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# ON $\Theta$-QUASI CONTINUOUS MULTIFUNCTIONS

## 1. Introduction

In 1963, Levine [8] has introduced the notion of semi-continuity between topological spaces. Since then several variations of continuity have been defined and investigated in the literature. Arya and Bhamini [1] introduced the notion of $\theta$-semi-continuity as a generalization of semi-continuity. Recently, the second author [12] of the present paper has further investigated properties of $\theta$-semi-continuity. In the present paper, we define and investigate upper (lower) $\theta$-quasi continuous multifunctions. In Section 3, we shall obtain several characterizations of upper (lower) $\theta$-quasi continuous multifunctions. In Section 4, we shall investigate several properties of such multifunctions.

## 2. Preliminaries

Throughout the present paper, $X$ and $Y$ always represent topological spaces. Let $A$ be a subset of $X$. By $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, we denote the closure of $A$ and the interior of $A$, respectively. The *$\theta$-closure* [21] of $A$, denoted by $\mathrm{Cl}_\theta(A)$, is defined to be the set of all $x \in X$ such that $A \cap \mathrm{Cl}(U) \neq \emptyset$ for every open neighborhood $U$ of $x$. The *$\theta$-interior* [9] of $A$, denoted by $\mathrm{Int}_\theta(A)$, is defined to be the set of all $x \in A$ such that $\mathrm{Cl}(U) \subset A$ for some open neighborhood $U$ of $x$. It is shown in [21] that $\mathrm{Cl}_\theta(A)$ is closed in $X$ and that $\mathrm{Cl}(U) = \mathrm{Cl}_\theta(U)$ for each open set $U$ of $X$. A subset $A$ of $X$ is said to be *semi-open* [8] if there exists an open set $U$ such that $U \subset A \subset \mathrm{Cl}(U)$, or equivalently if $A \subset \mathrm{Cl}(\mathrm{Int}(A))$. A subset $A$ is said to be *regular open* (resp. *preopen* [10]) if $A = \mathrm{Int}(\mathrm{Cl}(A))$ (resp. $A \subset \mathrm{Int}(\mathrm{Cl}(A))$)). The family of all semi-open (resp. regular open, preopen) sets of $X$ is denoted by $\mathrm{SO}(X)$ (resp. $\mathrm{RO}(X), \mathrm{PO}(X)$). For each $x \in X$, the family of semi-open sets containing $x$ is denoted by $\mathrm{SO}(X, x)$. The complement of a semi-open (resp. regular open) set is said to be *semi-closed* [2] (resp. *regular closed*). The family of all *semi-closed* (resp. *regular closed*) sets of $X$ is denoted by $\mathrm{SC}(X)$ (resp.

RC($X$)). The intersection of all semi-closed sets containing a subset $A$ is called the *semi-closure* [2] of $A$ and is denoted by sCl($A$). The *semi-interior* of $A$, denoted by sInt($A$), is defined by the union of all semi-open sets contained in $A$. The *semi θ-closure* of $A$ and the *semi θ-interior* of $A$ are respectively defined in [3] as follows:

$$\mathrm{sCl}_\theta(A) = \{x \in X \mid A \cap \mathrm{sCl}(U) \neq \emptyset \text{ for every } U \in \mathrm{SO}(X,x)\} \quad \text{and}$$
$$\mathrm{sInt}_\theta(A) = \{x \in A \mid \mathrm{sCl}(U) \subset A \text{ for some } U \in \mathrm{SO}(X,x)\}.$$

Throughout the present paper, $F : X \to Y$ (resp. $f : X \to Y$) represents a multifunction (resp. single valued function). For a multifunction $F : X \to Y$, we shall denote the upper and lower inverse of a subset $B$ of $Y$ by $F^+(B)$ and $F^-(B)$, respectively, that is

$$F^+(B) = \{x \in X \mid F(x) \subset B\} \quad \text{and} \quad F^-(B) = \{x \in X \mid F(x) \cap B \neq \emptyset\}.$$

DEFINITION 1. A multifunction $F : X \to Y$ is said to be

(a) *upper weakly quasi continuous* [16] if for each $x \in X$, each open set $U$ containing $x$ and each open set $V$ containing $F(x)$, there exists a nonempty open set $G$ of $X$ such that $G \subset U$ and $F(G) \subset \mathrm{Cl}(V)$;

(b) *lower weakly quasi continuous* [16] if for each $x \in X$, each open set $U$ containing $x$ and each open set $V$ such that $F(x) \cap V \neq \emptyset$, there exists a nonempty open set $G$ of $X$ such that $G \subset U$ and $F(g) \cap \mathrm{Cl}(V) \neq \emptyset$ for every $g \in G$.

LEMMA 1 (Noiri and Popa [15]). *A multifunction $F : X \to Y$ is upper (resp. lower) weakly quasi continuous if and only if for each $x \in X$ and each open set $V$ of $Y$ such that $F(x) \subset V$ (resp. $F(x) \cap V \neq \emptyset$), there exists $U \in \mathrm{SO}(X,x)$ such that $F(U) \subset \mathrm{Cl}(V)$ (resp. $U \subset F^-(\mathrm{Cl}(V))$).*

DEFINITION 2. A multifunction $F : X \to Y$ is said to be

(a) *upper almost quasi continuous* [18] if for each $x \in X$, each open set $U$ containing $x$ and each open set $V$ containing $F(x)$, there exists a nonempty open set $G$ of $X$ such that $G \subset U$ and $F(G) \subset \mathrm{sCl}(V)$;

(b) *lower almost quasi continuous* [18] if for each $x \in X$, each open set $U$ containing $x$ and each open set $V$ such that $F(x) \cap V \neq \emptyset$, there exists a nonempty open set $G$ of $X$ such that $G \subset U$ and $F(g) \cap \mathrm{sCl}(V) \neq \emptyset$ for every $g \in G$.

LEMMA 2 (Popa and Noiri [18]). *The following are equivalent for a multifunction $F : X \to Y$:*

(1) *$F$ is upper (resp. lower) almost quasi continuous;*
(2) *$F^+(V) \in \mathrm{SO}(X)$ (resp. $F^-(V) \in \mathrm{SO}(X)$ for every $V \in \mathrm{RO}(Y)$;*
(3) *$F^-(K) \in \mathrm{SC}(X)$ (resp. $F^+(K) \in \mathrm{SC}(X)$) for every $K \in \mathrm{RC}(Y)$;*

(4) *for each* $x \in X$ *and each open set* $V$ *of* $Y$ *such that* $F(x) \subset V$ *(resp.* $F(x) \cap V \neq \emptyset$*), there exists* $U \in \mathrm{SO}(X,x)$ *such that* $F(U) \subset \mathrm{sCl}(V) = \mathrm{Int}(\mathrm{Cl}(V))$ *(resp.* $U \subset F^-(\mathrm{sCl}(V)) = F^-(\mathrm{Int}(\mathrm{Cl}(V)))$*).*

## 3. Characterizations

Definition 3. A multifunction $F : X \to Y$ is said to be

(a) *upper θ-quasi continuous* (briefly u.θ.q.c) for each point $x \in X$ and each open set $V$ of $Y$ containing $F(x)$, there exists $U \in \mathrm{SO}(X,x)$ such that $F(\mathrm{sCl}(U)) \subset \mathrm{Cl}(V)$;

(b) *lower θ-quasi continuous* (briefly l.θ.q.c.) if for each point $x \in X$ and each open set $V$ of $Y$ such that $F(x) \cap V \neq \emptyset$, there exists $U \in \mathrm{SO}(X,x)$ such that $F(u) \cap \mathrm{Cl}(V) \neq \emptyset$ for every $u \in \mathrm{sCl}(U)$.

In this section we obtain several characterizations of upper (lower) θ-quasi continuous multifunctions.

Theorem 1. *The following are equivalent for a multifunction* $F{:}X \to Y$:

(1) $F$ *is u.θ.q.c.*;
(2) $\mathrm{sCl}_\theta(F^-(\mathrm{Int}(\mathrm{Cl}_\theta(B)))) \subset F^-(\mathrm{Cl}_\theta(B))$ *for every subset* $B$ *of* $Y$;
(3) $\mathrm{sCl}_\theta(F^-(\mathrm{Int}(\mathrm{Cl}(V)))) \subset F^-(\mathrm{Cl}(V))$ *for every open set* $V$ *of* $Y$;
(4) $\mathrm{sCl}_\theta(F^-(\mathrm{Int}(R))) \subset F^-(R)$ *for every* $R \in \mathrm{RC}(X)$;
(5) $F^+(V) \subset \mathrm{sInt}_\theta(F^+(\mathrm{Cl}(V)))$ *for every open set* $V$ *of* $Y$;
(6) $\mathrm{sCl}_\theta(F^-(\mathrm{Int}(K))) \subset F^-(K)$ *for every closed set* $K$ *of* $Y$;
(7) $\mathrm{sCl}_\theta(F^-(V)) \subset F^-(\mathrm{Cl}(V))$ *for every open set* $V$ *of* $Y$.

Proof. (1)⇒(2): Let $B$ be any subset of $Y$. Suppose that $x \notin F^-(\mathrm{Cl}_\theta(B))$. Then $x \in X - F^-(\mathrm{Cl}_\theta(B))$ and $F(x) \subset Y - \mathrm{Cl}_\theta(B)$. Since $\mathrm{Cl}_\theta(B)$ is closed in $Y$, there exists $U \in \mathrm{SO}(X,x)$ such that $F(\mathrm{sCl}(U)) \subset \mathrm{Cl}(Y - \mathrm{Cl}_\theta(B)) = Y - \mathrm{Int}(\mathrm{Cl}_\theta(B))$. Thus, we have $F(\mathrm{sCl}(U)) \cap \mathrm{Int}(\mathrm{Cl}_\theta(B)) = \emptyset$ and $\mathrm{sCl}(U) \cap F^-(\mathrm{Int}(\mathrm{Cl}_\theta(B))) = \emptyset$. This shows that $x \notin \mathrm{sCl}_\theta(F^-(\mathrm{Int}(\mathrm{Cl}_\theta(B))))$. Therefore, we obtain $\mathrm{sCl}_\theta(F^-(\mathrm{Int}(\mathrm{Cl}_\theta(B)))) \subset F^-(\mathrm{Cl}_\theta(B))$.

(2)⇒(3): This is obvious since $\mathrm{Cl}(V) = \mathrm{Cl}_\theta(V)$ for every open set $V$ of $Y$.

(3)⇒(4): Let $R \in \mathrm{RC}(Y)$, then we have $\mathrm{sCl}_\theta(F^-(\mathrm{Int}(R))) = \mathrm{sCl}_\theta(F^-(\mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(R))))) \subset F^-(\mathrm{Cl}(\mathrm{Int}(R))) = F^-(R)$.

(4)⇒(5): Let $V$ be any open set of $Y$. Then we have

$$X - \mathrm{sInt}_\theta(F^+(\mathrm{Cl}(V))) = \mathrm{sCl}_\theta(X - F^+(\mathrm{Cl}(V))) = \mathrm{sCl}_\theta(F^-(Y - \mathrm{Cl}(V))),$$
$$Y - \mathrm{Cl}(V) = \mathrm{Int}(Y - \mathrm{Cl}(V)) \subset \mathrm{Int}(Y - \mathrm{Int}(\mathrm{Cl}(V))), \quad \text{and}$$
$$Y - \mathrm{Int}(\mathrm{Cl}(V)) \in \mathrm{RC}(Y).$$

Therefore, we obtain

$$\mathrm{sCl}_\theta(F^-(\mathrm{Int}(Y-\mathrm{Int}(\mathrm{Cl}(V))))) \subset F^-(Y-\mathrm{Int}(\mathrm{Cl}(V))) = \\ = X - F^+(\mathrm{Int}(\mathrm{Cl}(V))) \subset X - F^+(V).$$

Consequently, we obtain $F^+(V) \subset \mathrm{sInt}_\theta(F^+(\mathrm{Cl}(V)))$.

(5)⇒(6): Let $K$ be any closed set of $Y$. Then by (5) we have

$$X - F^-(K) = F^+(Y-K) \subset \mathrm{sInt}_\theta(F^+(\mathrm{Cl}(Y-K))) = \\ = \mathrm{sInt}_\theta(F^+(Y-\mathrm{Int}(K))) = \mathrm{sInt}_\theta(X - F^-(\mathrm{Int}(K))) = \\ = X - \mathrm{sCl}_\theta(F^-(\mathrm{Int}(K))).$$

Therefore, we obtain $\mathrm{sCl}_\theta(F^-(\mathrm{Int}(K))) \subset F^-(K)$.

(6)⇒(7): Let $V$ be any open set of $Y$, then $\mathrm{Cl}(V)$ is closed and we have $\mathrm{sCl}_\theta(F^-(V)) \subset \mathrm{sCl}_\theta(F^-(\mathrm{Int}(\mathrm{Cl}(V)))) \subset F^-(\mathrm{Cl}(V))$.

(7)⇒(1): Let $x \in X$ and $V$ be any open set of $Y$ containing $F(x)$. Then $F(x) \cap \mathrm{Cl}(Y - \mathrm{Cl}(V)) = \emptyset$ and $x \notin F^-(\mathrm{Cl}(Y-\mathrm{Cl}(V)))$. It follows from (7) that $x \notin \mathrm{sCl}_\theta(F^-(Y-\mathrm{Cl}(V)))$. Then there exists $U \in \mathrm{SO}(X,x)$ such that $\mathrm{sCl}(U) \cap F^-(Y - \mathrm{Cl}(V) = \emptyset$; hence $F(\mathrm{sCl}(U)) \subset \mathrm{Cl}(V)$. This shows that $F$ is u.$\theta$.q.c.

LEMMA 3. *If $F : X \to Y$ is l.$\theta$.q.c., then for each $x \in X$ and each subset $B$ of $Y$ with $F(x) \cap \mathrm{Int}_\theta(B) \neq \emptyset$ there exists $U \in \mathrm{SO}(X,x)$ such that* $\mathrm{sCl}(U) \subset F^-(B)$.

Proof. Since $F(x) \cap \mathrm{Int}_\theta(B) \neq \emptyset$, there exists an open set $V$ of $Y$ such that $V \subset \mathrm{Cl}(V) \subset B$ and $F(x) \cap V \neq \emptyset$. Since $F$ is l.$\theta$.q.c., there exists $U \in \mathrm{SO}(X,x)$ such that $F(u) \cap \mathrm{Cl}(V) \neq \emptyset$ for every $u \in \mathrm{sCl}(U)$ and hence $\mathrm{sCl}(U) \subset F^-(B)$.

THEOREM 2. *The following are equivalent for a multifunction $F{:}X \to Y$:*

(1) *$F$ is l.$\theta$.q.c.;*
(2) $\mathrm{sCl}_\theta(F^+(B)) \subset F^+(\mathrm{Cl}_\theta(B))$ *for every subset $B$ of $Y$;*
(3) $\mathrm{sCl}_\theta(F^+(V)) \subset F^+(\mathrm{Cl}(V))$ *for every open set $V$ of $Y$;*
(4) $F^-(V) \subset \mathrm{sInt}_\theta(F^-(\mathrm{Cl}(V)))$ *for every open set $V$ of $Y$;*
(5) $F(\mathrm{sCl}_\theta(A)) \subset \mathrm{Cl}_\theta(F(A))$ *for every subset $A$ of $X$;*
(6) $\mathrm{sCl}_\theta(F^+(\mathrm{Int}(\mathrm{Cl}_\theta(B)))) \subset F^+(\mathrm{Cl}_\theta(B))$ *for every subset $B$ of $Y$;*
(7) $\mathrm{sCl}_\theta(F^+(\mathrm{Int}(\mathrm{Cl}(V)))) \subset F^+(\mathrm{Cl}(V))$ *for every open set $V$ of $Y$;*
(8) $\mathrm{sCl}_\theta(F^+(\mathrm{Int}(R))) \subset F^+(R)$ *for every $R \in \mathrm{RC}(Y)$;*
(9) $\mathrm{sCl}_\theta(F^+(\mathrm{Int}(K))) \subset F^+(K)$ *for every closed set $K$ of $Y$.*

Proof. (1)⇒(2): Let $B$ be any subset of $Y$. Suppose that $x \notin F^+(\mathrm{Cl}_\theta(B))$. Then $x \in F^-(Y - \mathrm{Cl}_\theta(B)) = F^-(\mathrm{Int}_\theta(Y-B))$. Since $F$ is l.$\theta$.q.c., by Lemma 3 there exists $U \in \mathrm{SO}(X,x)$ such that $\mathrm{sCl}(U) \subset$

$F^-(Y - B) = X - F^+(B)$. Thus we have $\mathrm{sCl}(U) \cap F^+(B) = \emptyset$ and hence $x \notin \mathrm{sCl}_\theta(F^+(B))$.

(2)⇒(3): This is obvious since $\mathrm{Cl}(V) = \mathrm{Cl}_\theta(V)$ for every open set $V$ of $Y$.

(3)⇒(4): Let $V$ be any open set of $Y$. Then, we have

$$X - \mathrm{sInt}_\theta(F^-(\mathrm{Cl}(V))) = \mathrm{sCl}_\theta(X - F^-(\mathrm{Cl}(V))) = \mathrm{sCl}_\theta(F^+(Y - \mathrm{Cl}(V))) \subset F^+(\mathrm{Cl}(Y - \mathrm{Cl}(V))) \subset F^+(\mathrm{Cl}(Y - V)) = F^+(Y - V) = X - F^-(V).$$

Therefore, we obtain $F^-(V) \subset \mathrm{sInt}_\theta(F^-(\mathrm{Cl}(V)))$.

(4)⇒(1): Let $x \in X$ and $V$ be any open set such that $F(x) \cap V \neq \emptyset$. Then $x \in F^-(V) \subset \mathrm{sInt}_\theta(F^-(\mathrm{Cl}(V)))$. Therefore, there exists $U \in \mathrm{SO}(X, x)$ such that $\mathrm{sCl}(U) \subset F^-(\mathrm{Cl}(V))$; hence $F(u) \cap \mathrm{Cl}(V) \neq \emptyset$ for every $u \in \mathrm{sCl}(U)$. This shows that $F$ is l.$\theta$.q.c.

(2)⇒(5): Let $A$ be any subset of $X$. By replacing $B$ in (2) by $F(A)$, we have $\mathrm{sCl}_\theta(A) \subset \mathrm{sCl}_\theta(F^+(F(A))) \subset F^+(\mathrm{Cl}_\theta(F(A)))$. Thus we obtain $F(\mathrm{sCl}_\theta(A)) \subset \mathrm{Cl}_\theta(F(A))$.

(5)⇒(2): Let $B$ be any subset of $Y$. Replacing $A$ in (5) by $F^+(B)$, we have $F(\mathrm{sCl}_\theta(F^+(B))) \subset \mathrm{Cl}_\theta(F(F^+(B))) \subset \mathrm{Cl}_\theta(B)$. Thus we obtain $\mathrm{sCl}_\theta(F^+(B)) \subset F^+(\mathrm{Cl}_\theta(B))$.

(3)⇒(6): Let $B$ be any subset of $Y$. Put $V = \mathrm{Int}(\mathrm{Cl}_\theta(B))$ in (3). Then, since $\mathrm{Cl}_\theta(B)$ is closed in $Y$, we have

$$\mathrm{sCl}_\theta(F^+(\mathrm{Int}(\mathrm{Cl}_\theta(B)))) \subset F^+(\mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}_\theta(B)))) \subset F^+(\mathrm{Cl}_\theta(B)).$$

(6)⇒(7): This is obvious since $\mathrm{Cl}(V) = \mathrm{Cl}_\theta(V)$ for any open set $V$ of $Y$.

(7)⇒(8): If $R \in \mathrm{RC}(Y)$, then by (7) we have

$$\begin{aligned}\mathrm{sCl}_\theta(F^+(\mathrm{Int}(R))) &= \mathrm{sCl}_\theta(F^+(\mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(R))))) \\ &\subset F^+(\mathrm{Cl}(\mathrm{Int}(R))) = F^+(R).\end{aligned}$$

(8)⇒(9): Let $K$ be any closed set of $Y$. Since $\mathrm{Cl}(\mathrm{Int}(K)) \in \mathrm{RC}(Y)$, we have

$$\begin{aligned}\mathrm{sCl}_\theta(F^+(\mathrm{Int}(K))) &= \mathrm{sCl}_\theta(F^+(\mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(K))))) \\ &\subset F^+(\mathrm{Cl}(\mathrm{Int}(K))) \subset F^+(K).\end{aligned}$$

(9)⇒(4): Let $V$ be any open set of $Y$. Then $Y - V$ is closed in $Y$ and by (9) we have $\mathrm{sCl}_\theta(F^+(\mathrm{Int}(Y - V))) \subset F^+(Y - V) = X - F^-(V)$. Moreover, we have $\mathrm{sCl}_\theta(F^+(\mathrm{Int}(Y - V))) = \mathrm{sCl}_\theta(F^+(Y - \mathrm{Cl}(V))) = \mathrm{sCl}_\theta(X - F^-(\mathrm{Cl}(V))) = X - \mathrm{sInt}_\theta(F^-(\mathrm{Cl}(V)))$. Therefore, we obtain $F^-(V) \subset \mathrm{sInt}_\theta(F^-(\mathrm{Cl}(V)))$.

COROLLARY 1 (Noiri [12]). *The following are equivalent for a function* $f : X \to Y$:

(1) $f$ *is θ-semi-continuous*;

(2) $\mathrm{sCl}_\theta(f^{-1}(B)) \subset f^{-1}(\mathrm{Cl}_\theta(B))$ *for every subset* $B$ *of* $Y$;
(3) $\mathrm{sCl}_\theta(f^{-1}(V)) \subset f^{-1}(\mathrm{Cl}(V))$ *for every open set* $V$ *of* $Y$;
(4) $f^{-1}(V) \subset \mathrm{sInt}_\theta(f^{-1}(\mathrm{Cl}(V)))$ *for every open set* $V$ *of* $Y$;
(5) $f(\mathrm{sCl}_\theta(A)) \subset \mathrm{Cl}_\theta(f(A))$ *for every subset* $A$ *of* $X$.

For a multifunction $F : X \to Y$, the *graph multifunction* $G_F : X \to X \times Y$ is defined as follows: $G_F(x) = \{x\} \times F(x)$ for every $x \in X$.

LEMMA 4. (Noiri and Popa [14]). *The following hold for a multifunction* $F : X \to Y$:

(a) $G_F^+(A \times B) = A \cap F^+(B)$ *and* (b) $G_F^-(A \times B) = A \cap F^-(B)$ *for every subsets* $A \subset X$ *and* $B \subset Y$.

THEOREM 3. *Let* $F : X \to Y$ *be a multifunction such that* $F(x)$ *is compact for each* $x \in X$. *Then* $F$ *is u.θ.q.c. if and only if* $G_F : X \to X \times Y$ *is u.θ.q.c..*

Proof. *Necessity.* Suppose that $F : X \to Y$ is u.$\theta$.q.c. Let $x \in X$ and $W$ be any open set of $X \times Y$ containing $G_F(x)$. For each $y \in F(x)$ there exist open sets $U(y) \subset Y$ and $V(y) \subset Y$ such that $(x, y) \in U(y) \times V(y) \subset W$. The family $\{V(y) \mid y \in F(x)\}$ is an open cover of $F(x)$ and there exists a finite number of points, says, $y_1, y_2, \ldots, y_n$ in $F(x)$ such that $F(x) \subset \bigcup\{V(y_i) \mid i = 1, 2, \ldots, n\}$. Set $U = \bigcap\{U(y_i) \mid i = 1, 2, \ldots, n\}$ and $V = \bigcup\{V(y_i) \mid i = 1, 2, \ldots, n\}$. Then $U$ and $V$ are open in $X$ and $Y$, respectively, and $\{x\} \times F(x) \subset U \times V \subset W$. Since $F$ is u.$\theta$.q.c., there exists $U_0 \in \mathrm{SO}(X, x)$ such that $F(\mathrm{sCl}(U_0)) \subset \mathrm{Cl}(V)$. It follows from [11, Lemma 1] that $G = U \cap U_0 \in \mathrm{SO}(X, x)$. By Lemma 4 we have

$$\begin{aligned}\mathrm{sCl}(G) &= \mathrm{sCl}(U \cap U_0) \subset \mathrm{sCl}(U) \cap \mathrm{sCl}(U_0) \subset \mathrm{Cl}(U) \cap F^+(\mathrm{Cl}(V)) \\ &= G_F^+(\mathrm{Cl}(U) \times \mathrm{Cl}(V)) = G_F^+(\mathrm{Cl}(U \times V)) \subset G_F^+(\mathrm{Cl}(W)).\end{aligned}$$

Thus $G_F(\mathrm{sCl}(G)) \subset \mathrm{Cl}(W)$. This shows that $G_F$ is u.$\theta$.q.c.

*Sufficiency.* Suppose that $G_F : X \to X \times Y$ is u.$\theta$.q.c. Let $x \in X$ and $V$ be any open set of $Y$ containing $F(x)$. Since $X \times V$ is open in $X \times Y$ and $G_F(x) \subset X \times Y$, there exists $U \in \mathrm{SO}(X, x)$ such that $G_F(\mathrm{sCl}(U)) \subset \mathrm{Cl}(X \times V) = X \times \mathrm{Cl}(V)$. Therefore by Lemma 4 we obtain $\mathrm{sCl}(U) \subset G_F^+(X \times \mathrm{Cl}(V)) = F^+(\mathrm{Cl}(V))$ and hence $F(\mathrm{sCl}(U)) \subset \mathrm{Cl}(V)$. This shows that $F$ is u.$\theta$.q.c.

THEOREM 4. *A multifunction* $F : X \to Y$ *is l.θ.q.c. if and only if* $G_F : X \to X \times Y$ *is l.θ.q.c.*

Proof. *Necessity.* Suppose that $F$ is l.$\theta$.q.c. Let $x \in X$ and $W$ be any open set of $X \times Y$ such that $G_F(x) \cap W \neq \emptyset$. There exists $y \in F(x)$ such that $(x, y) \in W$ and hence we have $(x, y) \in U \times V \subset W$ for some open sets $U \subset X$ and $V \subset Y$. Since $F$ is l.$\theta$.q.c. and $y \in F(x) \cap V$, there exists $U_0 \in \mathrm{SO}(X,x)$ such that $\mathrm{sCl}(U_0) \subset F^-(\mathrm{Cl}(V))$. It follows from [11, Lemma 1] that $G = U \cap U_0 \in \mathrm{SO}(X,x)$. By Lemma 4, we have

$$\begin{aligned}\mathrm{sCl}(G) &= \mathrm{sCl}(U \cap U_0) \subset \mathrm{sCl}(U) \cap \mathrm{sCl}(U_0) \subset \mathrm{Cl}(U) \cap F^-(\mathrm{Cl}(V)) \\ &= G_F^-(\mathrm{Cl}(U) \times \mathrm{Cl}(V)) = G_F^-(\mathrm{Cl}(U \times V)) \subset G_F^-(\mathrm{Cl}(W)).\end{aligned}$$

Therefore, $G_F(u) \cap \mathrm{Cl}(W) \neq \emptyset$ for every $u \in \mathrm{sCl}(G)$. This shows that $G_F$ is l.$\theta$.q.c.

*Sufficiency.* Suppose that $G_F$ is l.$\theta$.q.c. Let $x \in X$ and $V$ be an open set in $Y$ such that $F(x) \cap V \neq \emptyset$. Then $X \times V$ is open in $X \times Y$ and $G_F(x) \cap (X \times V) = (\{x\} \times F(x)) \cap (X \times V) = \{x\} \times (F(x) \cap V) \neq \emptyset$. There exists $U \in \mathrm{SO}(X,x)$ such that $G_F(u) \cap \mathrm{Cl}(X \times V) \neq \emptyset$ for each $u \in \mathrm{sCl}(U)$. By Lemma 4, we obtain $\mathrm{sCl}(U) \subset G_F^-(\mathrm{Cl}(X \times V)) = F^-(\mathrm{Cl}(V))$. This shows that $F$ is l.$\theta$.q.c.

COROLLARY 2 (Noiri [12]). *A function $f : X \to Y$ is θ-semi-continuous if and only if the graph function $g : X \to X \times Y$ is θ-semi-continuous.*

For a multifunction $F : X \to Y$, a multifunction $\mathrm{sCl}\, F : X \to Y$ is defined in [17] as follows: $(\mathrm{sCl}\, F)(x) = \mathrm{sCl}(F(x))$ for each $x \in X$.

LEMMA 5. (Noiri and Popa [14]). *Let $F : X \to Y$ be a multifunction. Then $(\mathrm{sCl}\, F)^-(V) = F^-(V)$ for every $V \in \mathrm{SO}(Y)$.*

THEOREM 5. *A multifunction $F : X \to Y$ is l.θ.q.c. if and only if* $\mathrm{sCl}\, F$ : *$X \to Y$ is l.θ.q.c.*

Proof. *Necessity.* Suppose that $F$ is l.$\theta$.q.c. Let $x \in X$ and $V$ be any open set of $Y$ such that $(\mathrm{sCl}\, F)(x) \cap V \neq \emptyset$. By Lemma 5, we have $F(x) \cap V \neq \emptyset$. Since $F$ is l.$\theta$.q.c., there exists $U \in \mathrm{SO}(X,x)$ such that $F(u) \cap \mathrm{Cl}(V) \neq \emptyset$ for every $u \in \mathrm{sCl}(U)$. Since $\mathrm{Cl}(V) \in \mathrm{SO}(Y)$, by Lemma 5 we have $\mathrm{sCl}(U) \subset F^-(\mathrm{Cl}(V)) = (\mathrm{sCl}\, F)^-(\mathrm{Cl}(V))$ and hence $(\mathrm{sCl}\, F)(u) \cap \mathrm{Cl}(V) \neq \emptyset$ for any $u \in \mathrm{sCl}(U)$. This shows that $\mathrm{sCl}\, F$ is l.$\theta$.q.c.

*Sufficiency.* Suppose that $\mathrm{sCl}\, F$ is l.$\theta$.q.c. Let $x \in X$ and $V$ be any open set of $Y$ such that $F(x) \cap V \neq \emptyset$. Then $\mathrm{sCl}(F(x)) \cap V \neq \emptyset$ and there exists $U \in \mathrm{SO}(X,x)$ such that $(\mathrm{sCl}\, F)(u) \cap \mathrm{Cl}(V) \neq \emptyset$ for every $u \in \mathrm{sCl}(U)$. Since $\mathrm{Cl}(V) \in \mathrm{SO}(Y)$, by Lemma 5 $\mathrm{sCl}(U) \subset (\mathrm{sCl}\, F)^-(\mathrm{Cl}(V)) = F^-(\mathrm{Cl}(V))$ and hence $F(u) \cap \mathrm{Cl}(V) \neq \emptyset$ for every $u \in \mathrm{sCl}(U)$. Therefore, $F$ is l.$\theta$.q.c.

## 4. Some results

First, we obtain some sufficient conditions for an upper (resp. lower) weakly quasi continuous multifunction to be u.$\theta$.q.c. (resp. l.$\theta$.q.c.).

THEOREM 6. *If a multifunction $F : X \to Y$ is upper weakly quasi continuous and lower almost quasi continuous, then $F$ is u.θ.q.c.*

Proof. Let $x \in X$ and $V$ be any open set of $Y$ containing $F(x)$. Then, by Lemma 1, there exists $U \in \mathrm{SO}(X,x)$ such that $F(U) \subset \mathrm{Cl}(V)$ and hence $U \subset F^+(\mathrm{Cl}(V))$. Since $\mathrm{Cl}(V) \in \mathrm{RC}(Y)$, by Lemma 2 we have $F^+(\mathrm{Cl}(V)) \in \mathrm{SC}(X)$. Therefore, we obtain $\mathrm{sCl}(U) \subset F^+(\mathrm{Cl}(V))$ and hence $F(\mathrm{sCl}(U)) \subset \mathrm{Cl}(V)$. This shows that $F$ is u.$\theta$.q.c.

THEOREM 7. *If a multifunction $F : X \to Y$ is lower weakly quasi continuous and upper almost quasi continuous, then $F$ is l.$\theta$.q.c.*

Proof. Let $x \in X$ and $V$ be any open set such that $F(x) \cap V \neq \emptyset$. Since $F$ is lower weakly quasi continuous, by Lemma 1 there exists $U \in \mathrm{SO}(X,x)$ such that $U \subset F^-(\mathrm{Cl}(V))$. Since $\mathrm{Cl}(V) \in \mathrm{RC}(Y)$, by Lemma 2 $F^-(\mathrm{Cl}(V)) \in \mathrm{SC}(X)$ and hence $\mathrm{sCl}(U) \subset F^-(\mathrm{Cl}(V))$. This implies that $F(u) \cap \mathrm{Cl}(V) \neq \emptyset$ for every $u \in \mathrm{sCl}(U)$. Thus $F$ is l.$\theta$.q.c.

COROLLARY 3. (Arya and Bhamini [1]). *Every almost semi-continuous function is $\theta$-semi-continuous.*

DEFINITION 4. A topological space $X$ is said to be *semi-regular* [4] if for each semi-closed set $A$ and each point $x \notin A$, there exist disjoint semi-open sets $U$ and $V$ such that $x \in U$ and $A \subset V$.

THEOREM 8. *If a multifunction $F : X \to Y$ is upper weakly quasi continuous and $X$ is semi-regular, then $F$ is u.$\theta$.q.c.*

Proof. Let $x \in X$ and $V$ be an open set of $Y$ containing $F(x)$. Then, by Lemma 1, there exists $G \in \mathrm{SO}(X,x)$ such that $F(G) \subset \mathrm{Cl}(V)$. By [4, Theorem 2.1], there exists $U \in \mathrm{SO}(X,x)$ such that $x \in U \subset \mathrm{sCl}(U) \subset G$. Therefore, we obtain $F(\mathrm{sCl}(U)) \subset \mathrm{Cl}(V)$ and hence $F$ is u.$\theta$.q.c.

THEOREM 9. *If a multifunction $F : X \to Y$ is lower weakly quasi continuous and $X$ is semi-regular, then $F$ is l.$\theta$.q.c.*

Proof. The proof is similar to that of Theorem 8.

DEFINITION 5. A subset of $X$ is said to be *$\alpha$-almost regular* [7] if for any point $a \in A$ and any $U \in \mathrm{RO}(X)$ containing $a$, there exists an open set $G$ such that $a \in G \subset \mathrm{Cl}(G) \subset U$.

DEFINITION 6. A subset $A$ of $X$ is said to be *$\alpha$-nearly paracompact* [6] if every cover of $A$ by regular open sets of $X$ has an $X$-open $X$-locally finite refinement which covers $A$.

LEMMA 6 (Kovačević [7]). *If $A$ is an $\alpha$-almost regular $\alpha$-nearly paracompact subset of $X$ and $U$ is a regular open neighborhood of $A$, then there exists an open neighborhood $G$ of $A$ such that $A \subset G \subset \mathrm{Cl}(G) \subset U$.*

THEOREM 10. *If a multifunction $F : Y \to Y$ is upper weakly quasi continuous and $F(x)$ is $\alpha$-almost regular $\alpha$-nearly paracompact in $Y$ for each $x \in X$, then $F$ is upper almost quasi continuous.*

Proof. Let $V$ be any regular open set of $Y$ and $F(x) \subset V$. Since $F(x)$ is $\alpha$-almost regular $\alpha$-nearly paracompact, by Lemma 6 there exists an open set $W$ of $Y$ such that $F(x) \subset W \subset \mathrm{Cl}(W) \subset V$. Since $F$ is upper weakly quasi continuous, by Lemma 1 there exists $U \in \mathrm{SO}(X, x)$ such that $F(U) \subset \mathrm{Cl}(W) \subset V$. Therefore, we have $x \in U \subset F^+(V)$ and hence $F^+(V) \in \mathrm{SO}(X)$. It follows from Lemma 2 that $F$ is upper almost quasi continuous.

COROLLARY 4. *If $F : X \to Y$ is an u.θ.q.c. multifunction and $F(x)$ is $\alpha$-almost regular $\alpha$-nearly paracompact in $Y$ for each $x \in X$, then $F$ is upper almost quasi continuous.*

LEMMA 7 (Noiri and Ahmad [13]). *Let $A$ and $X_0$ be subsets of $X$. Then the following hold*:

(a) *If $A \in \mathrm{SO}(X)$ and $X_0 \in \mathrm{PO}(X)$, then $A \cap (X_0) \in \mathrm{SO}(X_0)$.*

(b) *If $A \subset X_0 \in \mathrm{PO}(X)$, then $X_0 \cap \mathrm{sCl}(A) = \mathrm{sCl}_{X_0}(A)$, where $\mathrm{sCl}_{X_0}(A)$ denotes the semi-closure of $A$ in the subspace $X_0$ of $X$.*

THEOREM 11. *If a multifunction $F : X \to Y$ is u.θ.q.c. and $X_0 \in \mathrm{PO}(X)$, then the restriction $F \mid X_0 : X_0 \to Y$ is u.θ.q.c.*

Proof. Let $x \in X_0$ and $V$ be any open set of $Y$ containing $F(x)$. There exists $U \in \mathrm{SO}(X, x)$ such that $F(\mathrm{sCl}(U)) \subset \mathrm{Cl}(V)$. Since $X_0 \in \mathrm{PO}(X)$, by Lemma 7 we have $x \in U \cap X_0 \in \mathrm{SO}(X_0)$ and $\mathrm{sCl}_{X_0}(U \cap X_0) = X_0 \cap \mathrm{sCl}(U \cap X_0) \subset X_0 \cap \mathrm{sCl}(U)$. Therefore, we obtain

$$\begin{aligned}(F \mid X_0)(\mathrm{sCl}_{X_0}(U \cap X_0)) &\subset (F \mid X_0)(X_0 \cap \mathrm{sCl}(U)) \\ &= F(X_0 \cap \mathrm{sCl}(U)) \subset \mathrm{Cl}(V).\end{aligned}$$

This shows that $F \mid X_0$ is u.θ.q.c.

THEOREM 12. *If a multifunction $F : X \to Y$ is l.θ.q.c. and $X_0 \in \mathrm{PO}(X)$, then the restriction $F \mid X_0 : X_0 \to Y$ is l.θ.q.c.*

Proof. Let $x \in X_0$ and $V$ be any open set of $Y$ such that $F(x) \cap V \neq \emptyset$. There exists $U \in \mathrm{SO}(X, x)$ such that $F(u) \cap \mathrm{Cl}(V) \neq \emptyset$ for each $u \in \mathrm{sCl}(U)$. Since $X_0 \in \mathrm{PO}(X)$, we have $x \in U \cap X_0 \in \mathrm{SO}(X_0)$ and $\mathrm{sCl}_{X_0}(U \cap X_0) \subset X_0 \cap \mathrm{sCl}(U)$. For any $u \in \mathrm{sCl}_{X_0}(U \cap X_0), u \in \mathrm{sCl}(U) \cap X_0$ and $\emptyset \neq F(u) \cap \mathrm{Cl}(V) = (F \mid X_0)(u) \cap \mathrm{Cl}(V)$. Therefore, we have $(F \mid X_0)(u) \cap \mathrm{Cl}(V) \neq \emptyset$ for every $u \in \mathrm{sCl}_{X_0}(U \cap X_0)$. This shows that $F \mid X_0$ is l.θ.q.c.

COROLLARY 5 (Noiri [12]). *If a function $f : X \to Y$ is a $\theta$-semi-continuous and $X_0 \in \mathrm{PO}(X)$, then the restriction $f \mid X_0 : X_0 \to Y$ is $\theta$-semi-continuous.*

DEFINITION 7. A topological space $X$ is said to be *quasi H-closed* [19] (resp. *S-closed* [20], *s-closed* [3]) if for every open (resp. semi-open) cover $\{U_\alpha \mid \alpha \in \nabla\}$ of $X$, there exists a finite subset $\nabla_0$ of $\nabla$ such that $X = \bigcup\{\mathrm{Cl}(U_\alpha) \mid \alpha \in \nabla_0\}$ (resp. $X = \bigcup\{\mathrm{Cl}(U_\alpha) \mid \alpha \in \nabla_0\}, X = \bigcup\{\mathrm{sCl}(U_\alpha) \mid \alpha \in \nabla_0\}$).

It is well known that $s$-closedness implies $S$-closedness and $S$-closedness implies quasi $H$-closedness but none of these implications is reversible.

THEOREM 13. *Let $F : X \to Y$ be a surjective multifunction and $F(x)$ compact for each $x \in X$. If $F$ is u.$\theta$.q.c. and $X$ is s-closed, then $Y$ is quasi $H$-closed.*

Proof. Let $\{V_\alpha \mid \alpha \in \nabla\}$ be any open cover of $Y$. Let $x$ be any point of $X$. Since $F(x)$ is compact, there exists a finite subset $\nabla(x)$ of $\nabla$ such that $F(x) \subset \bigcup\{V_\alpha \mid \alpha \in \nabla(x)\}$. Put $V(x) = \bigcup\{V_\alpha \mid \alpha \in \nabla(x)\}$, then $F(x) \subset V(x)$ and $V(x)$ is open in $Y$. Since $F$ is u.$\theta$.q.c., there exists $U(x) \in \mathrm{SO}(X, x)$ such that $F(\mathrm{sCl}(U(x))) \subset \mathrm{Cl}(V(x))$. The family $\{U(x) \mid x \in X\}$ is a semi-open cover of $X$. Since $X$ is $s$-closed, there exists a finite number of points, says, $x_1, x_2, \ldots, x_n$ in $X$ such that $X = \bigcup\{\mathrm{sCl}(U(x_i)) \mid i = 1, 2, \ldots, n\}$. Since $F$ is surjective, we obtain

$$Y = F(X) \subset \bigcup_{i=1}^{n} F(\mathrm{sCl}(U(x_i))) \subset \bigcup_{i=1}^{n} \mathrm{Cl}(V(x_i)) = \bigcup_{i=1}^{n} \bigcup_{\alpha \in \nabla(x_i)} \mathrm{Cl}(\nabla_\alpha).$$

This shows that $Y$ is quasi $H$-closed.

THEOREM 14. *Let $F : X \to Y$ be a surjective multifunction and $F(x)$ compact for each $x \in X$. If $F$ is upper weakly quasi continuous and lower almost quasi continuous and $X$ is $S$-closed, then $Y$ is quasi $H$-closed.*

Proof. Let $\{V_\alpha \mid \alpha \in \nabla\}$ be any open cover of $Y$. For each $x \in X, F(x)$ is compact and there exists a finite subset $\nabla(x)$ of $\nabla$ such that $F(x) \subset \bigcup\{V_\alpha \mid \alpha \in \nabla(x)\}$. Now, put $V(x) = \bigcup\{V_\alpha \mid \alpha \in \nabla(x)\}$, then $V(x)$ is open in $Y$ and $F(x) \subset V(x)$. It follows from Theorem 6 that $F$ is u.$\theta$.q.c. By Theorem 1, we have $x \in F^+(V(x)) \subset \mathrm{sInt}_\theta(F^+(\mathrm{Cl}(V(x)))) \in \mathrm{SO}(X)$ and hence $\{\mathrm{sInt}_\theta(F^+(\mathrm{Cl}(V(x)))) \mid x \in X\}$ is a semi-open cover of $X$. Since $X$ is $S$-closed, there exists a finite number of points, says, $x_1, x_2, \ldots, x_n$ in $X$ such that

$$X = \bigcup_{i=1}^{n} \mathrm{Cl}(\mathrm{sInt}_\theta(F^+(\mathrm{Cl}(V(x_i))))) =$$

$$= \bigcup_{i=1}^{n} \mathrm{Cl}(F^+(\mathrm{Cl}(V(x_i)))) = \mathrm{Cl}\Big(\bigcup_{i=1}^{n} F^+(\mathrm{Cl}(V(x_i)))\Big).$$

It follows from [5, Theorem 2.4] that

$$X = \mathrm{sCl}\Big(\bigcup_{i=1}^{n} F^+(\mathrm{Cl}(V(x_i)))\Big) \subset \mathrm{sCl}\Big(F^+\Big(\bigcup_{i=1}^{n} \mathrm{Cl}(V(x_i))\Big)\Big)$$

$$= \mathrm{sCl}\Big(F^+\Big(\bigcup_{i=1}^{n} V(x_i)\Big)\Big).$$

Since $\mathrm{Cl}(\bigcup\{V(x_i) \mid 1 \le i \le n\})$ is regular closed in $Y$, by Lemma 2 $F^+(\mathrm{Cl}(\bigcup\{V(x_i) \mid 1 \le i \le n\}))$ is semi-closed in $X$. Therefore, we have

$$Y = F(X) = F(F^+(\mathrm{Cl}(\bigcup_{i=1}^{n} V(x_i)))) \subset \mathrm{Cl}(\bigcup_{i=1}^{n} V(x_i)) =$$

$$= \bigcup_{i=1}^{n} \mathrm{Cl}(V(x_i)) = \bigcup_{i=1}^{n} \bigcup_{\alpha \in V(x_i)} \mathrm{Cl}(V_\alpha).$$

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