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ON THE CONVOLUTION EQUATION $\check{\mu} \star \rho \star \mu = \rho$

Introduction

It has been resently proved in [B] that for any probability measure μ on a countable (discrete) group G the existence of nontrivial (i.e. nonzero) solutions ρ of the convolution equation ($\diamond$) $\check{\mu} \star \rho \star \mu = \rho$ is equivalent to the concentration (see definition below) of the measure μ. By ${}_\star P_\star(\mu)$ we denote the convex set of all probabilities ρ on G which solve ($\diamond$). Our definitions and notation follow [B]. For the reader's convenience we briefly recall some of them. By the support of a measure μ on G we mean the set $S(\mu) = \{g \in G : \mu(g) > 0\}$. If $S(\mu)$ is finite we say that the measure μ is finitary. The convolution of measures μ, ν is defined

$$(1) \qquad \mu \star \nu(g) = \sum_{h \in G} \mu(gh^{-1})\nu(h) = \sum_{h \in G} \mu(h)\nu(h^{-1}g).$$

Clearly $\mu \star \nu$ belongs to the set $P(G)$ of all probabilities on G if both μ and ν are from $P(G)$. Moreover $(P(G), \star)$ is an associative semigroup. It follows from (1) that $S(\mu \star \nu) = S(\mu)S(\nu)$. By $\check{\mu}$ we denote the symmetric reflection of a measure μ (i.e. $\check{\mu}(g) = \mu(g^{-1})$) and $\nu_1 \wedge \nu_2$ stands for the minimum of ν_1 and ν_2. For a fixed probability measure μ on G we define a positive linear operator P_μ acting on real (or complex) functions f on G

$$(2) \qquad P_\mu f(g) = \sum_{h \in G} f(gh)\,\mu(h).$$

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It is well known that each $\mu \in P(G)$ defines a (right) random walk $\{\xi_n\}_{n\geq 0}$ on the group G. The transition probabilities are:

$$\text{(3)} \qquad \operatorname{Prob}(\xi_{n+1} = g | \xi_n = h) = \mu(h^{-1}g) \quad g, h \in G.$$

Thus for any natural n, $A \subseteq G$ and $h \in G$ we have

$$\text{(4)} \qquad \operatorname{Prob}(\xi_n \in A | \xi_0 = h) = \mu^{\star n}(h^{-1}A).$$

In this note we continue investigations, originated in [B], of the asymptotic behaviour of $\sup\{\mu^{\star n}(hA) : h \in G\}$, where A are finite subsets of G.

DEFINITION 1. A concentration function of a probability measure $\mu \in P(G)$ is the set function $\mathbb{K}_\mu$ defined

$$\mathbb{K}_\mu(A) = \sup_{h \in G} \mu(hA).$$

We say that a measure $\mu \in P(G)$ is *concentrated* if there exist a finite set $A \subseteq G$ and a sequence $g_n \in G$ such that

$$\mathbb{K}_{\mu^{\star n}}(A) = \mu^{\star n}(g_n^{-1}A) \equiv 1.$$

We say that a measure $\mu \in P(G)$ is *not scattered* if there exists a finite set $A \subseteq G$ such that

$$\overline{\lim_{n\to\infty}} \mathbb{K}_{\mu^{\star n}}(A) > 0.$$

We say that a measure $\mu \in P(G)$ is *scattered* if for each finite set $A \subseteq G$ we have

$$\lim_{n\to\infty} \mathbb{K}_{\mu^{\star n}}(A) = 0.$$

Concentration functions of random walks have been investigated for almost forty years. Crucial papers for our considerations are [DL] and [B]. In the second paper it is proved that random walks are either concentrated or scattered. Moreover, (see the Theorem below) it is established that the classes of concentrated and non-scattered random walks coincide. Several conditions eqiuvalent to concentration are given there. In this note we add new eqiuvalent conditions in the case when the measure μ is adapted.

DEFINITION 2. A probability measure μ on G is said to be *adapted* if the smallest subgroup $\mathfrak{G}(\mu)$ containing $S(\mu)$ is the whole group G. By $\mathfrak{H}(\mu)$ we denote the smallest normal subgroup H of $\mathfrak{G}(\mu)$ such that for all $g \in S(\mu)$ we have $S(\mu) \subseteq gH$.

It has been discovered in [B] that if μ is concentrated then

$$\mathfrak{H}(\mu) = \bigcup_{n=1}^{\infty} S(\check{\mu}^{\star n} \star \mu^{\star n}) = \bigcup_{n=1}^{\infty} (S(\check{\mu}^{\star n} \star \mu^{\star n}) \cup S(\mu^{\star n} \star \check{\mu}^{\star n}))$$

is a finite subgroup of G. It is even true that for n being large enough we have $\mathfrak{H}(\mu) = S(\check{\mu}^{\star n} \star \mu^{\star n})$. This yields $a^{-1}\mathfrak{H}(\mu)b = \mathfrak{H}(\mu)$ for all $a, b \in S(\mu)$. The above property of concentrated measures brings our attention to the following family of permutations of the group G. Let $\Phi_{x,y}(g) = xgy^{-1}$ where $x, y, g \in G$.

DEFINITION 3. Given a probability measure $\mu \in P(G)$ by $\mathcal{A}(\mu)$ we denote the group of 1–1 and onto transformations of G generated by all $\Phi_{a,b}$ where $a, b \in S(\mu)$. A set $D \subseteq G$ is called *forward-back* (shortly f–b) invariant if $\Phi(D) = D$ for all $\Phi \in \mathcal{A}(\mu)$. A f–b invariant set D is called *forward-back minimal* if there are no f–b invariant sets included in D other than D.

Since $\mathcal{A}(\mu)$ is a group thus there exist f–b minimal sets and all of them have the form $\{\Phi(g) : \Phi \in \mathcal{A}(\mu)\}$, for some $g \in G$. Let $\mathcal{D}(\mu)$ denote the partition of G onto f–b minimal sets.

DEFINITION 4. Given a probability measure $\mu \in P(G)$ by $\mathcal{A}_\dagger(\mu)$ we denote the group of 1–1 and onto transformations of G generated by all $\Phi_{x,y}$ such that $x = a_n^{\varepsilon_n} a_{n-1}^{\varepsilon_{n-1}} \dots a_1^{\varepsilon_1}$, $y = b_n^{\sigma_n} b_{n-1}^{\sigma_{n-1}} \dots b_1^{\sigma_1}$ where $a_j, b_j \in S(\mu)$, $\varepsilon_j, \sigma_j \in \{-1, 1\}$ and $\sum_{j=1}^{n} \varepsilon_j = \sum_{j=1}^{n} \sigma_j$. By $\mathcal{D}_\dagger(\mu)$ we denote the partition of G onto minimal sets defined by the group $\mathcal{A}_\dagger(\mu)$ and we call them f–b $\dagger$ minimal.

Since $\mathcal{A}(\mu) \subseteq \mathcal{A}_\dagger(\mu)$ thus the partition $\mathcal{D}_\dagger(\mu)$ should be finer. However for concentrated μ we find the partitions $\mathcal{D}(\mu), \mathcal{D}_\dagger(\mu)$ are same. The existence at least one finite set of the partition $\mathcal{D}(\mu)$ or $\mathcal{D}_\dagger(\mu)$ is equivalent to concentration of μ and this fact is the main point of our *Theorem 1*. The partitions $\mathcal{D}(\mu)$ or $\mathcal{D}_\dagger(\mu)$ are also used to describe the geometry of ${}_\star P_\star(\mu)$. We find out that either there is only one trivial solution $\rho = 0$ of ($\diamond$) or if G is infinite and μ is adapted then the set ${}_\star P_\star(\mu)$ is infinite dimensional. Since solutions of ($\diamond$) form a Banach sublattice of $\ell^1(G)$ thus in the second case ${}_\star P_\star(\mu)$ is an affine and isometric copy of $\{(t_n)_{n=1}^{\infty} : \sum_{n=1}^{\infty} t_n = 1,\ t_n \geq 0\}$ (so an affine and isometric copy of $P(G)$). We finish the introductory part with the following *Theorem* which comes from [B].

THEOREM (see [B]). *Let μ be a probability measure on a countable group G. Then the following conditions are equivalent:*

(i) *μ is concentrated*
(ii) *μ is not scattered*
(iii) *there exists a function $f \in \ell^2(G)$ such that* $\lim_{n\to\infty} \|P_\mu^n f\|_2 > 0$

(iv) *there exists a probability measure* ρ *on* G *such that* $\check{\mu} \star \rho \star \mu = \rho$
(v) $\lim_{n\to\infty} \operatorname{card}(S(\mu^{\star n})) < \infty$
(vi) $\mathfrak{H}(\mu)$ *is finite.*

Results

The above *Theorem* gives us a convenient tool in investigating of random walks on discrete groups. Let us notice that the question whether a probability measure μ on countable G is concentrated may be studied using computers. By [B] nonfinitary measures may be excluded since they are scattered. For a concrete discrete group G and a finitary measure $\mu \in P(G)$ we may build an algorithm describing the sequence $S(\check{\mu}^{\star n})S(\mu^{\star n}) = \mathfrak{H}_n(\mu)$. If at some moment $n+1$ the sets $\mathfrak{H}_{n+1}(\mu)$ and $\mathfrak{H}_n(\mu)$ coincide then the procedure may be stopped with the conclusion that μ is concentrated. Moreover in this case the group $\mathfrak{H}(\mu)$ is exactly $\mathfrak{H}_n(\mu)$. To prove *Theorem 1* we need

LEMMA 1. *If* μ *is adapted and concentrated then* $\mathcal{D}(\mu)$ *and* $\mathcal{D}_{\dagger}(\mu)$ *coincide with the family of cosets of* $\mathfrak{H}(\mu)$.

Proof. Since for concentrated μ the subgroup $\mathfrak{H}(\mu)$ may be represented as $\bigcup_{n=1}^{\infty} S(\mu)^{-n}S(\mu)^{n} = S(\mu)^{-n_\mu}S(\mu)^{n_\mu}$ for some natural n_μ thus $\mathfrak{H}(\mu)$ is f–b invariant set. It is also a normal subgroup, so for any $a, a_1, \dots, a_n \in S(\mu)$ and $\varepsilon_1, \dots, \varepsilon_n \in \{-1, 1\}$ we have $a_n^{\varepsilon_n} \cdots a_1^{\varepsilon_1}\mathfrak{H}(\mu) = a^{\sum_{j=1}^{n}\varepsilon_j}\mathfrak{H}(\mu)$. In particular $\mathfrak{H}(\mu)$ is f–b $\dagger$ invariant. Clearly it is f–b minimal since $\mathfrak{H}(\mu) = \{\Phi(e) : \Phi \in \mathcal{A}(\mu)\}$ where e denotes the neutral element of G.This implies that the subgroup $\mathfrak{H}(\mu)$ is f–b $\dagger$ minimal. Since μ is assumed to be adapted on the same way we may prove f–b or f–b $\dagger$ minimality of any coset $g\mathfrak{H}(\mu)$. It follows that the partitions $\mathcal{D}(\mu)$ and $\mathcal{D}_{\dagger}(\mu)$ coincide with classes $\{g\mathfrak{H}(\mu)\}_{g\in G}$ and the proof of lemma is completed. ∎

Now we are in a position to prove:

THEOREM 1. *Let* μ *be an adapted measure on a countable group* G. *Then the following conditions are equivalent:*

(i) μ *is concentrated*
(vii) *there exists a finite f-b* $\dagger$ *invariant set*
(viii) *there exists a finite f-b invariant set*
(ix) *all sets of the partition* $\mathcal{D}_{\dagger}(\mu)$ *are finite and coincide with classes of* $\mathfrak{H}(\mu)$
(x) *all sets of the partition* $\mathcal{D}(\mu)$ *are finite and coincide with classes of* $\mathfrak{H}(\mu)$.

P r o o f. Implications (vii)$\Longrightarrow$(viii), (ix)$\Longrightarrow$(x)$\Longrightarrow$(viii) and (ix)$\Longrightarrow$(vii) are obvious. (i)$\Longrightarrow$(ix) and (i)$\Longrightarrow$(x) follow *Lemma 1*. So we only have to prove that (viii) implies (i). To show this we prove that existence of a finite f–b invariant set $D \subseteq G$ implies nontrivial solutions of ($\diamond$). For this let us take τ_D, the uniform distribution on D. We have

$$\begin{aligned}\check{\mu} \star \tau_D \star \mu(g) &= \sum_{a,b \in S(\mu)} \tau_D(agb^{-1})\mu(a)\mu(b) \\ &= \begin{cases} 0 & \text{if } g \notin D \\ \sum\limits_{a,b \in S(\mu)} \frac{1}{\operatorname{card}(D)}\mu(a)\mu(b) & \text{otherwise} \end{cases} \\ &= \begin{cases} 0 & \text{if } g \notin D \\ \frac{1}{\operatorname{card}(D)} & \text{otherwise} \end{cases} = \tau_D(g). \end{aligned}$$

Thus $\tau_D \in {}_\star P_\star(\mu)$ and the proof of the *Theorem 1* is completed. ∎

R e m a r k 1. Notice that by the same arguments $\tau_D \in {}_\star P_\star(\check{\mu})$.

Given a probability measure μ on G let $\{\xi_n\}_{n=1}^{\infty}$ denote the random walk generated by μ and $\{\tilde{\xi}_n\}_{n=1}^{\infty}$ its independent copy. Consider a Markov process $\{\eta_n\}_{n=1}^{\infty}$ on G defined as

$$\eta_n = \tilde{\xi}_n^{-1} \cdot \eta_0 \cdot \xi_n \tag{5}$$

where η_0 is independent of $\tilde{\xi}$ and ξ. If ρ_0 is a distribution of η_0 then the distribution of η_n is $\check{\mu}^{\star n} \star \rho_0 \star \mu^{\star n}$. For symmetric μ processes like (5) belong to the class of so called bilateral random walks. In this note we obtain a satisfactory description of their asymptotic distributions. We begin with:

THEOREM 2. *Let μ be an adapted probability measure on a countable group G. Then the convex set ${}_\star P_\star(\mu)$ i s either empty (if μ is scattered) or consists of all probabilities $\rho \in P(G)$ having the representation*

$$\rho = \sum_{D \in \mathcal{D}(\mu)} \alpha_D \tau_D \tag{6}$$

(if μ is concentrated) where $\alpha_D \geq 0$, $\sum_{D \in \mathcal{D}(\mu)} \alpha_D = 1$ and $\tau_D(\cdot) = \frac{\operatorname{card}\,(\cdot \cap D)}{\operatorname{card}\,(\mathfrak{H}(\mu))}$ is the uniform distribution on the set $D \in \mathcal{D}(\mu)$. Moreover extreme points of ${}_\star P_\star(\mu)$ coincide with measures τ_D.

P r o o f. We may assume that μ is concentrated. It is noticed in the proof of *Theorem 1* that $\tau_D \in {}_\star P_\star(\mu)$. First we prove (6). For $\rho \in_\star P_\star(\mu)$ we set $E_1 = \{g_1 \in G : \rho(g_1) = \alpha_1\}$ where $\alpha_1 = \sup\{\rho(g) : g \in G\}$. Clearly E_1 is nonempty and finite. If $g_1 \in E_1$ then

$$\rho(g_1) = \check{\mu} \star \rho \star \mu(g_1) = \sum_{a,b \in S(\mu)} \rho(a g_1 b^{-1}) \mu(a) \mu(b),$$

so for all $a, b \in S(\mu)$ the points $a g_1 b^{-1}$ belong to E_1. It implies E_1 is f–b invariant, so may be decomposed on finite many sets of the partition $\mathcal{D}(\mu)$. Since ρ is uniformly distributed on E_1 thus $\rho|_{E_1} = \sum_{D \subseteq E_1} \alpha_D \tau_D$, where $\alpha_D \equiv \alpha_1 \mathrm{card}\,(\mathfrak{H}(\mu))$ does not depend on $D \subseteq E_1$.

Assume that there are pairwise disjoint sets $E_1, \ldots, E_{k-1}$ each of them is a finite union of elements of the partition $\mathcal{D}(\mu)$, and that for any $g_j \in E_j$ ($1 \le j \le k-1$) we have

$$\rho(g_j) = \sup_{g \in G \setminus (E_1 \cup \ldots E_{j-1})} \rho(g) = \alpha_j.$$

This implies $\rho|_{E_1 \cup \ldots E_{k-1}} = \sum_{D \subseteq E_1 \cup \ldots E_{k-1}} \alpha_D \tau_D$, where $\alpha_D = \alpha_j \mathrm{card}(\mathfrak{H}(\mu))$ for $D \subseteq E_j$. Now we set $\alpha_k = \sup_{g \in G \setminus (E_1 \cup \ldots \cup E_{k-1})} \rho(g) < \alpha_{k-1}$ and define $E_k = \{g \in G : \rho(g) = \alpha_k\}$. If $\alpha_k = 0$ our procedure may be stopped. If not, we show that again E_k is a finite union of f–b minimal sets disjoint from $E_1, \ldots, E_{k-1}$. Since the last sets are f–b invariant thus $\Phi(E_k) \subseteq G \setminus (E_1 \cup \ldots \cup E_{k-1})$ for all $\Phi \in \mathcal{A}(\mu)$. On the other hand if $g_k \in E_k$ then we have $\rho(g_k) = \sum_{a,b \in S(\mu)} \rho(a g_k b^{-1}) \mu(a) \mu(b)$ so $\rho(\Phi_{a,b}(g_k)) \equiv \alpha_k$. This implies $\Phi_{a,b}(E_k) \subseteq E_k$ for all $a, b \in S(\mu)$. Since the set E_k is finite and the transformations Φ are 1–1, thus E_k is f–b invariant. The same arguments as before lead us to the representation $\rho|_{E_k} = \sum_{D \subseteq E_k} \alpha_D \tau_D$ where $\alpha_D = \alpha_k \mathrm{card}(\mathfrak{H}(\mu))$. Now by induction the decomposition (6) is easily seen.

In order to prove that extreme points of ${}_\star P_\star(\mu)$ are exactly measures τ_D it is sufficient to show that the support $S(\rho)$ of extremal $\rho \in \mathrm{ex}_\star P_\star(\mu)$ is f–b minimal, and that two distinct extremal solutions of ($\diamond$) have disjoint supports. If $S(\rho)$ is not f–b minimal then by the decomposition (6) $\rho = \sum_{D \in \mathcal{A}(\mu)} \alpha_D \tau_D$ has at least two nonzero coefficients α_D. Clearly such ρ is not extremal. Now let $\rho_1 \neq \rho_2$ be from ex ${}_\star P_\star(\mu)$. Since

$$\check{\mu} \star (\rho_1 \wedge \rho_2) \star \mu \le (\check{\mu} \star \rho_1 \star \mu) \wedge (\check{\mu} \star \rho_2 \star \mu) = \rho_1 \wedge \rho_2$$

and

$$\sum_{g \in G} \check{\mu} \star (\rho_1 \wedge \rho_2) \star \mu(g) = \sum_{g \in G} \rho_1 \wedge \rho_2(g) = \beta$$

thus $\rho_1 \wedge \rho_2 = 0$ (it holds if $\beta = 0$ and then $S(\rho_1) \cap S(\rho_2) = \emptyset$) or $\beta > 0$ and then $\frac{\rho_1 \wedge \rho_2}{\beta} \in_\star P_\star(\mu)$. Clearly $\beta < 1$ since $\rho_1 \neq \rho_2$. But $0 < \beta < 1$ gives

$$\rho_j = \beta \frac{\rho_1 \wedge \rho_2}{\beta} + (1-\beta) \frac{\rho_j - \rho_1 \wedge \rho_2}{1-\beta}.$$

For extreme ρ_1 and ρ_2 the above is possible only if $\rho_1 = \rho_2$. As a result we get that different extremal solutions ρ_1 and ρ_2 of ($\diamond$) satisfy $S(\rho_1) \cap S(\rho_2) = \emptyset$, and the proof of *Theorem 2* is completed. ∎

COROLLARY 1. *Let μ be an adapted probability measure on a countable group G. If μ is concentrated then ${}_\star P_\star(\mu)$ is an affine and isometric copy of $\{(t_j)_{j=1}^N : \sum_{j=1}^N t_j = 1, t_j \geq 0\}$ where $N = \frac{\text{card } (G)}{\text{card } (\mathfrak{H}(\mu))}$ if G is finite and $N = \infty$ if G is infinite.*

COROLLARY 2. *For any adapted probability measure μ on a countable group G we have ${}_\star P_\star(\mu) =_\star P_\star(\check{\mu})$.*

Proof. By *Theorem* from [B] μ is concentrated if and only if $\check{\mu}$ is concentrated. Since $\mathcal{A}(\mu) = \mathcal{A}(\check{\mu})$ thus $\mathcal{D}(\mu) = \mathcal{D}(\check{\mu})$, so for concentrated μ the decomposition (6) gives ${}_\star P_\star(\mu) =_\star P_\star(\check{\mu})$. ∎

Now we will study asymptotic behaviour of distributions of η_n. Obviously we may drop the case of scattered μ. In fact, for such measures we have:

$$\begin{aligned}
&\sup_{\nu \in P(G)} \sup_{g \in G} \check{\mu}^{\star n} \star \nu \star \mu^{\star n}(gA) = \\
&\sup_{\nu \in P(G)} \sup_{g \in G} \sum_{a,c \in G} \mu^{\star n}(c^{-1}agA)\nu(c)\mu^{\star n}(a) \leq \\
&\sup_{\nu \in P(G)} \sum_{a,c \in G} (\sup_{g \in G} \mu^{\star n}(gA))\nu(c)\mu^{\star n}(a) = \\
&\sup_{g \in G} \mu^{\star n}(gA) \overset{n \to \infty}{\longrightarrow} 0
\end{aligned}$$

for any finite set $A \subseteq G$ and $\nu \in P(G)$. Thus the process η_n is scattered as well. On the other hand if μ is concentrated and adapted the situation is different. For any initial distribution ν the distribution of η_n becomes exponentially stationary. Namely, we have

THEOREM 3. *Let μ be an adapted and concentrated measure on a countable discrete group G. Then for some $C > 0$ and $\gamma > 0$ the following estimation*

$$(7) \qquad \sup_{\nu \in P(G)} \|\check{\mu}^{\star n} \star \nu \star \mu^{\star n} - \sum_{D \in \mathcal{D}(\mu)} \nu(D)\tau_D\| \leq Ce^{-\gamma n}$$

holds, where $\|\cdot\|$ stands for $\ell^1(G)$ (or equivalently variation) norm.

Proof. For an element $D = g\mathfrak{H}(\mu)$ of the partition $\mathcal{D}(\mu)$ let $\{\eta_{n,D}\}_{n\geq 0}$ denote the restriction of the Markov chain to the f–b invariant set D. Now $\{\eta_{n,D}\}_{n\geq 0}$ is a finite Markov chain with transition probabilities

$$p^D_{g_1,g_2} = \sum_{a^{-1}g_1b=g_2} \mu(a)\mu(b).$$

It is proved in *Theorem 1* that the measure τ_D is a stationary probabilistic vector for this transition matrix. Let h $_\mu$ = card(D) = card$(\mathfrak{H}(\mu))$. With this notation $[p^D_{g_1,g_2}]_{h_\mu\times h_\mu}$ is a $h_\mu \times h_\mu$ doubly stochastic. We show that the family $\{P^{(D)}\}_{D\in\mathcal{D}(\mu)}$ is "uniformly irreducible". Let us fix for a while $D \in \mathcal{D}(\mu)$. For any $g \in D$ the states which may be reached in time n starting from g are exactly $S(\check{\mu}^{\star n})gS(\mu^{\star n}) = D_{g,n} \subseteq D$. Let $g = a_m^{\varepsilon_m}\ldots a_1^{\varepsilon_1}$ for some $a_m,\ldots,a_1 \in S(\mu)$ and $\varepsilon_m,\ldots,\varepsilon_1 \in \{-1,1\}$. By the same arguments as in the proof of *Theorem* in [B] we get

$$S(\check{\mu}^{\star n})gS(\mu^{\star n}) = S(\check{\mu}^{\star n})S(\mu^{(n+\sum_{j=1}^m \varepsilon_j)}) = \mathfrak{H}(\mu)g = D$$

if n is large enough. Thus $D_{g,n} = D$ for all $n \geq L(g)$.Let $L(D) = \sup_{g\in D} L(g)$ and $L = \sup_{D\in\mathcal{D}(\mu)} L(D)$.We show that $L < \infty$. Firstly we notice that for any fixed $D \in \mathcal{D}(\mu)$ and arbitrary $g \in D$ if $D_{g,n} = D_{g,k}$ for some $k > n$ then $D_{g,n} = D$. In fact, for any $j > n$ we have

$$D_{g,j} = S(\check{\mu}^{\star(j-n)})D_{g,n}S(\mu^{\star(j-n)}),$$

so $D_{g,n} = D_{g,k}$ implies that the sequence $\{D_{g,j}\}_{j\geq n}$ is periodic. Since for large j the sets $D_{g,j}$ are stabilized as D, by periodicity for some $n \leq j \leq k$ we have $D_{g,j} = D$. This means that $D = D_{g,j} = D_{g,k} = D_{g,n}$. By our *Lemma 1* all sets D of the partition $\mathcal{D}(\mu)$ have exactly h $_\mu$ elements. Obviously there are only finite many 1–1 sequences of subsets of a finite set D, and the amount of all such sequences is a function of h $_\mu$. It follows $L < \infty$.

In particular we get that for all $D \in \mathcal{D}(\mu)$ and $g_1, g_2 \in D$ the transition probabilities $p^{D,n}_{g_1,g_2} = \sum_{a^{-1}g_1b=g_2} \mu^{\star n}(a)\mu^{\star n}(b)$ are strictly positive if $n \geq L$. This implies that for some α we have $p^{D,n}_{g_1,g_2} \geq \alpha = \inf_{g\in S(\mu^{\star n})}(\mu^{\star n}(g))^2 > 0$. So the matrix of transition probabilities at the time n satisfies

$$(P^{(D)})^n = [p^{D,n}_{g_i,g_j}]_{h_\mu\times h_\mu} \geq \alpha[1]_{h_\mu\times h_\mu}$$

where $[1]_{h_\mu\times h_\mu}$ denotes the $h_\mu \times h_\mu$ matrix with 1's as entries. From the theory of doubly stochastic matrices $(P^{(D)})^n$ convereges to $[1/h_\mu]_{h_\mu\times h_\mu}$

(with exponential rate). Thus for some $C > 0$ and $\gamma > 0$ the inequality $\|(P^{(D)})^n - [1/h_\mu]_{h_\mu \times h_\mu}\| \leq Ce^{-\gamma n}$ holds. The constants C and γ depend on α and h $_\mu$ but not on any particular $D \in \mathcal{D}(\mu)$. As a result we get the estimation

$$\sup_{D \in \mathcal{D}(\mu)} \sup_{A \subseteq D} \sup_{\nu \in P(D)} |\check{\mu}^{\star n} \star \nu \star \mu^{\star n}(A) - \tau_D(A)| \leq Ce^{-\gamma n},$$

where $P(D)$ denotes the set of all probabilities ν such that $S(\nu) \subseteq D$.

Let $\nu \in P(G)$ be an initial distribution of the Markov chain $\{\eta_n\}_{n \geq 0}$ and $A \subseteq G$ be arbitrary. By ν_D we denote the conditional probability of ν on D (i.e. $\nu_D(\cdot) = \frac{\nu(\cdot \cap D)}{\nu(D)}$) if $\nu(D) > 0$ or something if $\nu(D) = 0$. Now we have

$$\begin{aligned} &\Big|\mathrm{Prob}_\nu(\eta_n \in A) - \sum_{D \in \mathcal{D}(\mu)} \nu(D)\tau_D(A)\Big| = \\ &\Big|\sum_{D \in \mathcal{D}(\mu)} (\mathrm{Prob}_\nu(\eta_n \in D \cap A) - \nu(D)\tau_D(A))\Big| \leq \\ &\sum_{D \in \mathcal{D}(\mu)} \nu(D)|\mathrm{Prob}_{\nu_D}(\eta_{n,D} \in D \cap A) - \tau_D(D \cap A)| \leq \\ &\sum_{D \in \mathcal{D}(\mu)} \nu(D)Ce^{-\gamma n} = Ce^{-\gamma n}. \end{aligned}$$

Here $A \subseteq G$ and $\nu \in P(G)$ are arbitrary so we obtain (7) and the proof of *Theorem 3* is completed. ∎

COROLLARY 3. *Let μ be an adapted probability measure on countable G and T_μ denote the stochastic operator on the Banach lattice $\ell^1(G)$ defined as $T_\mu(\nu) = \check{\mu} \star \nu \star \mu$. Then the following two conditions are equivalent:*

(i) *μ is concentrated*

(xi) *there exist a stochastic projection Q_μ and constants $C > 0$, $\gamma > 0$ such that*

$$\|T_\mu^n - Q_\mu\|_{\mathrm{oper}} \leq Ce^{-\gamma n}$$

where $\|\cdot\|_{oper}$ is the operator norm on $\mathcal{L}(\ell^1(G))$.

Moreover if the above hold then $Q_\mu = \sum_{D \in \mathcal{D}(\mu)} \lambda_D \otimes \tau_D$ where $\lambda_D(\nu) = \nu(D)$.

Proof. Any finite signed measure ν on G may be represented as $\nu = s\nu_1 + t\nu_2$ where ν_1, ν_2 are orthogonal probabilities and $\|\nu\| = |s| + |t|$. ∎

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