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A MEASURE OF NONCOMPACTNESS IN SEQUENCE BANACH SPACES

In this paper we define a measure of noncompactness (m.n.c.), in some sequence Banach spaces, which is equivalent to the Hausdorff m.n.c.. In [3] another m.n.c. in those spaces is considered, but it is not equivalent to the Hausdorff m.n.c.. The notion of a m.n.c. turns out to be an useful tool in many branches of mathematical analysis. The current state of this theory and its applications can be found in the books [1] and [2].

Assume that X is a Banach space. The unit closed ball will be denoted by B_X. Moreover $Conv(A)$ denotes the convex closure of a set $A \subset X$ and $\|A\| = sup\{\|x\| : x \in A\}$. Finally, denote by $P_b(X)$ the family of all nonempty and bounded subsets of X. We use the following definition [2]: a function

$$\mu : P_b(X) \to [0, \infty[$$

will be called a m.n.c. in X if it satisfies the following conditions:

(1) $\mu(A) = 0 \Leftrightarrow A$ is relatively compact,

(2) $A \subset B \Rightarrow \mu(A) \leq \mu(B)$,

(3) $\mu(\mathrm{Conv}\,(A)) = \mu(A)$,

(4) $\mu(A \cup B) = \max\{\mu(A), \mu(B)\}$,

(5) $\mu(A + B) \leq \mu(A) + \mu(B)$,

(6) $\mu(cA) = |c|\mu(A)$ (for every scalar c).

A m.n.c. that it seems to be the most convenient in the applications is the Hausdorff m.n.c. $h : P_b(X) \to [0, \infty[$, defined in the following way, for $A \subset X$ nonempty and bounded

$$h(A) := \inf\{r > 0 : \exists F \subset X \text{ finite}, \ A \subset F + rB_X\}.$$

This m.n.c. verifies $h(B_X) = 1$.

Assume that (X_i) is a sequence of Banach spaces. Denote by $l^p(X_i)$, $1 \leq p < \infty$, the space of all sequences $x = (x_i)$, $x_i \in X_i$ for $i = 1, 2, \ldots$ such that

$$\sum_{i=1}^{\infty} \|x_i\|^p < \infty.$$

It is well known that $l^p(X_i)$ is a Banach space under the norm

$$\|(x_i)\| = \Big(\sum_{i=1}^{\infty} \|x_i\|^p\Big)^{1/p}.$$

Denote by h^p the Hausdorff m.n.c. on the space $l^p(X_i)$ and h_i the Hausdorff m.n.c. on the space X_i. We consider the following operators

$$\pi_n : (x_i) \in l^p(X_i) \to x_n \in X_n,$$
$$\tau_n : (x_i) \in l^p(X_i) \to (0, \ldots, 0, x_n, x_{n+1}, \ldots) \in l^p(X_i),$$

and the following quantities, for $A \in P_b(l^p(X_i))$:

$$a^p(A) := \sup\{h_n(\pi_n(A)) : n = 1, 2, \ldots\},$$
$$b^p(A) := \inf\{\|\tau_n(A) : n = 1, 2, \ldots\}.$$

In [3], based on a theorem of [4], it is proved that

$$\mu^p(A) := \max\{a^p(A), b^p(A)\}$$

defines a m.n.c. on the space $l^p(X_i)$, which is not equivalent to the Hausdorff m.n.c. h^p. Now we consider the operator

$$\sigma_n : (x_i) \in l^p(X_i) \to (x_1, \ldots, x_n, 0, 0, \ldots) \in l^p(X_i)$$

and the quantity

$$c^p(A) := \sup\{h(\sigma_n(A)) : n = 1, 2, \ldots\}.$$

Note that $h(\sigma_n(A))$ agrees with the Hausdorff m.n.c. of $\sigma_n(A)$ in the space $l^p(X_1, X_2, \ldots, X_n)$. In the following theorem we define a m.n.c. η^p in the space $l^p(X_i)$, which is equivalent to Hausdorff m.n.c. h^p.

THEOREM. *In the space $l^p(X_i)$ we define the quantity*

$$\eta^p(A) := \max\{c^p(A), b^p(A)\},$$

for A nonempty and bounded subset of $l^p(X_i)$. Then η^p is a measure of noncompactness and moreover

$$\eta^p \leq h^p \leq 2\eta^p(A).$$

Proof. The proof of the first part is very simple and is omitted. To prove the second part denote $r := h^p(A)$. Then, for $\varepsilon > 0$, we can find a finite set $F \subset l^p(X_i)$ such that

$$A \subset F + (r + \varepsilon) B_{l^p(X_i)}.$$

Using the equality

$$\eta^p(B_{l^p(X_i)}) = 1$$

and the properties of a m.n.c., we infer that $\eta^p(A) \leq r+\varepsilon$. The arbitrariness of ε completes a part of the proof: $\eta^p \leq h^p$.

In what follows we show that $h^p \leq 2\eta^p$. Obviously, we have, for any $A \subset l^p(X_i)$ nonempty and bounded,

$$A \subset \sigma_n(A) + \tau_{n+1}(A) \qquad (n = 1, 2, \ldots).$$

We put $r := \eta^p(A)$. Consequently, we have $b^p(A) \leq r$ and $c^p(A) \leq r$. Given any $\varepsilon > 0$, by definition of $b^p(A)$, choose m such that $\|\tau_m(A)\| < r+\varepsilon$. On the other hand, by definition of $c^p(A)$, we have $h^p(\sigma_n(A)) \leq r$, for $n = 1, 2, \ldots$ Since

$$A \subset \sigma_m(A) + \tau_{m+1}(A),$$

and by the properties of a m.n.c., we obtain that

$$h^p(A) \leq h^p(\sigma_m(A)) + h^p(\tau_{m+1}(A)).$$

Trivially $h^p(A) \leq \|A\|$, hence

$$h^p(A) \leq r + \|\tau_{m+1}(A)\| < 2r + \varepsilon.$$

The arbitrariness of ε gives

$$h^p(A) \leq 2r = 2\eta^p(A)$$

and it completes the proof.

References

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