_{\vec{k}^i(n)}(x^{(i)}) +$$

$$+ \sum_{t=0}^{n-1} \sum_{k_1,\dots,k_t=0}^{m} \sum_{k_{t+1}=0}^{n-2} \sum_{k_{t+2},\dots,k_n=n-1}^{m} f_i(x^{(i)}_{\vec{k}^i(n),m}) w_{\vec{k}^i(n)}(x^{(i)}) \Big\}$$

$(k_{s_1}, k_{s_1+1,\dots,}k_{s_2} = \emptyset$ for $s_1 > s_2)$.

Denote the terms on the right-hand side of (2.6) by $e_1^m(x^{(i)})$ and $e_2^m(x^{(i)})$, successively, and let $\varepsilon > 0$ be arbitrarily fixed.

It is well known (cf [9], p. 152) that there is a number $m_\varepsilon^* \in N$ such that

$$e_1^m(x^{(i)}) < \frac{\varepsilon}{2} \tag{2.7}$$

when $m > m_\varepsilon^*$.

For the term $e_2^m(x^{(i)})$ we have (cf. (1.5))

$$e_2^m(x^{(i)}) \le K_1 \sum_{s=1}^{n} \sum_{k_s=0}^{n-2} \binom{m}{k_s} \left(A_s \frac{k_s}{m} \right)^{n-1} x_s^{k_s} (A_s - x_s)^{m-k_s} \le \tag{2.8}$$

$$\le K_1 (n-2)^{n-1} \sum_{s=1}^{n} A_s^{n-1} m^{1-n} \le K_1 (n-1)^n A^{n-1} m^{1-n}$$

$(s \ne i)$, where $A = \max_{1 \le i \le n} A_i$, and as a consequence we can assert that there is a number $\widetilde{m}_\varepsilon \in N$ such that

$$e_2^m(x^{(i)}) < \frac{\varepsilon}{2} \tag{2.9}$$

when $m > \tilde{m}_\varepsilon$.

On joining (2.6), (2.7) and (2.9) we get relation (2.4). Q.E.D.

LEMMA 2.2. *The surfaces S_i^m, of equations $x_i = f_i^m(x^{(i)})$, respectively $(i = 1, 2, \dots, n;\ m \in N;\ m \geq n-1)$ satisfy the following relation*

$$S_k \cap S_l = \{x \in \Omega : x_k = x_l = 0\}$$

$(k, l = 1, 2, \dots, n;\ k \neq l)$.

Proof. Suppose that S_k and S_l $(k \neq l)$ intersect at a point $\mathring{x} = (\mathring{x}_s) \in \Omega$ where $0 < \mathring{x}_k \leq A_k$ or $0 < \mathring{x}_l \leq A_l$. Then

$$\mathring{x}_k = f_k^m(\mathring{x}^{(k)}|_{\mathring{x}_l = f_l^m(\mathring{x}^{(l)})}) \quad \text{and} \quad \mathring{x}_l = f_l^m(\mathring{x}^{(l)}|_{\mathring{x}_k = f_k^m(\mathring{x}^{(k)})}). \tag{2.10}$$

We are going to prove that

$$f_k^m(x^{(k)}|_{x_l = f_l^m(x^{(l)})}) < x_k \tag{2.10'}$$

when $0 < x_k \leq A_k$.

To this end let us observe that formula (2.2) and Assumption I yield

$$f_i^m(x^{(i)}) \leq K_1 B_i^{-m} m^{1-n} \sum_{k_v=1}^{m} \prod_{s=1}^{n} A_s k_s \binom{m}{k_s} x_s^{k_s}(A_s - x_s)^{m-k_s}$$

$(v, s \neq i)$, whence we get

$$f_i^m(x^{(i)}) \leq K_1 m^{1-n} \prod_{s=1}^{n} A_s \sum_{k_s=1}^{m} k_s \binom{m}{k_s} \left(\frac{x_s}{A_s}\right)^{k_s} (1 - x_s)^{m-k_s}$$

$(s \neq i)$, and using the well known equality (cf [9], p. 150).

$$\alpha = m_\beta^{-1} \sum_{\beta=1}^{m} \beta \binom{m}{\beta} \alpha^\beta (1-\alpha)^{m-\beta}$$

we have

$$f_i^m(x^{(i)}) \leq K_1 \prod_{s=1}^{n-1} x_s^{(i)} \quad (i = 1, 2, \dots, n). \tag{2.11}$$

Basing on (1.5), (1.6) and (2.11), we obtain

$$f_k^m(x^{(k)}|_{x_l = f_l^m(x^{(l)})}) \leq K_1 \prod_{s=1}^{n-1} x_s^{(k)} f_l(x^{(l)}) \leq K_1^2 \prod_{s=1}^{n-1} x_s^{(k)} \prod_{r=1}^{n-1} x_r^{(l)} \leq$$
$$\leq (K_1 A^{n-2})^2 x_k < x_k$$

$(s \neq l;\ 0 < x_k \leq A_k)$, as required.

It is clear that inequality (2.10′) contradicts relations (2.10) and so Lemma 2.2 is valid. Q.E.D.

We have the following corollary whose validity follows from (1.5), (1.6) and (2.11).

COROLLARY 2.1. *The inequality*

$$\max(f_i^m(x^{(i)}),\ f_i(x^{(i)})) \le \vartheta \min_{1\le s\le n-1} x_s^{(i)} \tag{2.12}$$

$(i = 1,2,\ldots,n)$ *holds good.*

Now, let us consider the expressions $a_{r,j}^{i,m} : \Omega_i \to \mathbb{R}$ given by the formulae (cf. [7], [8])

$$a_{r,j}^{i,m}(x^{(i)}) = \begin{cases} x_r^{(j)} & \text{for } r \ne i \\ f_i^m(x^{(i)}) & \text{for } r = i \end{cases} \tag{2.13}$$

when $i < j$;

$$a_{r,j}^{i,m}(x^{(i)}) = \begin{cases} x_r^{(j)} & \text{for } r \ne i-1 \\ f_i^m(x^{(i)}) & \text{for } r = i-1 \end{cases} \tag{2.14}$$

when $i > j (x^{(i)} \in \Omega_i;\ 1 \le i,j \le n;\ r = 1,2,\ldots,n-1)$, and the sequences $(z_{\vec{k}(t)}^{v,m})$ and $(u_{\vec{k}(t),j}^{v,m})$ defined by

$$z_{\vec{k}(t)}^{v,m}(x^{(v)}) = (z_{s,\vec{k}(t)}^{v,m}(x^{(v)})) \tag{2.15}$$

$(s = 1,2,\ldots,n-1)$, where

$$z_{s,\vec{k}(t)}^{v,m}(x^{(v)}) = a_{s,k_t}^{k_{t-1},m}(z_{\vec{k}(t-1)}^{v,m}(x^{(v)}))$$

for $t = 2,3,\ldots,;\ s = 1,2,\ldots,n-1$

$$z_{s,\vec{k}(1)}^{v,m}(x^{(v)}) = a_{s,k_1}^{v,m}(x^{(v)}) \quad \text{for } s = 1,2,\ldots,n-1 \tag{2.16}$$

$(\vec{k}(t) = (k_l)$ where $l = 1,2,\ldots,t;\ t \in N;\ 1 \le k_l \le n;\ k_l \ne k_{l-1};\ k_0 = v;$ $v = 1,2,\ldots,n)$;

$$u_{\vec{k}(t),j}^{v,m}(x^{(v)}) = (u_{s,\vec{k}(t),j}^{v,m}(x^{(v)})) \tag{2.17}$$

$(s = 1,2,\ldots,n-1)$, where

$$u_{s,\vec{k}(t),j}^{v,m}(x^{(v)}) = a_{s,j}^{k_t,m}(z_{\vec{k}(t)}^{v,m}(x^{(v)})) \quad \text{for } t = 1,2,\ldots, \tag{2.18}$$

$(\vec{k}(t)$ is understood as in (2.16), $k_t \ne j;\ j = 1,2,\ldots,n;\ s = 1,2,\ldots,n-1;$ $v = 1,2,\ldots,n)$.

It is easily observed that

$$z_{\vec{k}(t)}^{v,m}(x^{(v)}) = u_{\vec{k}(t-1),k_t}^{v,m}(x^{(v)}) \tag{2.19}$$

$(v = 1,2,\ldots,n;\ t = 2,3,\ldots)$.

LEMMA 2.3. *For each number $\eta > 0$ there is a positive integer $m_* = m_*(\eta)$ such that the inequalities*

$$(2.20)\quad \begin{aligned} &\max_{1\le v\le n}\max_{1\le s\le n-1}\sup_{\Omega_v}|z^{v,m}_{s,\vec{k}(t)}(x^{(v)}) - z^{v}_{s,\vec{k}(t)}(x^{(v)})| < t\eta;\\ &\max_{1\le v\le n}\max_{1\le s\le n-1}\max_{1\le j\le n}\sup_{\Omega_v}|u^{v,m}_{s,\vec{k}(t),j}(x^{(v)}) - u^{v}_{s,\vec{k}(t),j}(x^{(v)})| < t\eta \end{aligned}$$

(*where $z^{v}_{s,\vec{k}(t)}$ and $u^{v}_{s,\vec{k}(t),j}$ are given by formulae analogous to* (2.16), (2.18), *respectively, with m being omitted) hold good for $t \in N$ and $m \in N$; $m > m_*(\eta)$.*

Proof of Lemma 2.3 is similar to that of Lemma 7 in [2].

Now, let us consider the following truncated Bernstein polynomials (cf. (1.7) and (2.1))

$$(2.21)\qquad N^m_{i,r}(x^{(i)}) = B_i^{-m}\sum_{k_v=c_r}^{m} N_{i,r}(x^{(i)}_{\vec{k}^i(n),m})w_{\vec{k}^i(n)}(x^{(i)})$$

($v = 1,2,\dots,n$; $v \ne i$), where $i = 1,2,\dots,n$; $r = 0,1,\dots,p-1$; $m \in N$; $m \ge n+p$.

LEMMA 2.4. *The following relations hold good*

(2.22) $N^m_{i,r} : \Omega_i \to Y$; $\quad N^m_i \in C^\infty(\Omega_i)$;

(2.23) $N^m_{i,r} \rightrightarrows N_{i,r}$ *when* $m \to \infty$;

(2.24) $D^l N^m_{i,r}(x^{(i)}) = 0 \quad$ *when* $\prod_{s=1}^{n-1} x^{(i)}_s = 0; \quad 0 \le |l| \le n-r+p-1$

(D^l *is understood as in* (2.5′));

$$(2.25)\qquad |||N^{m(l)}_{i,r}(x^{(i)})|||_l \le C(m)\prod_{s=1}^{n-1} x^{(i)}_s$$

when $l = n+p-r-2$, $C(m)$ being a positive constant dependent on m. Above, $|||\cdot|||_l$ denotes the norm in the space of l-linear continuous functions from $\mathbb{R}^{n-1}$ into Y.

Proof. The proof of (2.23) is analogous to that of (2.4), and (2.24) follows from (2.1) and (2.21). It is also clear that $N^m_{i,r} \in C^\infty(\Omega_i)$. Thus, it suffices to prove (2.25). To this end let us observe that by (1.7) an (2.21) we have (cf. (2.5′))

$$\|D^l N^m_{i,r}(x^{(i)})\| \le \text{const}\, B_i^{-m}\sum_{k_v=c_r}^{m}\prod_{s=1}^{n}\binom{m}{k_s}\left[\min\left(A_v\frac{k_v}{m}\right)\right]^{c_r}\times$$

$$\times \sum_{\alpha_s=0}^{\tilde{m}_s} \binom{l_s}{\alpha_s} \frac{k_s!}{(k_s - l_s + \alpha_s)!} \frac{(m-k_s)!}{(m-k_s-\alpha_s)!} x_s^{k_s - l_s + \alpha_s} (A_s - x_s)^{m-k_s-\alpha_s}$$

($v, s \neq i$; $\tilde{m}_s = \min(l_s, m - k_s)$ and $|l| = n + p - r - 2$), whence

$\|D^l N_{i,r}^m(x^{(i)})\| \leq$

$$\leq \widetilde{C}(m) \prod_{s=1}^{n} x_s^{c_r - l_s} \sum_{k_s=c_r}^{m} \sum_{\alpha_s=0}^{\tilde{m}_s} \binom{m}{k_s} \left(\frac{x_s}{A_s}\right)^{k_s - c_r + \alpha_s} \left(1 - \frac{x_s}{A_s}\right)^{m-k_s-\alpha_s}$$

($s \neq i$ and $\widetilde{C}(m)$ is a positive constant dependent on m), and as a consequence we obtain

$$|||N_{i,r}^{m(l)}(x^{(i)})|||_l \leq C(m) \prod_{s=1}^{n-1} x_s^{(i)},$$

as required.

LEMMA 2.5. *The following inequality is valid*

$$\|N_{i,r}^m(x^{(i)})\| \leq K_2 C_* [\min_{1 \leq s \leq n-1} x_s^{(i)}]^{c_r} \tag{2.26}$$

where $C_* = (c_r)^{c_r - 1}$.

Proof. By (1.7) and (2.21) we can write

$$\|N_{i,r}^m(x^{(i)})\| \leq K_2 B_i^{-m} \sum_{k_v=c_r}^{m} \prod_{s=1}^{n} \binom{m}{k_s} \left[\min\left(A_v \frac{k_v}{m}\right)\right]^{c_r} x_s^{k_s} (A_s - x_s)^{m-k_s}$$

($v, s \neq i$), whence

$$\|N_{i,r}^m(x^{(i)})\| \leq K_2 m^{-c_r} A_{v_0}^{c_r} \sum_{k_{v_0}=c_r}^{m} \binom{m}{k_{v_0}} k_{v_0}^{c_r} \left(\frac{x_{v_0}}{A_{v_0}}\right)^{k_{v_0}} \left(1 - \frac{x_{v_0}}{A_{v_0}}\right)^{m-k_{v_0}}$$

where v_0 is an arbitrarily fixed positive integer such that $1 \leq v_0 \leq n$; $v_0 \neq i$.

Now, it suffices to repeat the argument used in paper [2], p. 636 (with the replacement of $2p$ by c_r) to obtain the inequality

$$\|N_{i,r}^m(x^{(i)})\| \leq K_2 C_* x_{v_0}^{c_r}. \tag{2.27}$$

As v_0 ($1 \leq v_0 \leq n$; $v_0 \neq i$) has been arbitrarily fixed, (2.27) yields the thesis (2.26). Q.E.D.

We shall end this section with the examination of the Bernstein polynomials

$$F^m(x) = B^{-m} \sum_{k_v=0}^{m} F(x_{\vec{k}(n),m}) \widetilde{w}_{\vec{k}(n)}(x) \tag{2.28}$$

$(v = 2, 3, \ldots, n)$ where $\vec{k}(n) = (k_s)$ with $s = 1, 2, \ldots, n$; $x_{\vec{k}(n),m} = x$ with $x_s = A_s \frac{k_s}{m}$;

$$\widetilde{w}_{\vec{k}(n)}(x) = \prod_{s=1}^{n} \binom{m}{k_s} x_s^{k_s} (A_s - k_s)^{m-k_s}; \quad B = \prod_{s=1}^{n} A_s. \tag{2.29}$$

It is evident that $F^m \in C^\infty(\Omega)$, and well known that

$$F^m \rightrightarrows F \quad \text{when } m \to \infty, \tag{2.30}$$

as a consequence of which the function

$$R_p^m(x) = [(p-1)!]^{-n} \int_0^{x_1} \ldots \int_0^{x_n} \prod_{v=1}^{n} (x_v - \eta_v)^{p-1} F(\eta)\, d\eta \tag{2.31}$$

tends uniformly in Ω to the limit

$$R_p(x) = [(p-1)!]^{-n} \int_0^{x_1} \ldots \int_0^{x_n} \prod_{v=1}^{n} (x_v - \eta_v)^{p-1} F(\eta)\, d\eta \tag{2.32}$$

when m tends to infinity.

3. The $(\mathfrak{G}^m)$-problems

It follows from the results of Section 2 (cf. Lemmas 2.1, 2.2, 2.4 and 2.5, and the properties of F^m) that the functions f_i^m, $N_{i,r}^m$ and F^m given by (2.2), (2.21) and (2.28), respectively satisfy the assumptions of paper [8] (cf. [8]], pp. 492, 493), and so, for each $(\mathfrak{G}^m)$-problem, i.e. the $(\mathfrak{G})$-problem generated by the said functions f_i^m, $N_{i,r}^m$ and F^m where $m > m_0$ with $m_0 \in N$ being sufficiently large, Theorem 2 of [8] concerning the existence of c.s. of this problem can be applied.

According to the said theorem, for each $m \in N$, $m > m_0$ the corresponding $(\mathfrak{G}^m)$-problem has a c.s. given by the formula

$$u^m(x) = R_p^m(x) + \sum_{j=1}^{p} \sum_{i=1}^{n} (x_i)^{p-j} \psi_{i,p-j}^m(x^{(i)}) \tag{3.1}$$

$(x \in \Omega)$, where

$$\psi_{i,p-j}^m(x^{(i)}) = \{(p-j)![(p-j-1)!]^{n-1}\}^{-1} \cdot \int_0^{x_1^{(i)}} \ldots \int_0^{x_{n-1}^{(i)}} \prod_{s=1}^{n-1} (x_s^{(i)} - \eta_s^{(i)})^{p-j-1} \phi_{i,p-j}^m(\eta_1^{(i)}, \ldots, \eta_{n-1}^{(i)})\, d\eta_1^{(i)} \ldots d\eta_{n-1}^{(i)} \tag{3.2}$$

for $j = 1,2,\ldots,p-1$; $\psi^m_{i,0} = \phi^m_{i,0}$ with $\phi^m_{i,p-j}$ defined by

$$(3.3) \qquad \phi^m_{i,p-j}(x^{(i)}) = W^m_{i,p-j}(x^{(i)}) + \sum_{t=1}^{\alpha} V^{t,m}_{i,p-j}(x^{(i)})$$

$(i = 1,2,\ldots,n;\ j = 1,2,\ldots,p)$. Above,

$$(3.4) \qquad W^m_{i,p-j}(x^{(i)}) = N^m_{i,p-j}(x^{(i)}) - \overset{*m}{R}_{i,j}(x^{(i)});$$

$$(3.5) \qquad V^{t,m}_{i,p-j}(x^{(i)}) = (-1)^t \sum_{\vec{k}(t)} W^m_{k_t,p-j[z^{i,m}_{\vec{k}(t)}(x^{(i)})]};$$

$$(3.6) \qquad \overset{*m}{R}_{i,j}(x^{(i)}) = \overset{*m}{R}_{i,j}(x)|_{x_i=f_i(x^{(i)})}$$

with*)

$$(3.7) \quad \overset{*}{R}{}^m_j(x) = R^m_j(x) + \sum_{r=1}^{j-1}\{(j-r)![(j-r-1)!]^{n-1}\}^{-1}\sum_{k=1}^{n}(x_k)^{j-r}\cdot$$

$$\cdot \int_0^{x^{(k)}_1} \cdots \int_0^{x^{(k)}_{n-1}} \prod_{s=1}^{n-1}(x^{(k)}_s - \eta^{(k)}_s)^{j-r-1}\phi_{k,p-r}(\eta^{(k)})\, d\eta^{(k)}_1 \ldots d\eta^{(k)}_{n-1}.$$

This solution is unique in the set of all solutions of equation (1.1) (with F replaced by F^m) in Ω, which (cf. Lemma 1 in [8]) are given by formula (3.1), such that the functions $\phi^m_{i,p-j}$ $(i = 1,2,\ldots,n;\ j = 1,2,\ldots,p)$ appearing in (3.2) satisfy the condition

$$|||\phi^{m(l)}_{i,p-j}(x^{(i)})|||_l \le C(\min_{1\le s\le n-1} x^{(i)}_s)^{j+n-r-1}$$

(C is a positive constant depending in general on $\phi^m_{i,p-j}$) for $l = 0,1,\ldots,j+n-2$.

4. Generalized solutions of the $(\mathfrak{G})$-problem

We shall prove the following theorem

THEOREM 4.1. *If Assumptions I-III are satisfied, then there is a g.s. of the $(\mathfrak{G})$-problem given by the following formula*

$$(4.1) \qquad u(x) = R_p(x) + \sum_{j=1}^{p}\sum_{i=1}^{n} x_i^{p-j}\psi_{i,p-j}(x^{(i)})$$

*) The functions $R^m_j(x)$ $(j = 1,2,\ldots,p)$ are given by formula (2.31) with p replaced by j.

($x \in \Omega$; $x^{(i)} \in \Omega_i$; $i = 1,2,\dots,n$) in which $R_p(x)$ is defined by (2.32) and the functions $\psi_{i,p-j}$ ($i = 1,2,\dots,n$; $j = 1,2,\dots,p$) are given by the relations (3.2)–(3.7) with m being omitted.

Proof. Let $N \ni m > m_0$, $m_0 \in N$ being sufficiently large, and consider the sequences of functions $\{f_i^m\}$, $\{N_{i,r}^m\}$ and $\{F^m\}$ given by (2.2), (2.21) and (2.28), respectively, and the sequnce of Goursat problems $\{(\mathfrak{G}^m)\}$ generated by these functions (i.e. such that, for each $N \ni m > m_0$, the said functions f_i^m, $N_{i,r}^m$ and F^m are the given functions appearing in $(\mathfrak{G}^m)$).

We know from Lemmas 2.1 and 2.4, and formula (2.28), that the aforesaid functions f_i^m, $N_{i,r}^m$ and F^m ($i = 1,2,\dots,n$; $r = 0,1,\dots,$) satisfy relations (1.3), respectively.

We also know from Section 3 that each of the $(\mathfrak{G}^m)$-problems has a solution u^m given by formula (3.1) together with (3.2)–(3.7).

Thus, in order to prove Theorem 4.1 it is sufficient to show that relation (1.4) is satisfied, where u^m and u are given by (3.1) and (4.1), respectively.

Let $\varepsilon > 0$ be a given positive number and observe that (cf. (3.1) and (4.1))

$$\|u^m(x) - u(x)\| \le E_1^m(x) + E_2^m(x) \tag{4.2}$$

where

$$E_1^m(x) = [(p-1)!]^{-n} \int_0^{x_1} \dots \int_0^{x_n} \prod_{r=1}^{n} (x_r - \eta_r)^{p-1} \|F^m(\eta) - F(\eta)\|\, d\eta; \tag{4.3}$$

$$E_2^m(x) = \sum_{j=1}^{p} \sum_{i=1}^{n} x_i^{p-j} \|\psi_{i,p-j}^m(x^{(i)}) - \psi_{i,p-j}(x^{(i)})\| \tag{4.4}$$

($x \in \Omega$).

It is evident (cf. (2.30)) that there is a number $N \ni m_1 = m_1(\varepsilon) > m_0$ such that

$$E_1^m(x) < \frac{\varepsilon}{2} \tag{4.5}$$

for $x \in \Omega$; $N \ni m > m_1$.

In order to estimate the expression $E_2^m(x)$ we apply the method of mathematical induction.

Set $j = 1$. In this case (cf. (3.4)–(3.7))

$$\begin{aligned} W_{i,p-1}^m(x^{(i)}) &= N_{i,p-1}^m(x^{(i)}) - R_1^m(x)|_{x_i = f_i^m(x^{(i)})}; \\ W_{i,p-1}(x^{(i)}) &= N_{i,p-1}(x^{(i)}) - R_1(x)|_{x_i = f_i(x^{(i)})}. \end{aligned} \tag{4.6}$$

Let $\theta \in (0,1)$. Basing on (4.6), and using Assumptions I–III and inequality (2.11), we get

$$(4.7)\qquad \|W^m_{i,p-1}(x^{(i)}) - W_{i,p-1}(x^{(i)})\| \le \mathrm{const}\,\rho^\theta_m(\min_{1\le s\le n-1} x^{(i)}_s)^{2(1-\theta)}$$

where ρ_m is given by

$$(4.8)\qquad \rho_m = \max_{1\le i\le n}\max\{\sup_\Omega \|F^m(x) - F(x)\|,\ \sup_{\Omega_i}|f^m_i(x^{(i)}) - f_i(x^{(i)})|,$$
$$\sup_{\Omega_i}\|N^m_{i,p-1}(x^{(i)}) - N_{i,p-1}(x^{(i)})\|\}.$$

Let us observe that, by (4.6) and Assumptions I–III, we have

$$(4.9)\qquad W_{i,p-1}(x^{(i)}) - W_{i,p-1}(\overline{x}^{(i)})\| \le \mathrm{const}[\max_{1\le s\le n-1}|\overline{x}^{(i)}_s - x^{(i)}_s|]^{h_*\theta}\cdot$$
$$\cdot[\max(\min_{1\le s\le n-1}\overline{x}^{(i)}_s,\ \min_{1\le s\le n-1}x^{(i)}_s]^{1-\theta}$$

($h_* = \min(h_f, h_N)$), where $x^{(i)}$, $\overline{x}^{(i)} \in \Omega_i$; $i = 1,2,\ldots,n$.

Using (4.7) and (4.9), we get

$$\|W^m_{k_t,p-1}(z^{i,m}_{\vec{k}(t)}(x^{(i)})) - W_{k_t,p-1}(z^i_{\vec{k}(t)}(x^{(i)}))\| \le$$
$$\le \|W^m_{k_t,p-1}(z^{i,m}_{\vec{k}(t)}(x^{(i)})) - W_{k_t,p-1}(z^{i,m}_{\vec{k}(t)}(x^{(i)}))\| +$$
$$+\|W^m_{k_t,p-1}(z^{i,m}_{\vec{k}(t)}(x^{(i)})) - W_{k_t,p-1}(z^i_{\vec{k}(t)}(x^{(i)}))\| \le$$
$$\le \mathrm{const}\{\rho^\theta_m[\max_{\vec{k}(t)}\min_{1\le r\le n-1} z^{i,m}_{r,\vec{k}(t)}(x^{(i)})]^{2(1-\theta)} + [\max_{\vec{k}(t)}\max_{1\le i\le n-1}\max_{1\le r\le n-1}$$
$$\sup_{\Omega_i}|z^{i,m}_{r,\vec{k}(t)}(x^{(i)}) - z^i_{r,\vec{k}(t)}(x^{(i)})|]^{k_*\theta}\cdot$$
$$\cdot[\max_{\vec{k}(t)}\max(\min_{1\le r\le n-1} z^{i,m}_{r,\vec{k}(t)}(x^{(i)}),\ \min_{1\le r\le n-1} z^i_{r,\vec{k}(t)}(x^{(i)}))]^{1-\theta}\},$$

whence, and by inequality (2.12), Corollary 1 in [7] and Lemma 2.3 above, we obtain

$$(4.10)\qquad \|W^m_{k_t,p-1}(z^{i,m}_{\vec{k}(t)}(x^{(i)})) - W_{k_t,p-1}(z^i_{\vec{k}(t)}(x^{(i)}))\| \le \mathrm{const}(\vartheta)^{t(1-\theta)}t\frac{\varepsilon}{\varkappa_0}$$

($\varkappa_0$ is a positive integer to be chosen later — cf. (4.19)), on condition that $N \ni m > m^{(1)} = m^{(1)}(\varepsilon, \varkappa_0)$.

As $\vartheta \in (0,1)$, formulas (3.5) and (4.10) yield

$$(4.11) \qquad \sum_{t=1}^{\infty} \|V_{i,p-1}^{t,m}(x^{(i)}) - V_{i,p-1}^{t}(x^{(i)})\| \le \text{const}\,\frac{\varepsilon}{\varkappa_0}$$

and as a consequence of (3.3), (4.7) and (4.11) we obtain the inequality

$$(4.12) \qquad \|\phi_{i,p-1}^{m}(x^{(i)}) - \phi_{i,p-1}(x^{(i)})\| \le C_1 \frac{\varepsilon}{\varkappa_0}$$

whence (cf. (3.2))

$$(4.13) \qquad \|\psi_{i,p-1}^{m}(x^{(i)}) - \psi_{i,p-1}(x^{(i)})\| \le \widetilde{C}_1 \frac{\varepsilon}{\varkappa_0}$$

($N \ni m > m^{(1)}$), C_1 and $\widetilde{C}_1$ being positive constant.

Now, let $j_0 \in N$ be arbitrarily fixed so that $1 \le j_0 \le p-1$ and assume that

$$(4.14) \qquad \|\phi_{i,p-j}^{m}(x^{(i)}) - \phi_{i,p-j}(x^{(i)})\| \le C_j \frac{\varepsilon}{\varkappa_0}$$

for $j = 1,2,\ldots,j_0$ when $N \ni m > \max_{1\le v\le j_0} m^{(v)}$, $m^{(v)} = m^{(v)}(\varepsilon,\varkappa_0)$ ($v = 1,2,\ldots,j_0$) being sufficiently large positive integers and $C_1,\ldots,C_{j_0}$ positive constants.

Evidently (cf. relation (3.2) satisfied by the functions $\psi_{i,p-j}^{m}$ and $\psi_{i,p-j}$), the said assumption yields

$$(4.15) \qquad \|\psi_{i,p-j}^{m}(x^{(i)}) - \psi_{i,p-j}(x^{(i)})\| \le \widetilde{C}_j \frac{\varepsilon}{\varkappa_0}$$

(m as above, $j = 1,2,\ldots,j_0$), where $\widetilde{C}_1,\ldots,\widetilde{C}_{j_0}$ are positive constants.

Basing on (3.3)–(3.7) and (4.14), and using an argument similar to that in the proof of (4.12), we get

$$(4.16) \qquad \|\phi_{i,p-(j_0+1)}^{m}(x^{(i)}) - \phi_{i,p-(j_0+1)}(x^{(i)})\| \le C_{j_0+1} \frac{\varepsilon}{\varkappa_0}$$

when

$$N \ni m > \max_{1\le v\le j_0+1} m^{(v)} m^{(j_0+1)} = m^{(j_0+1)}(\varepsilon,\varkappa_0)$$

is a sufficiently large positive integer), whence and by (3.2) we obtain

$$(4.17) \qquad \|\psi_{i,p-(j_0+1)}^{m}(x^{(i)}) - \psi_{i,p-(j_0+1)}(x^{(i)})\| \le \widetilde{C}_{j_0+1} \frac{\varepsilon}{\varkappa_0}$$

(C_{j_0+1} and $\widetilde{C}_{j_0+1}$ are positive constant).

Thus, by (4.13), (4.15), (4.17) and the induction principle, we can assert that the inequality (4.15) holds good for $j = 1, 2, \ldots, p$ when $N \ni m > \max_{1 \le v \le p} m^{(v)} (m^{(v)} = m^{(v)}(\varepsilon, \varkappa_0)$; $v = 1, 2, \ldots, p$ are sufficiently large positive integers, with $\widetilde{C}_1, \ldots, \widetilde{C}_p$ being positive constants.

As a consequence of the aforesaid result and equality (4.4), we have

$$E_2^m(x) \le \widehat{C} \frac{\varepsilon}{\varkappa_0} \tag{4.18}$$

(where $\widehat{C}$ is a positive constant) when $N \ni m > \widetilde{m}_2 = m_2(\varepsilon, \varkappa_0)$.

Choosing $\varkappa_0 \in N$ so that $\frac{\widehat{C}}{\varkappa_0} < \frac{1}{2}$, we can conclude that there is a number $N \ni m_2 = m_2(\varepsilon) > m_0$ such that

$$E_2^m(x) < \frac{\varepsilon}{2} \tag{4.19}$$

for $x \in \Omega$; $N \ni m > m_2$.

Inequalities (4.2), (4.5) and (4.19) yield

$$\|u^m(x) - u(x)\| < \varepsilon \tag{4.20}$$

for $x \in \Omega$; $N \ni m > \max(m_1, m_2)$, which ends the proof of Theorem 4.1.

Now, we shall prove the following theorem

THEOREM 4.2. *Let us assume that for $N \ni m > m_0$ (cf. the proof of Theorem 4.1) the following conditions concerning the $(\mathfrak{G}^m)$-problems are fulfilled*

1°. *The functions $f_i^m : \Omega_i \to [0, A_i]$ $(i = 1, 2, \ldots, n)$ have the properties expressed by Lemmas 2.1–2.5;*

2°. *The functions $N_{i,r}^m : \Omega_i \to Y$ $(i = 1, 2, \ldots, n;\ r = 0, 1, \ldots, p-1)$ have the properties expressed by Lemmas 2.4, 2.5 (with C_* in (2.26) replaced by any positive constant independent of m);*

3°. *The functions $F^m : \Omega \to Y$ have the properties mentioned on p. 11 and are equibounded together with their first-order partial derivatives;*

4°. *The functions $\psi_{i,p-j}^m$ $(i = 1, 2, \ldots, n;\ j = 1, 2, \ldots, p)$ appearing in formula (3.1) for c.s. of the $(\mathfrak{G}^m)$-problems satisfy the inequality*

$$\|\psi_{i,p-j}^{m(v)}(x^{(i)})\| \le C_v \big(\min_{1 \le s \le n-1} x_s^{(i)}\big)^{n+p-v-1} \tag{4.21}$$

$(i = 1, 2, \ldots, n;\ j = 1, 2, \ldots, p;\ v = 0, 1, \ldots, p+n-2)$, *where C_v are positive constants independent of m.*

Then, there is at most one g.s. of the ($\mathfrak{G}$)-problem

Proof. It is our aim to show that if $\{f_i^{\mu,m}\}$, $\{N_{i,r}^{\mu,m}\}$, $\{F^{\mu,m}\}$ and $\{u_\mu^m\}$ ($\mu = 1, 2,$) satisfy the conditions of Definition 1.1 and the assumptions of Theorem 4.2, then the corresponding generalized solutions u_μ ($\mu = 1.2$) of the ($\mathfrak{G}$)-problem are identical in Ω.

To this end let us observe that

$$\|u_2(x) - u_1(x)\| \leq \|u_2(x) - u_2^m(x)\| + \|u_1(x) - u_1^m(x)+ + \|u_2^m(x) - u_1^m(x)\| \tag{4.22}$$

($x \in \Omega$) and that, for an arbitrary $\varepsilon > 0$ there is a sufficiently large number $N \ni m_0' = m_0'(\varepsilon)$ such that $N \ni m > m_0'$ implies

$$\|u_2(x) - u_2^m(x)\| + \|u_1(x) - u_1^m(x)\| < \frac{\varepsilon}{2}. \tag{4.23}$$

Furthermore, due to the present assumptions and Theorem in [8], we can assert that the functions u_μ^m ($\mu = 1, 2$) are of the form (3.1), where R_p^m and $\psi_{i,p-j}^m$ are as on p. 12.

Basing on (1.3) and using an argument analogous to that applied in the proof of (4.20), we get

$$\|u_2^m(x) - u_1^m(x)\| \leq \frac{\varepsilon}{2} \tag{4.24}$$

for $N \ni m > n_0''$, n_0'' ($n_0'' = n_0''(\varepsilon)$) being a sufficiently large positive integer.

On joining (4.22)–(4.24) we can conclude that $u_1 = u_2$ in Ω. Q.E.D.

Finally, we have the following theorem

THEOREM 4.3. *If Assumptions* I–III *of the present paper are replaced by those in paper* [8], *then the g.s. of the* ($\mathfrak{G}$)*-problem given by Theorem* 4.1 *is a c.s of this problem.*

The validity of this theorem follows from the results obtained in [8].

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