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ON GAUGE–NATURAL OPERATORS OF CURVATURE TYPE ON PAIRS OF CONNECTIONS

Recently, Kolář has determined all gauge–natural operators for any Lie group G of curvature type transforming every principal connection into modified curvature operator, [3]. In the case the structure group is the general linear group $GL(m, \mathbf{R})$ in an arbitrary dimension m he has obtained two parameter family of all $GL(m, \mathbf{R})$–natural operators of curvature type consisting of a curvature of connection and a contracted curvature of connection and generated by linear adjoint invariant maps of the Lie algebra $\mathbf{g}l(m, \mathbf{R})$ into itself of the form id and $A \mapsto \operatorname{tr} A \cdot \mathrm{id}$, [4].

Using a general method by Kolář, [3], [4], [5], we determine all gauge–natural operators for any Lie group G defined on the bundle $\mathbf{Q} \oplus \mathbf{Q}$ of pairs of principal connections with values in $L \otimes \otimes^2 T^* B$. We deduce that all such operators form a system generated by two modified curvatures of both principal connections and by values on the difference of these connections of some map induced by bilinear adjoint invariant map of the Lie algebra $\mathbf{g}$ of G. In the case the structure group is the general linear group $GL(m, \mathbf{R})$ in an arbitrary dimension m we obtain 10–parameter family of all $GL(m, \mathbf{R})$–natural operators of curvature type consisting 4–parameter system generated by curvatures and contracted curvatures both connections and 6–parameter system generated by values on the difference of these connections of some bilinear and adjoint invariant maps of the Lie algebra $\mathbf{g}l(m, \mathbf{R})$.

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1. Let Mf_n be a category of n–dimensional manifolds and their local diffeomorphisms. Let FM be a category of fibred manifolds and denote by B the base functor. Fix a Lie group G and define a category $P_n(G)$, whose objects are principal G–bundles and whose morphisms are the morphisms of principal G–bundles $f : P \to \overline{P}$ with the base map $Bf : BP \to B\overline{P}$ in Mf_n.

DEFINITION 1. A gauge–natural bundle over n–manifolds is a functor $F : P_n(G) \to FM$ such that

1. every principal bundle $p : P \to BP$ in $P_n(G)$ is transformed by F into fibred manifold $q : FP \to BP$ over the same base BP
2. every morphism $f : P \to \overline{P}$ in $P_n(G)$ is transformed by F into morphism $Ff : FP \to F\overline{P}$ over Bf in FM
3. for every open subset $U \subset BP$, the inclusion $\imath : p^{-1}(U) \to P$ is transformed into the inclusion $F\imath : q^{-1}(U) \to FP$.

Let F and E be two G–natural bundle over n–manifolds.

DEFINITION 2. A gauge–natural operator $A : F \to E$ is a system of operators $A_P : C^\infty FP \to C^\infty EP$ for every object P in $P_n(G)$ transforming every section $s \in C^\infty FP$ into section $A_P s \in C^\infty EP$ such that

1. $A_P(Ff \circ s \circ Bf^{-1}) = Ef \circ A_P s \circ Bf^{-1}$ for every isomorphism $f : P \to \overline{P}$ in $P_n(G)$
2. $A_{P^{-1}(U)}(s|U) = (A_P s)|U$ for every open subset $U \subset BP$
3. A_P transforms every smoothly parametrized family of sections into smoothly parametrized family of sections.

A gauge–natural bundle F over n-manifolds is said to be of order r, if for any two morphisms $f, h : P \to \overline{P}$ in $P_n(G)$ the condition $j^r_y f = j^r_y h$ at some point $y \in P_x$ of the fibre of P over $x \in BP$ implies $Ff|_{F_x P} = Fh|_{F_x P}$.

Let $W^r P$ be a space of all r–jets $j^r_{(0,e)}\varphi$, where $\varphi : \mathbf{R}^n \times G \to P$ is a morphism in $P_n(G)$. The space $W^r P$ is a principal bundle over BP with structure group $W^r_n G$, which is the group of all r–jets $j^r_{(0,e)}\psi$ of morphisms $\psi : \mathbf{R}^n \times G \to \mathbf{R}^n \times G$ in $P_n(G)$ satisfying $B\psi(0,e) = 0$. Every morphism $f : P \to \overline{P}$ in $P_n(G)$ is extended into a principal bundle morphism $W^r f : W^r P \to W^r \overline{P}$ defined by the jet composition $W^r f(j^r_{(0,e)}\varphi) = j^r_{(0,e)}(f \circ \varphi)$. Every smooth left action of $W^r_n G$ on a manifold S determines r–th order G–natural bundle over n–manifolds as a functor transforming every object P in $P_n(G)$ into the fibre bundle associated to $W^r P$ with standard fibre S and every morphism f in $P_n(G)$ into $(W^r f, \mathrm{id}_S)$.

Every r–th order gauge–natural bundle is a fibre bundle associated to bundle W^r. The k–th jet prolongation $J^k F$ of a gauge–natural bundle F of order r is a gauge–natural bundle of order $(k + r)$.

According to a general theory, [1], [2], [3], [4], [5], there is a canonical bijection between the k–th order G–natural operators $A : F \to E$ and the

$W_n^s G$-equivariant maps of standard fibres $A : J_0^k F(\mathbf{R}^n \times G) \to E_0(\mathbf{R}^n \times G)$, where s is maximum of the orders $J^k F$ and E.

2. The connection bundle $\mathbf{Q}P \to BP$ of P can be defined as the factor space $\mathbf{Q}P = J^1 P/G$ over BP. Clearly, the connection bundle $\mathbf{Q} : \mathcal{P}_n(G) \to \mathcal{FM}$, $\mathbf{Q} : P \mapsto \mathbf{Q}P$, is a gauge–natural bundle of the order 1.

Given a connection Γ on $P = \mathbf{R}^n \times G$, its value $\Gamma(0,e)$ is a 1-jet $j_0^1\gamma$ of a section $\gamma : \mathbf{R}^n \to \mathbf{R}^n \times G$, which is identified with a map $\gamma : \mathbf{R}^n \to G$, where $\gamma(0) = e$ is a unit of G.

The standard fibre of $\mathbf{Q}$ is of the form $\mathbf{Q}_0(\mathbf{R}^n \times G) = J_0^1(\mathbf{R}^n, G)_e = \mathbf{g} \otimes \mathbf{R}^{n*}$. Fix a basis e_p of the Lie algebra $\mathbf{g}$ of G, we have the coordinate expression of an element Γ of $\mathbf{g} \otimes \mathbf{R}^{n*}$ in the form $\Gamma = \Gamma_i^p e_p \, dx^i$.

Consider an isomorphism $\Phi : \mathbf{R}^n \times G \to \mathbf{R}^n \times G$ in $\mathcal{P}_n(G)$ of the form

$$\overline{x} = f(x), \qquad \overline{y} = \varphi(x) \cdot y \tag{2.1}$$

where $f : \mathbf{R}^n \to \mathbf{R}^n$ is a local diffeomorphism satisfying $f(0) = 0$ and $\varphi : \mathbf{R}^n \to G$ is a smooth map and the dot denotes the multiplication in G.

Then 2–jet $j_0^2\Phi \in W_n^2 G$ has coordinates

$$\begin{aligned} a &= \varphi(0) \in G \\ (a_i^p, a_{ij}^p) &= j_0^2(\varphi(x) \cdot a^{-1}) \in \mathbf{g} \otimes \mathbf{R}^{n*} \times \mathbf{g} \otimes S^2\mathbf{R}^{n*} \\ (a_j^i, a_{jk}^i) &= j_0^2 f \in G_n^2. \end{aligned} \tag{2.2}$$

Isomorphism Φ transforms a section γ generating $\Gamma(0,e)$ into a section $\varphi(f^{-1}(x) \cdot \gamma(f^{-1}(x))$ generating image $\Phi(\Gamma(0,e))$ as the 1–jet of the section

$$\Phi(\Gamma(0,e)) = j_0^1[\varphi(f^{-1}(x)) \cdot \gamma(f^{-1}(x)) \cdot a^{-1}]. \tag{2.3}$$

Let $A_q^p(a)$ be a coordinate expression of the adjoint representation of G in Lie algebra $\mathbf{g}$. Let tilda denote the inverse matrix. Then (2.3) gives the action of $W_n^1 G$ on the standard fibre $\mathbf{Q}_0 = \mathbf{g} \otimes \mathbf{R}^{n*}$ in the form

$$\overline{\Gamma}_i^p = A_q^p(a)(\Gamma_j^q + a_j^q)\widetilde{a}_i^j. \tag{2.4}$$

Let $\Gamma_{ij}^p = \frac{\partial \Gamma_i^p}{\partial x^j}$ be the induced coordinates on $J_0^1\mathbf{Q} = \mathbf{g} \otimes \mathbf{R}^{n*} \times \mathbf{g} \otimes S^2\mathbf{R}^{n*}$. Using a general prolongation procedure, [3], we deduce from (2.4) the action of $W_n^2 G$ on $J_0^1\mathbf{Q}$ in the form (2.4) and

$$\begin{aligned} \overline{\Gamma}_{ij}^p = {} & A_q^p(a)\Gamma_{kl}^q \widetilde{a}_i^k \widetilde{a}_j^l + A_q^p(a) a_{kl}^q \widetilde{a}_i^k \widetilde{a}_j^l + \\ & + B_{qr}^p(a)\Gamma_k^q a_l^r \widetilde{a}_i^k \widetilde{a}_j^l + E_{qr}^p(a) a_k^q a_l^r \widetilde{a}_i^k \widetilde{a}_j^l + \\ & + A_q^p(a)(\Gamma_k^q + a_k^q)\widetilde{a}_{ij}^k \ , \end{aligned} \tag{2.5}$$

where B_{qr}^p and E_{qr}^p are some functions on G.

The curvature of a principal connection can be considered as a section $C_P\Gamma : BP \to LP \otimes \Lambda^2 T^* BP$, where LP is an accociated bundle to P with standard fibre $\mathbf{g}$ with respect to adjoint action of Lie group G. The curvature operator $C : \mathbf{Q} \to L \otimes \Lambda^2 T^*B$ is a gauge–natural operator of the first order because of geometric definition of the curvature.

In order to obtain a coordinate expression of curvature, we write the equation of a connection Γ on $P = \mathbf{R}^n \times G$ in the form

$$\omega^p = \Gamma^p_i(x)\, dx^i, \tag{2.6}$$

where ω^p are the Maurer–Cartan forms on G with respect to the basis e_p of Lie algebra $\mathbf{g}$. Then, components of a connection form are

$$\eta^p = \omega^p - \Gamma^p_i(x)\, dx^i. \tag{2.7}$$

Using a structure equation of the connection Γ in the form

$$d\eta^p = C^p_{qr}\eta^q \wedge \eta^r + R^p_{ij} dx^i \wedge dx^j \tag{2.8}$$

and Maurer–Cartan equation

$$d\omega^p = C^p_{qr}\omega^q \wedge \omega^r, \tag{2.9}$$

we obtain a coordinate expression of curvature in the form

$$R^p_{ij} = \Gamma^p_{ij} - \Gamma^p_{ji} + C^p_{qr}\Gamma^q_i \Gamma^r_j, \tag{2.10}$$

where C^p_{qr} are a structure constant of G.

Let $Z \subset \mathrm{Lin}(\mathbf{g}, \mathbf{g})$ be the subspace of all linear maps commuting with the adjoint action of G. Since every $z \in Z$ is an equivariant linear map of the standard fibre $\mathbf{g}$ of the vector bundle LP, it induces a morphism $z_P : LP \to LP$. Hence, we can construct a modified curvature operator of the curvature operator C_P in the form $C(z)_P = (z_P \otimes \Lambda^2 T^* \mathrm{id}_{BP}) \circ C_P$.

We shall need some new essential relations concerning the function B^p_{qr} on G appearing in (2.5), which we will obtain in the detailed proof of the following theorem developed by I. Kolář in [3].

THEOREM 1. *All gauge–natural operators $\mathbf{Q} \to L \otimes \otimes^2 T^*B$ are the modified curvature operators*

$$C(z) = (z \otimes \Lambda^2 T^* \mathrm{id}_B) \circ C. \tag{2.11}$$

for all $z \in Z$ of the curvature operator C.

Proof. I. The first order gauge–natural operators $A : \mathbf{Q} \to L \otimes \otimes^2 T^*B$ are in bijection with $W^2_n G$ equivariant maps of standard fibres $A : J^1_0 \mathbf{Q} \to \mathbf{g} \otimes \otimes^2 \mathbf{R}^{n*}$. The group $W^2_n G$ acts on the standard fibre $J^1_0 \mathbf{Q}$ by formulas (2.4) and (2.5). On $\mathbf{g} \otimes \otimes^2 \mathbf{R}^{n*}$ we have the canonical coordinates y^p_{ij} and we have the action of $W^2_n G$ in the form

$$\overline{y}^p_{ij} = A^p_q(a) y^q_{kl} \widetilde{a}^k_i \widetilde{a}^l_j. \tag{2.12}$$

The equivariancy of A with respect to homotheties in G^1_n : $a = e$, $\widetilde{a}^i_j = k\delta^i_j$, $a^i_{jk} = 0$, $a^p_i = 0$, $a^p_{ij} = 0$, gives a homogeneity condition

$$k^2 f^p_{ij}(\Gamma^p_i, \Gamma^p_{ij}) = f^p_{ij}(k \cdot \Gamma^p_i, k^2 \Gamma^p_{ij}). \tag{2.13}$$

By the homogeneous function theorem, [5], we deduce that f^p_{ij} are linear in Γ^p_{ij} and quadratic in Γ^p_i. Using invariant tensor theorem for G^1_n, [5], we obtain f^p_{ij} in the form

$$f^p_{ij} = b^p_q \Gamma^q_{ij} + d^p_q \Gamma^q_{ji} + k^p_{qr} \Gamma^q_i \Gamma^r_j \tag{2.14}$$

with real coefficients.

Considering equivariancy of f^p_{ij} in the form (2.14) with respect to the subgroup in $W^2_n G$: $a = e$, $\widetilde{a}^i_j = \delta^i_j$ and a^i_{jk}, a^p_i, a^p_{ij} are arbitrary, we get conditions

$$\begin{gathered} b^p_q + d^p_q = 0 \\ b^p_q B^q_{rs}(e) + k^p_{rs} = 0, \qquad d^p_q B^q_{rs}(e) + k^p_{sr} = 0. \end{gathered} \tag{2.15}$$

From this, we obtain the following relations

$$d^p_q = -b^p_q, \qquad k^p_{sr} = -k^p_{rs}, \qquad B^q_{sr}(e) = -B^q_{rs}(e), \qquad k^p_{rs} = -b^p_q B^q_{rs}(e). \tag{2.16}$$

If we put $b^p_q = \delta^p_q$ into f^p_{ij} in (2.14) and if we use uniqueness of the curvature operator, we can put

$$B^q_{rs}(e) = -C^q_{rs}, \tag{2.17}$$

where C^q_{rs} are the constant structure of G.

The equivariancy of f^p_{ij} in (2,14) with respect to the canonical inclusion $\imath(G)$ into $W^2_n G$: $a \in G$ is arbitrary and $\widetilde{a}^i_j = \delta^i_j$, $a^i_{jk} = 0$, $a^p_i = 0$, $a^p_{ij} = 0$, gives the relation

$$A^p_q(a) b^q_r R^r_{ij} = b^p_q A^q_r(a) R^r_{ij}. \tag{2.18}$$

This means that b^p_q commutes with adjoint action.

II. The r–th order gauge–natural operators $\mathbf{Q} \to L \otimes \otimes^2 T^* B$ correspond to $W^{r+1}_n G$ equivariant maps of standard fibres $J^r_0 \mathbf{Q} \to \mathbf{g} \otimes \otimes^2 \mathbf{R}^{n*}$.

Let $\Gamma^p_{i\alpha}$ be the induced coordinates on $J^r_0 \mathbf{Q}$, where α is a multiindex of range n with $|\alpha| \le r$.

Equivariancy $f^p_{ij}(\Gamma^q_{k\alpha})$ with respect to homotheties in G^1_n gives a homogeneity condition

$$k^2 f^p_{ij}(\Gamma^q_{k\alpha}) = f^p_{ij}(k^{1+|\alpha|} \Gamma^q_{k\alpha}). \tag{2.19}$$

By homogeneous function theorem f^p_{ij} is independent on $\Gamma^q_{k\alpha}$ for $|\alpha| \ge 2$ and is linear in Γ^p_{ij} and quadratic in Γ^p_i. Hence, the r–th order operators are reduced to the case I for every $r \ge 2$. Moreover, every gauge–natural operator defined on the connection bundle has a finite order. This proves Theorem 1.

3. Consider a pair of principal connections Γ and Δ on the principal bundle $P = \mathbf{R}^n \times G$ as a section of the bundle $\mathbf{Q} \oplus \mathbf{Q}$. We are going to determine all gauge–natural operators $\mathbf{Q} \oplus \mathbf{Q} \to L \otimes \otimes^2 T^*B$.

Let $Y \subset \operatorname{Billin}(\mathbf{g} \times \mathbf{g}, \mathbf{g})$ be the subspace of all bilinear and adjoint equivariant maps with respect to adjoint action of G.

For every $y \in Y$ we define a bilinear map $\overline{y} : \mathbf{g} \otimes \mathbf{R}^{n*} \times \mathbf{g} \otimes \mathbf{R}^{n*} \to \mathbf{g} \otimes \otimes^2 \mathbf{R}^{n*}$ by formula

$$\overline{y}(A_1 \otimes X_1, A_2 \otimes X_2) = y(A_1, A_2) \otimes X_1 \otimes X_2. \tag{3.1}$$

Now, we prove the main

THEOREM 2. *All gauge–natural operators $\mathbf{Q} \oplus \mathbf{Q} \to L \otimes \otimes^2 T^*B$ form the following system*

$$(\Gamma, \Delta) \mapsto C(z)(\Gamma) + C(w)(\Delta) + \overline{y}(D, D), \tag{3.2}$$

where $C(z)(\Gamma)$ and $C(w)(\Delta)$ are modified curvature operators for all $z, w \in Z$ and $\overline{y}(D, D)$ are the values on the difference of connections $D = \Gamma - \Delta$ for all $y \in Y$.

Proof. I. The first order gauge–natural operators $\mathbf{Q} \oplus \mathbf{Q} \to L \otimes \otimes^2 T^*B$ correspond bijectively to the $W_n^2 G$ equivariant maps of the standard fibres $J_0^1(\mathbf{Q} \oplus \mathbf{Q}) \to \mathbf{g} \otimes \otimes^2 \mathbf{R}^{n*}$. The group $W_n^2 G$ acts on the standard fibre $J_0^1(\mathbf{Q} \oplus \mathbf{Q})$ by formulas (2.4) and (2.5) and by formulas

$$\begin{aligned} \overline{\Delta}_i^p &= A_q^p(a)(\Delta_j^q + a_j^q)\widetilde{a}_i^j \\ \overline{\Delta}_{ij}^p &= A_q^p(a)\Delta_{kl}^q \widetilde{a}_i^k \widetilde{a}_j^l + A_q^p(a) a_{kl}^q \widetilde{a}_i^k \widetilde{a}_j^l + \\ &\quad + B_{qr}^p(a)\Delta_k^q a_l^r \widetilde{a}_i^k \widetilde{a}_j^l + E_{qr}^p(a) a_k^q a_l^r \widetilde{a}_i^k \widetilde{a}_j^l + \\ &\quad + A_q^p(a)(\Delta_k^q + a_k^q)\widetilde{a}_{ij}^k. \end{aligned} \tag{3.3}$$

The action of $W_n^2 G$ on $\mathbf{g} \otimes \otimes^2 \mathbf{R}^{n*}$ is of the form (2.12).

Any map of standard fibres $A : J_0^1(\mathbf{Q} \oplus \mathbf{Q}) \to \mathbf{g} \otimes \otimes^2 \mathbf{R}^{n*}$ in coordinates is of the form

$$y_{ij}^p = f_{ij}^p(\Gamma_k^q, \Gamma_{kl}^q, \Delta_k^q, \Delta_{kl}^q). \tag{3.4}$$

The equivariancy of f_{ij}^p with respect to homotheties in G_n^1 : $a = e$, $\widetilde{a}_j^i = k\delta_j^i$, $a_{jk}^i = 0$, $a_i^p = 0$, $a_{ij}^p = 0$, gives a homogeneity condition

$$k^2 f_{ij}^p(\Gamma_i^p, \Gamma_{ij}^p, \Delta_i^p, \Delta_{ij}^p) = f_{ij}^p(k\Gamma_i^p, k^2\Gamma_{ij}^p, k\Delta_i^p, k^2\Delta_{ij}^p). \tag{3.5}$$

By the homogeneous function theorem, [5], we deduce that f_{ij}^p are linear in Γ_{ij}^p, Δ_{ij}^p and bilinear in Γ_i^p, Δ_i^p and quadratic in Γ_i^p, Δ_i^p. Using invariant tensor theorem for G^1_n, we obtain f^p_{ij} in the form

$$f^p_{ij} = b^p_q \Gamma^q_{ij} + c^p_q \Gamma^q_{ji} + d^p_q \Delta^q_{ij} + e^p_q \Delta^q_{ji} + k^p_{qr} \Gamma^q_i \Delta^r_j + \\ + l^p_{qr} \Delta^q_i \Gamma^r_j + m^p_{qr} \Gamma^q_i \Gamma^r_j + n^p_{qr} \Delta^q_i \Delta^r_j \tag{3.6}$$

with real coefficients.

Considering equivariancy of f^p_{ij} with respect to the subgroup in $W^2_n G$: $a = e$, $a^i_j = \delta^i_j$ and a^i_{jk}, a^p_i, a^p_{ij} are arbitrary, we get the following relations

$$\begin{aligned} c^p_q &= -b^p_q, \quad e^p_q = -d^p_q \\ k^p_{rs} &= -m^p_{rs} - b^p_q B^q_{rs}(e) \\ l^p_{rs} &= -m^p_{rs} + b^p_q B^q_{sr}(e) \\ k^p_{rs} &= -n^p_{rs} + d^p_q B^q_{sr}(e) \\ l^p_{rs} &= -n^p_{rs} - d^p_q B^q_{rs}(e). \end{aligned} \tag{3.7}$$

Using the relation (2.16), $B^q_{rs}(e) = -B^q_{sr}(e)$, we get $k^p_{rs} = l^p_{rs}$. If we put

$$h^p_{rs} = -k^p_{rs}, \quad h^p_{rs} = -l^p_{rs} \tag{3.8}$$

and take into account the relation (2.17) $B^q_{rs}(e) = -C^q_{rs}$, we obtain following relations

$$\begin{aligned} m^p_{rs} &= h^p_{rs} + b^p_q C^q_{rs} \\ n^p_{rs} &= h^p_{rs} + d^p_q C^q_{rs}. \end{aligned} \tag{3.9}$$

Finally, f^p_{ij} is of the form

$$f^p_{ij} = b^p_q \overset{\Gamma}{R}{}^q_{ij} + d^p_q \overset{\Delta}{R}{}^q_{ij} + h^p_{qr} D^q_i \cdot D^r_j \tag{3.10}$$

where we denote

$$\begin{aligned} \overset{\Gamma}{R}{}^q_{ij} &= \Gamma^q_{ij} - \Gamma^q_{ji} + C^q_{rs} \Gamma^r_i \Gamma^s_j, \\ \overset{\Delta}{R}{}^q_{ij} &= \Delta^q_{ij} - \Delta^q_{ji} + C^q_{rs} \Delta^r_i \Delta^s_j, \\ D^q_i &= \Gamma^q_i - \Delta^q_i. \end{aligned} \tag{3.11}$$

Considering equivariancy of the map (3.10), f^p_{ij}, with respect to the canonical inclusion $\imath(G)$ into $W^2_n G$: $a \in G$ is arbitrary and $a^i_j = \delta^i_j$, $a^i_{jk} = 0$, $a^p_i = 0$, $a^p_{ij} = 0$, we obtain the relation:

$$A^p_q(a) b^q_r \overset{\Gamma}{R}{}^r_{ij} + A^p_q(a) d^q_r \overset{\Delta}{R}{}^r_{ij} + A^p_q(a) h^q_{rs} D^r_i D^s_j = \\ = b^p_q A^q_r(a) \overset{\Gamma}{R}{}^r_{ij} + d^p_q A^q_r(a) \overset{\Delta}{R}{}^r_{ij} + h^p_{qr} A^q_s(a) A^r_t(a) D^s_i D^t_j. \tag{3.12}$$

This means that b^p_q and d^p_q defines linear maps of $\mathbf{g}$ into itself commuting with adjoint action $A^p_q(a)$ and h^p_{qr} define a bilinear map $\mathbf{g}\otimes\mathbf{R}^{n*}\times\mathbf{g}\otimes\mathbf{R}^{n*}\to\mathbf{g}\otimes\otimes^2\mathbf{R}^{n*}$ determined by some linear adjoint equivariant map $\mathbf{g}\times\mathbf{g}\to\mathbf{g}$.

II. The r–th order gauge–natural operators $\mathbf{Q}\oplus\mathbf{Q}\to L\otimes\otimes^2T^*B$ correspond bijectively to the W^{r+1}_nG equivariant maps of the standard fibres $J^r_0(\mathbf{Q}\oplus\mathbf{Q})\to\mathbf{g}\otimes\otimes^2\mathbf{R}^{n*}$. Let $(\Gamma^p_{i\alpha},\Delta^p_{i\alpha})$ be induced coordinates on $J^r_0(\mathbf{Q}\oplus\mathbf{Q})$, where α is a multiindex of range n with $|\alpha|\le r$.

Equivariancy of $y^p_{ij}=f^p_{ij}(\Gamma^p_{i\alpha},\Delta^p_{i\alpha})$ with respect to homotheties in G^1_n gives a homogeneity condition

$$k^2 f^p_{ij}(\Gamma^p_{i\alpha},\Delta^p_{i\alpha})=f^p_{ij}(k^{1+|\alpha|}\Gamma^p_{i\alpha},k^{1+|\alpha|}\Delta^p_{i\alpha}). \tag{3.13}$$

By homogeneous function theorem f^p_{ij} is independent on $\Gamma^p_{i\alpha}$, $\Delta^p_{i\alpha}$ for $|\alpha|\ge 2$ and is linear in Γ^p_{ij}, Δ^p_{ij} and is bilinear in Γ^p_i, Δ^p_i and is quadratic in Γ^p_i, Δ^p_i. Hence, the r–th order operators are reduced to the case I. Moreover, since every gauge–natural operator $\mathbf{Q}\to L\otimes\otimes^2T^*B$ defined on $\mathbf{Q}$ has a finite order so gauge–natural operator $\mathbf{Q}\oplus\mathbf{Q}\to L\otimes\otimes^2T^*B$ defined on $\mathbf{Q}\oplus\mathbf{Q}$ has a finite order, too. This proves our theorem.

4. Consider a principal bundle $P=\mathbf{R}^n\times G^1_m$ with the general linear group $G^1_m=GL(m,\mathbf{R})$ in an arbitrary dimension m as a structure group. Let x^i,x^p_q be the canonical coordinates on the product bundle $\mathbf{R}^n\times G^1_m$, where $i,j,\ldots=1,\ldots,n$ and $p,q,\ldots=1,\ldots,m$.

The equation of connection Γ on $\mathbf{R}^n\times G^1_m$ are

$$dx^p_q=\Gamma^p_{ri}(x)x^r_q\,dx^i, \tag{4.1}$$

where Γ^p_{ri} are smooth functions defined on $\mathbf{R}^n$.

The curvature of the connection Γ on $P=\mathbf{R}^n\times G^1_m$ is a section $C\Gamma:\mathbf{R}^n\to L(\mathbf{R}^n\times G^1_m)\otimes\Lambda^2T^*\mathbf{R}^n$ of the form in local coordinates:

$$C\Gamma=(\Gamma^p_{qij}+\Gamma^p_{ri}\Gamma^r_{qj})\frac{\partial}{\partial x^p_q}\otimes dx^i\wedge dx^j. \tag{4.2}$$

In the case of the structure group G^1_m, all linear adjoint equivariant maps of $\mathbf{g}l(m,\mathbf{R})=\mathbf{R}^m\otimes\mathbf{R}^{m*}$ into itself form the 2–parameter family generated by id and $A\mapsto(\operatorname{tr}A)\,\mathrm{id}$. This gives the 2–parameter family of G^1_m–natural operators $\mathbf{Q}\to L\otimes\otimes^2T^*B$ generated by the curvature operator C and by a contracted curvature operator $\overline{C}=(\tau\otimes\Lambda^2\,\mathrm{id})\circ C$, where τ is a linear map of $L(\mathbf{R}^n\times G^1_m)$ into itself defined by the linear adjoint equivariant map of the standard fibre $\mathbf{g}$ into itself of the form $A\mapsto(\operatorname{tr}A)\,\mathrm{id}$.

In a local coordinates on $P=\mathbf{R}^n\times G^1_m$ the contracted curvature of the connection Γ is of the form

$$\overline{C}\Gamma=\delta^p_q(\Gamma^r_{rij})\frac{\partial}{\partial x^p_q}\otimes dx^i\wedge dx^j. \tag{4.3}$$

We will use the following

LEMMA 3. *All bilinear and adjoint invariant maps* $gl(m,\mathbf{R})\times gl(m,\mathbf{R}) \to gl(m,\mathbf{R})$ *form the 6-parameter family*

$$w^p_q = k_1 y^p_s z^s_q + k_2 y^s_q z^p_s + k_3 y^p_q z^s_s + k_4 y^s_s z^p_q + + k_5 \delta^p_q y^r_r z^s_s + k_6 \delta^p_q y^r_s z^s_r \tag{4.4}$$

for any real parameters $k_1,\ldots,k_6$.

Proof. Using the invariant tensor theorem for G^1_m, [5], we obtain any bilinear and adjoint invariant map $\mathbf{R}^m\otimes\mathbf{R}^{m*}\times\mathbf{R}^m\otimes\mathbf{R}^{m*}\to\mathbf{R}^m\otimes\mathbf{R}^{m*}$ in the form

$$\begin{aligned} w^p_q = {} & k_1\delta^t_q\delta^p_s\delta^r_u y^s_r z^u_t + k_2\delta^r_q\delta^t_s\delta^p_u y^s_r z^u_t + \\ & + k_3\delta^r_q\delta^p_s\delta^t_u y^s_r z^u_t + k_4\delta^t_q\delta^r_s\delta^p_u y^s_r z^u_t + \\ & + k_5\delta^p_q\delta^t_s\delta^r_u y^s_r z^u_t + k_6\delta^p_q\delta^r_s\delta^t_u y^s_r z^u_t \end{aligned} \tag{4.5}$$

with any real parameter $k_1,\ldots,k_6$.

Consider a pair of connections Γ and Δ on $P=\mathbf{R}^n\times G^1_m$ as a section $(\Gamma,\Delta):\mathbf{R}^n\to\mathbf{Q}(\mathbf{R}^n\times G^1_m)\oplus\mathbf{Q}(\mathbf{R}^n\times G^1_m)$ with equations (4.1) and $dx^p_q=\Delta^p_{qi}(x)dx^i$.

We denote by

$$D^p_{qi}=\Gamma^p_{qi}-\Delta^p_{qi} \tag{4.6}$$

coordinates of difference of connections Γ and Δ and by

$$\overset{\Gamma}{R}{}^p_{qij}=\Gamma^p_{qij}-\Gamma^p_{qji}+\Gamma^p_{ri}\Gamma^r_{qj}-\Gamma^p_{rj}\Gamma^r_{qi}, \tag{4.7}$$

$$\overset{\Gamma}{R}{}^r_{rij}=\Gamma^r_{rij}-\Gamma^r_{rji} \tag{4.8}$$

coordinates of the curvature $C\Gamma$ and the contracted curvature $\overline{C}\Gamma$ of the connection Γ, respectively.

Immediately from Theorem 2 and Lemma 3 we obtain the following

COROLLARY 4. *All* $GL(m,\mathbf{R})$*-natural operators* $\mathbf{Q}\oplus\mathbf{Q}\to L\otimes\otimes^2T^*B$ *form the 10-parameter family*

$$\begin{aligned} (\Gamma,\Delta)\mapsto{} & a_1\overset{\Gamma}{R}{}^p_{qij}+a_2\delta^p_q\overset{\Gamma}{R}{}^r_{rij}+a_3\overset{\Delta}{R}{}^p_{qij}+a_4\delta^p_q\overset{\Delta}{R}{}^r_{rij}+ \\ & a_5D^p_{ri}D^r_{qj}+a_6D^p_{rj}D^r_{qi}+a_7D^r_{ri}D^p_{qj}+ \\ & a_8D^p_{qi}D^r_{rj}+a_9\delta^p_qD^r_{si}D^s_{rj}+a_{10}\delta^p_qD^r_{ri}D^s_{sj} \end{aligned} \tag{4.9}$$

with any real parameters $a_1,\ldots,a_{10}$.

This (4.9) the 10-parameter family of $GL(m,\mathbf{R})$-natural operators $\mathbf{Q}\oplus\mathbf{Q}\to L\otimes\otimes^2T^*B$ may be also obtained by direct calculation.

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