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STRONG APPROXIMATION IN GENERALIZED HÖLDER NORMS

In this paper we extend some results on approximation of 2π-periodic functions, given in [3]–[5], to the case of strong approximation by some means of Fourier series. Similarly as in [3], we present approximation results in Hölder norms based on general modulus-type functions.

1. Notations

1.1. Let $C = C_{2\pi}$ be the space of 2π-periodic real-valued functions continuous on $R = (-\infty, +\infty)$ with the norm

$$\|f\|_C := \max_x |f(x)|. \tag{1}$$

Let Ω be the set of modulus-type functions, i.e. Ω is the set of all functions Ω satisfying the following conditions:

a) ω is defined and continuous on $\langle 0, +\infty)$,
b) ω is increasing and $\omega(0) = 0$,
c) $\omega(h)h^{-1}$ is decreasing for $h > 0$.

For a given $\omega \in \Omega$ we define the class H^ω of all functions $f \in C$ for which

$$\|f\|_\omega := \sup_{h>0} \frac{\|\Delta_h f\|_C}{\omega(h)} < +\infty, \tag{2}$$

where

$$\Delta_h f(x) = f(x+h) - f(x). \tag{3}$$

In H^ω we define the norm

$$\|f\|_{H^\omega} := \|f\|_C + \|f\|_\omega. \tag{4}$$

It is known that H^ω $(\omega \in \Omega)$ with the norm (4) is a Banach space. H^ω is called generalized Hölder space. If $\omega(h) = h^\alpha$, $0 < \alpha \leq 1$, then H^ω is the classical Hölder space.

1.2. Similarly as in [3]–[5] we define subspace $\widetilde{H}^\omega \subset H^\omega$, $\omega \in \Omega$, as follows

$$\widetilde{H}^\omega := \left\{ f \in H^\omega : \lim_{h\to 0_+} \frac{\|\Delta_h f\|_C}{\omega(h)} = 0 \right\}$$

with the norm $\|\cdot\|_{\widetilde{H}^\omega}$ defined by (4).

If $\omega, \mu \in \Omega$ and $q(h) = \frac{\omega(h)}{\mu(h)}$, $h > 0$, is a non-decreasing function, then

$$H^\omega \subset H^\mu \quad \text{and} \quad \widetilde{H}^\omega \subset \widetilde{H}^\mu. \tag{5}$$

1.3. Denote, as usual, by $E_n(f;C)$, $n \in N = \{0,1,\ldots\}$, the best approximation of function $f \in C$ by trigonometric polynomials of order $\leq n$ in the sense of C. It is known ([6], [8]) that if $f \in H^\omega$, $\omega \in \Omega$, then

$$E_n(f;C) \leq 3\omega\left(\frac{1}{n+1}\right)\|f\|_\omega, \qquad n \in N. \tag{6}$$

If $f \in \widetilde{H}^\omega$, $\omega \in \Omega$, then

$$E_n(f;C) = o\left(\omega\left(\frac{1}{n+1}\right)\right) \qquad \text{as } n \to \infty. \tag{7}$$

1.4. For a given $f \in C$ let

$$\frac{a_o}{2} + \sum_{k=1}^{\infty}(a_k \cos kx + b_k \sin kx) \tag{8}$$

be its Fourier series. Let $S_n(x;f)$, $n \in N$, $x \in R$, be the n-th partial sum of (8).

In this paper we shall consider the strong approximation of function f belonging to the spaces H^ω by some means of Fourier series (8) in generalized Hölder norms (4).

Let Q be a set of real numbers, $r_o \notin Q$ be a accumulation point of Q and let $T = \{t_k(r)\}_{k=0}^{\infty}$ be a sequence of real-valued functions defined on Q and such that:

1° $t_k(r) \geq 0$ for all $r \in Q$ and $k \in N$,

2° $\lim_{\substack{r\to r_o\\ r\in Q}} t_k(r) = 0$ for every fixed $k \in N$,

3° the series $\sum_{k=1}^{\infty} t_k(r) log(k+1)$ is convergent on Q,

4° $\sum_{n=0}^{\infty} 2^n T_{\bar{n},2\bar{n}}(r) \leq M_1$, for all $r \in Q$, where $\bar{n} = 2^n - 1$, $M_1 =$ const. > 0 and

$$T_{p,q}(r) := \max_{p\leq k\leq q} t_k(r), \qquad p,q \in N. \tag{9}$$

We define the strong T-means of Fourier series (8) of $f \in C$:

$$L(r,x;f,T) := \sum_{k=0}^{\infty} t_k(r)|S_k(x;f) - f(x)| \tag{10}$$

for $x \in R$ and $r \in Q$. It is obvious that if $f \in C$, then for every fixed $r \in Q$ the function $L(r,\cdot;f,T)$ belongs also to C.

The purpose of this note is to estimate the generalized Hölder norms of $L(r,\cdot;f,T)$ for $f \in H^\omega$ and $f \in \widetilde{H}^\omega$.

By $M_k(\cdot)$, $k = 1,2,\ldots$, we shall denote suitable positive constants depending only on indicated parameters.

2. Auxiliary results

Let $P = \{p_k\}_{k=0}^{\infty}$ be a non-negative and bounded sequence,

$$P_{n,m} := \max_{n \le k \le m} p_k, \qquad n,m \in N, \tag{11}$$

and let $\{v_n\}$ be a monotone non-decreasing sequence of integers such that $0 \le v_n \le n$ and $n \le \lambda v_n$ for $n \in N$, where $\lambda = \text{const.} \ge 1$. For a given $f \in C$ we define two functions as follows:

$$U_{n,v_n}(x;f,P) := \frac{1}{v_n+1} \sum_{k=n-v_n}^{n} p_k|S_k(x;f)| \tag{12}$$

and

$$W_{n,v_n}(x;f,P) := \frac{1}{v_n+1} \sum_{k=n-v_n}^{n} p_k|S_k(x;f) - f(x)|, \tag{13}$$

$x \in R$. Clearly the functions (12) and (13) belong to C also.

Using the Leindler results given in [1] and [2] ([2], Th. 1.11), we immediately obtain

LEMMA 1. *If $f \in C$, then we have*

$$\|U_{n,v_n}(\cdot;f,P)\|_C \le M_1(\lambda)P_{n-v_n,n}\|f\|_C \tag{14}$$

and

$$\|W_{n,v_n}(\cdot;f,P)\|_C \le M_2(\lambda)P_{n-v_n,n}E_{n-v_n}(f;C) \tag{15}$$

for all $n \in N$.

Using Lemma 1, we shall prove two lemmas.

LEMMA 2. *If $f \in H^\omega$, $\omega \in \Omega$, then*

$$\|W_{n,v_n}(\cdot;f,P)\|_\omega \le M_4(\lambda)P_{n-v_n,n}\|f\|_\omega$$

for all $n \in N$, which proves that $W_{n,v_n}(\cdot;f,P) \in H^\omega$.

Proof. By (1)–(3) and (11)–(13), we have

$$\|W_{n,v_n}(\,\cdot\,;f,P)\|_\omega = \sup_{h>0}\{\|\Delta_h W_{n,v_n}(\,\cdot\,;f,P)\|_C/\omega(h)\}$$

and

$$\begin{aligned}|\Delta_h W_{n,v_n}(x;f,P)| &= \left|\frac{1}{v_n+1}\sum_{k=n-v_n}^{n} p_k\Delta_h|S_k(x;f)-f(x)|\right| \\ &\le \frac{1}{v_n+1}\sum_{k=n-v_n}^{n} p_k|\Delta_h S_k(x;f)-\Delta_h f(x)| \\ &= \frac{1}{v_n+1}\sum_{k=n-v_n}^{n} p_k|S_k(x;\Delta_h f)-\Delta_h f(x)| \\ &\le U_{n,v_n}(x;\Delta_h f,P)+P_{n-v_n,n}|\Delta_h f(x)|\end{aligned}$$

for all $x\in R$. Hence and by (14) we get

$$\|\Delta_h W_{n,v_n}(\,\cdot\,;f,P)\|_C \le (M_1(\lambda)+1)P_{n-v_n,n}\|\Delta_h f\|_C, \tag{16}$$

which implies

$$\|W_{n,v_n}(.;f,P)\|_\omega \le (M_1(\lambda)+1)P_{n-v_n,n}\|f\|_\omega.$$

Thus the proof is completed.

LEMMA 3. *Suppose that $f\in H^\omega$, $\omega\in\Omega$, and $\mu\in\Omega$ is a function such that $q(h)=\frac{\omega(h)}{\mu(h)}$ is non-decreasing for $h>0$. Then we have*

$$\|W_{n,v_n}(\,\cdot\,;f,P)\|_\mu \le M_5(\lambda)P_{n-v_n,n}\,q\left(\frac{1}{n-v_n+1}\right)\|f\|_\omega$$

for all $n\in N$.

Proof. From Lemma 2 and (5) it follows that $W_{n,v_n}(\,\cdot\,;f,P)\in H^\mu$. Hence and by (1)–(3) we have

$$\begin{aligned}\|W_{n,v_n}(\,\cdot\,;f,P)\|_\mu &= \sup_{h>0}(\|\Delta_h W_{n,v_n}(\,\cdot\,;f,P)\|_C/\mu(h)) \\ &\le \|W_{n,v_n}(\,\cdot\,;f,P)\|_{\mu,1}+\|W_{n,v_n}(\,\cdot\,;f,P)\|_{\mu,2},\end{aligned}$$

where

$$\|W_{n,v_n}(\,\cdot\,;f,P)\|_{\mu,1} := \sup_{h\in A_n}(\|\Delta_h W_{n,v_n}(\,\cdot\,;f,P)\|_C/\mu(h)),$$

$$\|W_{n,v_n}(\,\cdot\,;f,P)\|_{\mu,2} := \sup_{h\in B_n}(\|\Delta_h W_{n,v_n}(\,\cdot\,;f,P)\|_C/\mu(h)),$$

$$A_n=\{h:h>1/(n-v_n+1)\},\qquad B_n=\{h:0<h\le 1/(n-v_n+1)\},$$

By (3) we have

$$\|\Delta_h W_{n,v_n}(\,\cdot\,;f,P)\|_C \le 2\|W_{n,v_n}(\,\cdot\,;f,P)\|_C,$$

which, by (15) and (6), gives

$$\begin{aligned}\|W_{n,v_n}(\,\cdot\,;f,P)\|_{\mu,1} &\le 2\Big(\mu\Big(\frac{1}{n-v_n+1}\Big)\Big)^{-1}\|W_{n,v_n}(\,\cdot\,;f,P)\|_C\\ &\le 2M_2(\lambda)\Big(\mu\Big(\frac{1}{n-v_n+1}\Big)\Big)^{-1}P_{n-v_n,n}E_{n-v_n}(f;C)\\ &\le 6M_2(\lambda)P_{n-v_n,n}q\Big(\frac{1}{n-v_n+1}\Big)\|f\|_\omega.\end{aligned}$$

Using (16), we get

$$\begin{aligned}\|W_{n,v_n}(\,\cdot\,;f,P)\|_{\mu,2} &\le (M_1(\lambda)+1)P_{n-v_n,n}\sup_{h\in B_n}\frac{\|\Delta_h f\|_C}{\mu(h)}\\ &\le (M_1(\lambda)+1)P_{n-v_n,n}q\Big(\frac{1}{n-v_n+1}\Big)\|f\|_\omega.\end{aligned}$$

Summing up, we obtain our result.

Applying (4), (15), (6) and Lemma 3, we obtain the following

LEMMA 4. *Under the assumptions of Lemma 3 we have*

$$\|W_{n,v_n}(\,\cdot\,;f,P)\|_{H^\mu} \le M_5(\lambda,\mu)P_{n-v_n,n}q\Big(\frac{1}{n-v_n+1}\Big)\|f\|_\omega$$

for all $n\in N$.

Arguing as in the proofs of Lemmas 2–4 and using (1)–(7) we obtain

LEMMA 5. *If* $f\in\widetilde{H}^\omega$, $\omega\in\Omega$, *then the function* $W_{n,v_n}(\,\cdot\,;f,P)$, $n\in N$, $0\le v_n\le n$, *belongs to* $\widetilde{H}^\omega$ *also.*

LEMMA 6. *Suppose that* $f\in\widetilde{H}^\omega$, $\omega\in\Omega$ *and* $\mu\in\Omega$ *is such that* $q(h)=\frac{\omega(h)}{\mu(h)}$ *is a non-decreasing function for* $h>0$. *Then, if* $P_{n-v_n,n}>0$ *and* $n-v_n\to\infty$, *we have*

$$\|W_{n,v_n}(\,\cdot\,;f,P)\|_{\widetilde{H}^\mu} = o\Big(P_{n,v_n,n}q\Big(\frac{1}{n-v_n+1}\Big)\Big) \quad \text{as } n\to\infty.$$

In particular, if $P_{n,2n}>0$, then

$$\|W_{2n,n}(\,\cdot\,;f,P)\|_{\widetilde{H}^\mu} = o\Big(P_{n,2n}q\Big(\frac{1}{n+1}\Big)\Big) \quad \text{as } n\to\infty.$$

3. General theorems

In this part we shall give four theorems on the strong means $L(r, \cdot\,; f, T)$ of Fourier series of $f \in C$. We observe that if T is a sequence having the properties 1°–4° and $f \in C$, then formula (10) can be written in the form

$$L(r,x;f,T) = \sum_{n=0}^{\infty} \sum_{k=\bar{n}}^{2\bar{n}} t_k(r)|S_k(x;f) - f(x)| = \sum_{n=0}^{\infty} 2^n W_{2\bar{n},\bar{n}}(r,x;f,T), \tag{17}$$

for $x \in R$ and $r \in Q$, where $\overline{n} = 2^n - 1$ and $W_{2\bar{n},\bar{n}}(r,x;f,T)$ is defined by (13) with $P \equiv T$.

From (4) and (17) we get

$$\Delta_h L(r,x;f,T) = \sum_{n=0}^{\infty} 2^n \Delta_h W_{2\bar{n},\bar{n}}(r,x;f,T),$$

which, by (16), (9) and 4°, gives that

$$\begin{aligned} \|\Delta_h L(r, \cdot\,; f, T)\|_C &\le \sum_{n=0}^{\infty} 2^n \|\Delta_h W_{2\bar{n},\bar{n}}(r, \cdot\,; f, T)\|_C \\ &\le M_6 \|\Delta_h f\|_C \sum_{n=0}^{\infty} 2^n T_{\bar{n},2\bar{n}}(r) \\ &\le M_7 \|\Delta_h f\|_C, \quad r \in Q,\ h > 0. \end{aligned} \tag{18}$$

From (18) we immediately obtain

THEOREM 1. *If $f \in H^\omega$, $\omega \in \Omega$, then*

$$\|L(r, \cdot\,; f, T)\|_\omega \le M_8 \|f\|_\omega, \quad r \in Q,$$

which proves that $L(r, \cdot\,; f, T)\| \in H^\omega$ for every fixed $r \in Q$.

THEOREM 2. *If $f \in \widetilde{H}^\omega$, $\omega \in \Omega$, then for every fixed $r \in Q$ the function $L(r, \cdot\,; f, T)$ belongs to $\widetilde{H}^\omega$ also.*

Applying Lemma 1 and Lemma 4, we shall prove two theorems.

THEOREM 3. *If $f \in H^\omega$, $\omega \in \Omega$, then*

$$\|L(r, \cdot\,; f, T)\|_C \le M_9 \|f\|_\omega \sum_{n=0}^{\infty} 2^n T_{\bar{n},2\bar{n}}(r) \omega(2^{-n})$$

for all $r \in Q$ ($\overline{n} = 2^n - 1$).

Proof. Using Lemma 1 and (6) to (17), we obtain

$$\begin{aligned}\|L(r,\cdot;f,T)\|_C &\le \sum_{n=0}^{\infty} 2^n \|W_{2\bar{n},\bar{n}}(r,\cdot;f,T)\|_C \\ &\le M_2(2)\sum_{n=0}^{\infty} 2^n T_{\bar{n},2\bar{n}}(r) E_{\bar{n}}(f;C) \\ &\le 3M_2(2)\|f\|_\omega \sum_{n=0}^{\infty} 2^n T_{\bar{n},2\bar{n}}(r)\omega(2^{-n}) \quad \text{for } r\in Q.\end{aligned}$$

THEOREM 4. *Suppose that $f\in H^\omega$, $\omega\in\Omega$, and $\mu\in\Omega$ is a function such that $q(h)=\frac{\omega(h)}{\mu(h)}$ is non-decreasing for $h>0$. Then*

$$\|L(r,\cdot;f,T)\|_{H^\mu} \le M_{10}(\mu)\|f\|_\omega \sum_{n=0}^{\infty} 2^n T_{\bar{n},2\bar{n}}(r) q(2^{-n}) \tag{19}$$

for all $r\in Q$ ($\bar{n}=2^n-1$).

Proof. By Theorem 1 and (5) we have $L(r,\cdot;f,T)\in H^\mu$ for every fixed $r\in Q$. From this and from (17) we get

$$\|L(r,.;f,T)\|_{H^\mu} \le \sum_{n=0}^{\infty} 2^n \|W_{2\bar{n},\bar{n}}(r,\cdot;f,T)\|_{H^\mu}$$

($r\in Q$). Now using Lemma 4, we obtain (19).

4. Applications

4.1. Riesz method. Consider the following strong Riesz means of Fourier series of $f\in C$:

$$R_n(x;f,\beta) := \frac{1}{(n+1)^\beta}\sum_{k=0}^{n}((k+1)^\beta - k^\beta)|S_k(x;f)-f(x)|,$$

$n\in N$, $x\in R$, $\beta>0$. From Theorem 1 and Theorem 4 we obtain the following

COROLLARY 1. *Suppose that $f\in H^\omega$, $\omega\in\Omega$, and μ is a function as in Theorem 4. Then, for every fixed $\beta>0$, we have*

$$\|R_n(\cdot;f,\beta)\|_{H^\mu} \le M_{11}(\mu,\beta)\|f\|_\omega (n+1)^{-\beta}\sum_{k=0}^{m} 2^{k\beta} q(2^{-k})$$

for all $n\in N$, where m is integer such that $2^m\le n+1<2^{m+1}$.

In particular, if $q(h) \le M_{12}h^\gamma$ $(0<\gamma<1)$ for $h>0$, then

$$\|R_n(\cdot;f,\beta)\|_{H^\mu} \le M_{13}^*\|f\|_\omega \begin{cases} (n+1)^{-\gamma} & \text{if } \gamma<\beta, \\ (n+1)^{-\gamma}\log(n+1) & \text{if } \gamma=\beta, \\ (n+1)^{-\beta} & \text{if } \gamma>\beta, \end{cases}$$

for all $n \in N$, where $M_{13}^* = M_{13}(\mu,\beta,\gamma)$.

Using Theorem 2, (20) and Lemma 6 to (21), we obtain

COROLLARY 2. *Suppose that ω and μ are two functions as in Theorem 4 and $q(h) \le M_{12}h^\gamma$ $(0<\gamma<1)$ for $h>0$. If $f \in \widetilde{H}^\omega$, then*

$$\|R_n(\cdot;f,\beta)\|_{\widetilde{H}^\mu} = \begin{cases} o((n+1)^{-\gamma}) & \text{if } 0<\gamma<\beta, \\ o((n+1)^{-\gamma}\log(n+1)) & \text{if } \gamma=\beta, \\ O((n+1)^{-\beta}) & \text{if } \gamma>\beta, \end{cases}$$

as $n\to\infty$.

4.2. **Abel method.** Consider the strong Abel means of Fourier series of $f \in C$:

$$A_p(r,x;f) := (1-r)^{1+p}\sum_{k=0}^{\infty}\binom{k+p}{k}r^k|S_k(x;f)-f(x)|, \tag{22}$$

$x \in R$, $r \in (0,1)$, $r \to 1_-$ and $p \in N$.

It is easily verified that the sequence T defining the strong Abel means (22) satisfies the conditions 1°–4° and

$$\begin{aligned} T_{\bar{n},2\bar{n}}(r) &= \max_{\bar{n}\le k\le 2\bar{n}}(1-r)^{1+p}\binom{k+p}{k}r^k \\ &\le \frac{(p+1)^p}{p!}(1-r)^{1+p}(2\overline{n})^p r^{\bar{n}} \end{aligned} \tag{23}$$

for $n \in N$, $\overline{n} = 2^n-1$, $p \in N$ and $r \in (0,1)$.

Using Theorem 1, Theorem 4 and (23) to (22), we obtain

COROLLARY 3. *Under the assumptions of Theorem 4 we have*

$$\|A_p(r,\cdot;f)\|_{H^\mu} \le M_{14}^*\|f\|_\omega(1-r)^{1+p}\sum_{k=0}^{S}2^k(p+1)q(2^{-k}))$$

for all $r \in (0,1)$, where $S = [\log_2\frac{1}{1-r}]$ (i.e. S is the integral part of $\log_2\frac{1}{1-r}$) and $M_{14}^* = M_{14}(\mu,p)$.

In particular if $g(h) \le M_{12}h^\gamma$ for $h>0$ with $0<\gamma<1$, then

$$\|A_p(r,\cdot;f)\|_{H^\mu} \le M_{15}(\mu,p,\gamma)\|f\|_\omega(1-r)^\gamma$$

for all $r \in (0,1)$.

From Theorem 2, (5), (20)–(22) and Lemma 6 follows

COROLLARY 4. *Let ω and μ be a functions as in Corollary 2. If $f \in H^{\omega}$ and $p \in N$, then*

$$\|A_p(r,\cdot\,;f)\|_{\tilde{H}^{\mu}} = o((1-r)^{\gamma}) \quad \text{as } r \to 1_-.$$

4.3. **Riemann method.** Now we shall consider the strong Riemann means of Fourier series (8) of $f \in C$ defined by the formula

$$R(r,x;f) = \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{\sin^2 kr}{k^2 r}|S_k(x;f) - f(x)|,$$

$x \in R$, $0 < r < 1$, $r \to 0_+$ ([7], [8]).

The sequence T with $t_k(r) = \frac{\sin^2 kr}{k^2 r}$, $0 < r < 1$, $r \to 0_+$, satisfies the conditions 1°–4°, e.g. applying the inequality

$$\frac{\sin^2 kr}{k^2 r} \le \begin{cases} r & \text{if } 0 < kr \le 1, \\ \frac{1}{k^2 r} & \text{if } kr > 1, \end{cases}$$

and writing $S = [\log_2(\frac{1}{r} + 2)]$, $0 < r < 1$, we get

$$\begin{aligned}\sum_{n=1}^{\infty} 2^n T_{\bar{n},2\bar{n}}(r) &\le \Big(\sum_{n=1}^{S-1} + \sum_{n=S}^{\infty}\Big) 2^n T_{\bar{n},2\bar{n}}(r) \\ &\le r\sum_{n=1}^{S-1} 2^n + \frac{1}{r}\sum_{n=S}^{\infty} 2^n(2^n-1)^{-2} \le 35,\end{aligned}$$

which proves that the condition 4° is fulfilled.

From Theorems 1–4 and Lemma 1–6 we obtain the following estimations:

COROLLARY 5. *If $f \in H^{\omega}$, $\omega \in \Omega$, then*

$$\|R(r,\cdot\,;f)\|_C \le M_{16}\|f\|_{\omega} r \sum_{n=1}^{S} 2^n \omega(2^{-n})$$

for all $r \in (0,1)$, where $S = [\log_2(\frac{1}{r} + 2)]$.

COROLLARY 6. *If the assumptions of Theorem 4 are satisfied, then*

$$\|R(r,\cdot\,;f)\|_{H^{\mu}} \le M_{17}(\mu)\|f\|_{\omega} r \sum_{n=1}^{S} 2^n q(2^{-n})$$

for all $r \in (0,1)$, where $S = [\log_2(\frac{1}{r} + 2)]$.

In particular, if $g(h) \le M_{12} h^{\gamma}$, $0 < \gamma < 1$, for $h > 0$, then

$$\|R(r,\cdot\,;f)\| H^{\mu} \le M_{18}(\mu,\gamma)\|f\|_{\omega} r^{\gamma}$$

for all $r \in (0,1)$.

COROLLARY 10. *If the assumptions of Corollary 2 are satisfied, then for* $f \in \widetilde{H}^{\mu}$ *we have*

$$\|R(r,\cdot;f)\|_{\widetilde{H}^{\mu}} = o(r^{\gamma}) \qquad \text{as } r \to 0_{+}.$$

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