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## REMARK ON GLOBALLY LIPSCHITZIAN COMPOSITION OPERATORS

### Introduction

Let  $I \subseteq \mathbb{R}$  be an interval,  $f : I \times \mathbb{R} \rightarrow \mathbb{R}$  a fixed two-place function, and  $\mathcal{F}(I)$  the linear space of all the functions  $u : I \rightarrow \mathbb{R}$ . The function  $F : \mathcal{F}(I) \rightarrow \mathcal{F}(I)$  given by the formula

$$(F(u))(x) := f(x, u(x)), \quad x \in I, u \in \mathcal{F}(I),$$

is said to be a composition operator.

Let  $a \in I$  be fixed. Denote by  $\text{Lip}(I)$  the Banach space of all the functions  $u \in \mathcal{F}(I)$  with the norm

$$(1) \quad \|u\|_{\text{Lip}(I)} := |u(a)| + \sup \left\{ \frac{u(x_1) - u(x_2)}{x_1 - x_2} : x_1, x_2 \in I; x_1 \neq x_2 \right\}.$$

In [2] it is proved that if a composition operator  $F$  mapping  $\text{Lip}(I)$  into itself is globally Lipschitzian with respect to the  $\text{Lip}(I)$ -norm, then  $f(x, y) = g(x)y + h(x)$ , ( $x \in I; y \in \mathbb{R}$ ), for some  $g, h \in \text{Lip}(I)$ .

Next this result has been extended to some other function Banach spaces (cf. [1] for references). In particular, (cf. [3]) if  $F$  is a globally Lipschitzian selfmap of  $C_n(I)$ , i.e. there is an  $L \geq 0$  such that

$$(2) \quad \|F(u) - F(v)\|_{C_n(I)} \leq L\|u - v\|_{C_n(I)}, \quad u, v \in C_n(I),$$

where

$$(3) \quad \|u\|_{C_n(I)} := \sum_{i=0}^{n-1} |u^{(i)}(a)| + \sup\{|u^{(n)}(x)| : x \in I\},$$

then  $f(x, y) = g(x)y + h(x)$ , ( $x \in I; y \in \mathbb{R}$ ), for some  $g, h \in C_n(I)$ .

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In the present note we generalize these results. We show that the basic assumption of the global Lipschitz continuity of the composition operator can be essentially weakened. It turns out that the main result of [3] remains valid if the inequality (2) holds only for  $u, v$  being the polynomials at most of the degree  $n$ . Moreover, the argument presented here is much simpler than that in [3] where some complicated chain rule formulas are used.

### 1. Main result

We start with the following

**Remark 1.** Let  $x_1, x_2, u_1, u_2 \in \mathbb{R}$ ,  $x_1 \neq x_2$ , and  $n \in \mathbb{N}$ , be arbitrarily fixed. Then it is easy to verify that the polynomial  $u : \mathbb{R} \rightarrow \mathbb{R}$  given by

$$u(x) := a_n(x - a)^n + a_1x + a_0,$$

where

$$a_n = \frac{u_1 - u_2}{k!(x_1 - x_2)}, \quad a_1 = \frac{u_1 - u_2}{x_1 - x_2} \left( 1 - \frac{(x_1 - a)^n - (x_2 - a)^n}{n!(x_1 - x_2)} \right),$$

$$a_0 = u_1 - \frac{u_1 - u_2}{n!(x_1 - x_2)}(x_1 - a)^n - \frac{u_1 - u_2}{x_1 - x_2} \left( 1 - \frac{(x_1 - a)^n - (x_2 - a)^n}{n!(x_1 - x_2)} \right) x_1$$

has the following properties

$$u(x_1) = u_1, \quad u(x_2) = u_2; \quad \|u\|_{C_n[a,b]} = |a_0| + |a_1| + \left| \frac{u_1 - u_2}{x_1 - x_2} \right|.$$

In the same way, taking arbitrary  $v_1, v_2 \in \mathbb{R}$ , we can find a polynomial

$$v(x) := b_n(x - a)^n + b_1x + b_0$$

such that

$$v(x_1) = v_1, \quad v(x_2) = v_2; \quad \|v\|_{C_n[a,b]} = |b_0| + |b_1| + \left| \frac{v_1 - v_2}{x_1 - x_2} \right|.$$

where  $b_n, b_1, b_0$  are defined as  $a_n, a_1, a_0$  with  $u_1, u_2$  replaced by  $v_1, v_2$ .

From the formulas for  $a_0, a_1, b_0, b_1$  it is easy to observe that

$$\|u - v\|_{C_n[a,b]} = |a_0 - b_0| + |a_1 - b_1| + \left| \frac{u_1 - u_2 - v_1 + v_2}{x_1 - x_2} \right|,$$

and, if  $x_1$  and  $x_2$  tend to an  $x \in \mathbb{R}$ , then there exists a  $c(x) \in \mathbb{R}$  such that

$$(4) \quad \lim_{x_1, x_2 \rightarrow x} |x_1 - x_2| \|u - v\|_{C_n(I)} = c(x) |u_1 - u_2 - v_1 + v_2|.$$

Denote by  $P_n(I)$  the set of all the real polynomials of the degree at most  $n$ , restricted to the interval  $I$ .

**THEOREM.** Let  $F : \mathcal{F}(I) \rightarrow \mathcal{F}(I)$  be the composition operator generated by a function  $f : I \times \mathbb{R} \rightarrow \mathbb{R}$ , and suppose that  $n, m \in \mathbb{N}$  are fixed positive integers. If  $F$  maps  $\mathbf{P}_n(I)$  into  $\mathbf{C}_m(I)$  and there exists an  $L \geq 0$  such that

$$(5) \quad \|F(u) - F(v)\|_{\mathbf{C}_m(I)} \leq L\|u - v\|_{\mathbf{C}_n(I)}, \quad u, v \in \mathbf{P}_n(I),$$

then there exist  $g, h \in \mathbf{C}_m(I)$  such that

$$f(x, y) = g(x)y + h(x), \quad x \in I, y \in \mathbb{R}.$$

**Proof.** We have  $F(u) = f(\cdot, y)$  for each constant function  $u(x) := y \in \mathbb{R}$ . Since  $F$  maps  $\mathbf{P}_n(I)$  into  $\mathbf{C}_m(I)$ , it follows that, for every  $y \in \mathbb{R}$ , the function  $f(\cdot, y)$  is continuous in  $I$ .

From the definition of the norms (1) and (3) we get

$$\|u\|_{\text{Lip}(I)} \leq \|u\|_{\mathbf{C}_k(I)}, \quad u \in \mathbf{C}_k(I), \quad k \in \mathbb{N},$$

and inequality (5) implies

$$(6) \quad \|F(u) - F(v)\|_{\text{Lip}(I)} \leq L\|u - v\|_{\mathbf{C}_n(I)}, \quad u, v \in \mathbf{P}_n(I).$$

Let us fix arbitrary  $x_1, x_2 \in I$ ,  $x_1 \neq x_2$ ;  $u_1, u_2, v_1, v_2 \in \mathbb{R}$ , and take the polynomials  $u$  and  $v$  constructed in Remark 1. Making use of the definition of  $\text{Lip}(I)$ -norm and substituting  $u$  and  $v$  to the inequality (6), we obtain

$$\begin{aligned} & \left| \frac{f(x_1, u_1) - f(x_2, u_2) - f(x_1, v_1) + f(x_2, v_2)}{x_1 - x_2} \right| = \\ & \left| \frac{f(x_1, u(x_1)) - f(x_2, u(x_2)) - f(x_1, v(x_1)) + f(x_2, v(x_2))}{x_1 - x_2} \right| \leq \\ & \leq \|F(u) - F(v)\|_{\text{Lip}(I)} \leq L\|u - v\|_{\mathbf{C}_n(I)}, \end{aligned}$$

which implies that

$$|f(x_1, u_1) - f(x_2, u_2) - f(x_1, v_1) + f(x_2, v_2)| \leq L|x_1 - x_2|\|u - v\|_{\mathbf{C}_n(I)}.$$

Hence, letting  $x_1$  and  $x_2$  tend to an arbitrary fixed  $x \in I$ , and making use of (4) and the continuity of  $f(\cdot, y)$  for every  $y \in \mathbb{R}$ , we obtain

$$(7) \quad |f(x, u_1) - f(x, u_2) - f(x, v_1) + f(x, v_2)| \leq Lc(x)|u_1 - u_2 - v_1 + v_2|, \\ x \in I, u_1, u_2, v_1, v_2 \in \mathbb{R}.$$

Substituting here  $u_1 := y + w$ ,  $u_2 := y$ ,  $v_1 := w$ ,  $v_2 := 0$ , we get

$$f(x, y + w) - f(x, y) = [f(x, y) - f(x, 0)] + [f(x, w) - f(x, 0)], \quad y, w \in \mathbb{R}.$$

Put  $h(x) := f(x, 0)$ ,  $x \in I$ . It follows that, for each fixed  $x \in I$ , the function  $\alpha_x : \mathbb{R} \rightarrow \mathbb{R}$  given by

$$(8) \quad \alpha_x(y) := f(x, y) - h(x), \quad y \in \mathbb{R},$$

satisfies the Cauchy functional equation

$$\alpha_x(y + w) = \alpha_x(y) + \alpha_x(w), \quad y, w \in \mathbb{R}.$$

Taking  $v_1 = v_2 := 0$  in (7) we get

$$|\alpha_x(u_1) - \alpha_x(u_2)| \leq L c(x) |u_1 - u_2|, \quad u_1, u_2 \in \mathbb{R}.$$

Hence, for each  $x \in I$ ,  $\alpha_x$  is additive and continuous. Consequently, for each  $x \in I$ , there exists a  $g(x) \in \mathbb{R}$  such that  $\alpha_x(y) = g(x)y$ ,  $y \in \mathbb{R}$ . Now from (8) we have

$$f(x, y) - h(x) = g(x)y, \quad x \in I, y \in \mathbb{R}.$$

Since  $h(x) = f(x, 0) = F(0)$  and  $g(x) = f(x, 1) - f(x, 0) = F(1) - F(0)$ ,  $x \in I$ , we have  $g, h \in C_m(I)$ . This completes the proof.

**Remark 2.** It is easy to observe that the above Theorem remains true on replacing the norm  $\|\cdot\|_{C_n(I)}$  in (6) by any norm  $\|\cdot\|$  such that for some  $M > 0$  and all  $u \in P_n(I)$ , we have  $\|u\| \leq M\|u\|_{C_n(I)}$ .

## 2. Some Corollaries

As an immediate consequence of Theorem we obtain

**COROLLARY 1.** *Let  $f : I \times \mathbb{R} \rightarrow \mathbb{R}$  and let  $n, m$  be positive integer such that  $m \leq n$ . The composition operator  $F$  generated by  $f$  maps the space  $C_n(I)$  into  $C_m(I)$  and is globally Lipschitzian, i.e. there exists an  $L > 0$  such that*

$$\|F(u) - F(v)\|_{C_m(I)} \leq L\|u - v\|_{C_n(I)}, \quad u, v \in C_n(I),$$

*if and only if there exist  $g, h \in C_m(I)$  such that*

$$f(x, y) = g(x)y + h(x), \quad x \in I, y \in \mathbb{R}.$$

**Remark 3.** If in the above Corollary we have  $m > n$ , then  $f(x, y) = h(x)$ ,  $x \in I$ ,  $y \in \mathbb{R}$  (cf. [3], also [1], p. 211, Theorem 8.3).

Because  $\|u\|_{C_1(I)} = \|u\|_{\text{Lip}(I)}$  for all  $u \in P_1(I)$ , by an obvious change in the proof of Theorem 1, we obtain the following generalization of the result proved in [2] and quoted in the Introduction.

**COROLLARY 2.** *Let  $f : I \times \mathbb{R} \rightarrow \mathbb{R}$ . If the composition operator  $F$  generated by  $f$  maps  $P_1(I)$  into  $\text{Lip}(I)$  and there exists an  $L \geq 0$  such that*

$$\|F(u) - F(v)\|_{\text{Lip}(I)} \leq L\|u - v\|_{\text{Lip}(I)}, \quad u, v \in P_1(I),$$

*then there exist  $g, h \in \text{Lip}(I)$  such that*

$$f(x, y) = g(x)y + h(x), \quad x \in I, y \in \mathbb{R}.$$

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