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Jadwiga Korczak, Małgorzata Migda

ASYMPTOTIC PROPERTIES OF SOLUTIONS OF HIGHER ORDER LINEAR DIFFERENCE EQUATIONS

1. Introduction

In this paper we will investigate the asymptotic behaviour of solutions of certain forms of m-th order linear difference equations. Motivated by the work of Trench [4] on differential equations we will give some conditions under which the equation

$$(\mathrm{E})\qquad \Delta^m x_n + p_{1,n}\Delta^{m-1}x_n + \ldots + p_{m-1,n}\Delta x_n + p_{m,n}x_n = f_n, \qquad n \in N, m \geq 2,$$

where $p_1, \ldots, p_m, f : N \to R$ has a solution x_0 behaving for a sufficiently large n like a given polynomial q of degree $< m$.

Let us use the following notations: $x_n = x(n), R := (-\infty,\infty), R_+ := (0,\infty), N := \{n_0, n_0+1, \ldots\}$, where n_0 is a given nonnegative integer. For a function $x : N \to R$, we define the difference operator Δ^i as follows

$$\Delta^0 x_n = x_n, \Delta^k x_n = \Delta(\Delta^{k-1}x_n) = \Delta^{k-1}x_{n+1} - \Delta^{k-1}x_n, \quad k \geq 1.$$

Further, by $n^{(k)}$ we will denote the product

$$n^{(k)} = \prod_{j=0}^{k-1}(n-j) \quad \text{for } n \geq k,\ n^{(0)} = 1,$$

where k is a positive integer.

2. Main result

Throughout this paper q is a given polynomial of degree $< m$. For convenience, we note

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(1) $$Mx_n = \sum_{k=1}^{m} p_{k,n} \Delta^{m-k} x_n.$$

Hence, the equation (E) can be rewritten in the form

(2) $$\Delta^m x_n + Mx_n = f_n.$$

Now, we introduce the new unknown h defined by the formula

(3) $$h = x - q.$$

Because $\Delta^m q_n = 0$, it is obvious that x is a solution of (2) if and only if h is a solution of

(4) $$\Delta^m h_n = -Mh_n - g_n,$$

where

(5) $$g_n = -f_n + Mq_n = -f_n + \sum_{k=1}^{m} p_{k,n} \Delta^{m-k} q_n.$$

LEMMA. *Let $\Phi : N \to R_+$ be a nonincreasing sequence and $u : N \to R$. Let us assume that the series*

(6) $$\sum_{j=n_0}^{\infty} j^{m-1} u_j$$

is convergent and that

(7) $$\sum_{j=n}^{\infty} j^{m-1} u_j = O(\Phi_n).$$

Further, let us define

(8) $$\varrho_n = \sup_{\tau \geq n} \left| \Phi_\tau^{-1} \sum_{j=\tau}^{\infty} j^{m-1} u_j \right|.$$

Then for

(9) $$w_n = (-1)^{m-1} \sum_{j=n}^{\infty} \frac{(j+m-1-n)^{(m-1)}}{(m-1)!} u_j$$

the inequalities

(10) $$|\Delta^r w_n| \leq \frac{2\varrho_n \Phi_n n^{-r}}{(m-1-r)!}, \qquad 0 \leq r \leq m-1,$$

are satisfied. Moreover, if

(11) $$\lim_{n \to \infty} \varrho_n = 0,$$

then

(12) $$\Delta^r w_n = o(\Phi_n n^{-r}), \qquad 0 \leq r \leq m-1.$$

The proof of this Lemma is in section 3.

Now we can formulate the main result of the paper.

THEOREM. *Let us assume that series*

$$\sum_{j=n_0}^{\infty} j^{m-1} g_j \tag{13}$$

is convergent and that

$$\sum_{j=n}^{\infty} j^{m-1} g_j = O(\Phi_n), \tag{14}$$

where g is defined by (5) and Φ_n is a nonincreasing positive sequence. Moreover, let us assume that

$$\sum_{j=n_0}^{\infty} j^{k-1} |p_{k,j}| < \infty, \quad 1 \le k \le m. \tag{15}$$

Then equation (E) *has a solution x_0 such that*

$$\Delta^r x_0 = \Delta^r q_n + O(\Phi_n n^{-r}), \quad 0 \le r \le m-1. \tag{16}$$

Proof. Let us denote by $m(\Phi)$ the Banach space of sequences $h : N \to R$ satisfying the condition

$$\Delta^r h_n = O(\Phi_n n^{-r}), \quad 0 \le r \le m-1 \tag{17}$$

with the norm

$$\|h\| = \sup_{n \ge n_0} \left\{ \Phi_n^{-1} \sum_{r=0}^{m-1} n^r |\Delta^r h_n| \right\}. \tag{18}$$

We will show that the equation (4) has a solution with expected property and that it is a fix point of a contraction mapping of space $m(\Phi)$ into itself.

Let us note

$$\overline{n} = \sup \left\{ t \in N : \sum_{k=1}^{m} \sum_{j=t}^{\infty} j^{k-1} |p_{k,j}| \ge \sum_{r=0}^{m-1} \frac{(m-r-1)!}{4} \right\}, \tag{19}$$

Now we define the transformation L by

$$(Lh) = \begin{cases} 0, & n < \overline{n} \\ (-1)^{m-1} \sum_{j=n}^{\infty} \frac{(j+m-1-n)^{(m-1)}}{(m-1)!} M h_j, & n \ge \overline{n}. \end{cases} \tag{20}$$

With g as in (4), let

$$s_n = (-1)^{m-1} \sum_{j=n}^{\infty} \frac{(j+m-1-n)^{(m-1)}}{(m-1)!} g_n. \tag{21}$$

We will show that the mapping T define by

$$Th = s + Lh \tag{22}$$

maps $m(\Phi)$ into itself and is a contraction mapping.

At first we must prove that the sequences (20) and (21) are convergent. In accordance with (1), one can write

$$\sum_{j=n}^{\infty} \frac{(j+m-1-n)^{(m-1)}}{(m-1)!} Mh_j = \tag{23}$$

$$= \sum_{j=n}^{\infty} \frac{(j+m-1-n)^{(m-1)}}{(m-1)!} \Big[\sum_{k=1}^{m} p_{k,j} \Delta^{m-k} h_j \Big].$$

In the following, let $j \geq n \geq n_0 \geq m-1$. From (18) we get

$$\Phi_j^{-1} \sum_{r=0}^{m-1} j^r |\Delta^r h_j| \leq \|h\|.$$

Hence, for each $r = 0, 1, \dots, m-1$, we have $\Phi_j^{-1} j^r |\Delta^r h_j| \leq \|h\|$. Taking $r = m-k$, we obtain

$$j^{m-1} |\Delta^{m-k} h_j| \leq \|h\| \Phi_j j^{k-1}, \quad 1 \leq k \leq m.$$

From the above inequality and from monotonicity of Φ we get

$$\sum_{j=n}^{\infty} j^{m-1} |\Delta^{m-k} h_j| |p_{k,j}| \leq \Phi_n \|h\| \sum_{j=n}^{\infty} j^{k-1} |p_{k,j}|, \quad 1 \leq k \leq m, \tag{24}$$

and, by virtue of (15), we have the convergence of the series on the left-hand side of (24).

Let us consider the series on the right-hand side of (23). From (24) it follows that the series $\sum_{j=n}^{\infty} j^{m-1} p_{k,j} \Delta^{m-k} h_j$ is absolutely convergent. Convergent is the series $\sum_{k=1}^{m} j^{m-1} [\sum_{j=n}^{\infty} p_{k,j} \Delta^{m-k} h_j]$, too. So, by virtue of Abel's criterion, the series

$$\sum_{j=n}^{\infty} \frac{(j+m-1-n)^{(m-1)}}{(m-1)!} \Big[\sum_{k=1}^{\infty} p_{k,j} \Delta^{m-k} h_j \Big].$$

is convergent. Furthermore, the equality

$$\sum_{j=n}^{\infty} \frac{(j+m-1-n)^{(m-1)}}{(m-1)!} \Big[\sum_{k=1}^{m} p_{k,j} \Delta^{m-k} h_j \Big] =$$

$$= \sum_{j=n}^{\infty} \sum_{k=1}^{m} \frac{(j+m-1-n)^{(m-1)}}{(m-1)!} p_{k,j} \Delta^{m-k} h_j$$

holds. The convergence of the series (21) follows from Lemma for $u = g$.

Noting

$$I(n;h) = \sum_{k=1}^{m} \sum_{j=n}^{\infty} j^{m-1} p_{k,j} \Delta^{m-k} h_j,$$

by (24), we get the estimate

(25) $$|I(n;h)| \le \Phi_n \|h\| \sigma_n,$$

where $\sigma_n = \sum_{k=1}^{m} \sum_{j=n}^{\infty} j^{k-1} |p_{k,j}|$.

Now, we can apply Lemma with $u = Mh$ and $w = Lh$. Then (8) becomes $\varrho_n = \sup_{\tau \ge n} \Phi_\tau^{-1} |I(\tau;h)|$ which, with (25), implies that

(26) $$\varrho_n \le \|h\| \sup_{\tau \ge n} \sigma_\tau = o(1).$$

Further, we have (applying (10))

$$\Phi_n^{-1} n^r |\Delta^r (Lh)_n| \le \frac{2\varrho_n}{(m-1-r)!}, \qquad r = 0, 1, \ldots, m-1.$$

From the above and from (26) it follows that $Lh \in m(\Phi)$ and

(27) $$\|Lh\| \le K \|h\| \sup_{\tau \ge n_0} \sigma_\tau,$$

where $K = \sum_{r=0}^{m-1} \frac{2}{(m-r-1)!}$ is an universal constant.

By virtue of Lemma (for $u = g$), we have the inequality

$$|\Delta^r s_n| \le \frac{2\varrho_n \Phi_n n^{-r}}{(m-r-1)!}$$

implying $s \in m(\Phi)$, by the assumption (14). Hence, the transformation T defined by (22) is a mapping of $m(\Phi)$ into itself.

Let $h_1, h_2 \in m(\Phi)$. Then, using (19) and (27), we have

$$\|Th_1 - Th_2\| = \|L(h_1 - h_2)\| \le K \|h_1 - h_2\| \sup_{\tau \ge n_0} \sigma_\tau = \frac{1}{2} \|h_1 - h_2\|.$$

Hence, T is a contraction mapping of $m(\Phi)$ into itself. So it has a fixed point h_0 such that $Th_0 = h_0$. From (20) and (21) it follows that h_0 satisfies the equation

$$h_{0,n} = (-1)^{m-1} \sum_{j=n}^{\infty} \frac{(j+m-1-n)^{(m-1)}}{(m-1)!} (Mh_{0,j} + g_j).$$

Hence $\Delta^{m-1} h_{0,n} = \sum_{j=n}^{\infty} (Mh_{0,j} + g_j)$, and it means that $\Delta^m h_{0,n} = -Mh_{0,n} - g_n$. So, h_0 is a solution of the equation (4), too. Since $h_0 \in m(\Phi)$, the condition (17) is satisfied. From this fact and from (3) we get the condition (18) and this completes the proof of Theorem.

COROLLARY 1. *If in assumption* (14) *of Theorem we interchange „O" with „o", then Theorem still holds for* (16) *with „O" interchanged with „o".*

Proof. Since from (26) follows (11), from Lemma with $u = Mh$ and $w = Lh_0$ we get

(28) $$\Delta^r L h_{0,n} = o(\Phi_n n^{-r}), \qquad 0 \le r \le m-1.$$

Moreover, if in (14) we change „O" with „o", then from Lemma it follows

(29) $$\Delta^r s_n = o(\Phi_n n^{-r}), \quad 0 \le r \le m-1.$$

But $h_0 = s + Lh_0$, so, by means of (28) and (29), it follows that (17) can be replaced by $\Delta^r h_n = o(\Phi_n n^{-r})$.

EXAMPLE. Let us assume that the series

(30) $$\sum_{j=n_0}^{\infty} j^{k-1}|p_{k,j}|, \quad 1 \le k \le m,$$

are convergent. Let v be any integer in $\{0, 1, \ldots, m-1\}$ and let $q_n = \frac{n^{(v-r)}}{v!}$, $f_n \equiv 0$. Then the function g defined by the formula (5) has the form

$$g_n = \sum_{k=m-v}^{\infty} p_{k,n} \frac{n^{(v-m+k)}}{(v-m+k)!}.$$

From (28) it follows that

$$\sum_{j=n}^{\infty} j^{m-1} g_j = O(\Phi_n),$$

where $\Phi_n = n^{-v}$. By virtue of Theorem, the homogeneous equation

$$\Delta^m x_n + p_{1,n}\Delta^{m-1}x_n + \ldots + p_{m-1,n}\Delta x_n + p_{m,n}x_n = 0$$

has a fundamental system of solutions $x_0, x_1, \ldots, x_{m-1}$ such that

$$\Delta^r x_{v,n} = \begin{cases} \frac{n^{(v-r)}}{(v-r)!}, & 0 \le r \le v, \\ O(n^{v-r}), & v+1 \le r \le m-1. \end{cases}$$

3. Appendix. Proof of Lemma

Let us denote

(31) $$Q_n = \sum_{j=n}^{\infty} j^{m-1} u_j.$$

From definition of the difference we get $\Delta Q_n = -n^{m-1}u_n$. Hence

(32) $$u_n = -\frac{\Delta Q_n}{n^{m-1}}.$$

By virtue of (8) and (31), we have $\varrho_n \ge \Phi_n^{-1}|Q_n|$. So, we obtain

(33) $$\Phi_n \varrho \ge |Q_n|.$$

By (6), the series $\sum_{j=n}^{\infty} j^{m-1}u_j$, is convergent. The sequence

$$\left\{ \frac{(j+m-1-n)^{(m-1)}}{j^{m-1}} \right\}_{j=n}^{\infty}$$

is increasing and bounded; hence, by virtue of Abel's criterion, the series

$$\sum_{j=n}^{\infty} \frac{(j+m-1-n)^{(m-1)}}{(m-1)!} u_j = \frac{1}{(m-1)!} \sum_{j=n}^{\infty} j^{m-1} u_j \frac{(j+m-1-n)^{(m-1)}}{j^{m-1}}$$

is convergent, too.

Using definition (9) and the definition of difference Δ, we get

$$\Delta^r w_n = (-1)^{m-1-r} \sum_{j=1}^{\infty} \frac{(j+m-1-n-r)^{(m-1-r)}}{(m-1-r)!} u_j$$

for $j \geq n$ and $0 \leq r \leq m-1$.

The series $\sum_{j=n}^{\infty}(j+m-1-n-r)^{(m-1-r)} u_j$ can be written in the form

$$(34) \qquad \sum_{j=n}^{\infty}(j+m-1-n-r)^{(m-1-r)} u_j = \\ = \sum_{j=n}^{\infty} \left[\prod_{t=-m+1+n+r}^{n-1} \left(1 - \frac{t}{j}\right)\right] j^{m-r-1} u_j.$$

The series on the left-hand side of (34) is convergent; hence, that on the right-hand side is convergent, too.

Applying (32), we can write (34) in the form

$$\sum_{j=n}^{\infty}(j+m-1-n-r)^{(m-1-r)} u_j = \sum_{j=n}^{\infty} \left[\prod_{t=-m+1+n+r}^{n-1} \left(1 - \frac{t}{j}\right)\right] j^{-r}(-\Delta Q_j).$$

Further, applying the formula for summing by parts, we get

$$(35) \qquad \sum_{j=n}^{\infty}(j+m-1-n-r)^{(m-1-r)} u_j = \\ = \lim_{s\to\infty} \sum_{j=n}^{s}(j+m-1-n-r)^{(m-1-r)} u_j \\ = -\lim_{s\to\infty} (s+1)^{-r} Q_{s+1} \prod_{t=-m+1+n+r}^{n-1} \left(1 - \frac{t}{s+1}\right) + \\ + n^{-r} Q_n \prod_{t=-m+1+n+r}^{n-1} \left(1 - \frac{t}{n}\right) + \\ + \sum_{j=n}^{\infty} Q_{j+1} \Delta \left[j^{-r} \prod_{t=-m+1+n+r}^{n-1} \left(1 - \frac{t}{j}\right)\right].$$

Since, by virtue of (6), we have $Q_{s+1} \to 0$, as $s \to \infty$, and the sequences $\{(s+1)^{-r}\}$ and $\{\prod_{t=-m+1+n+r}^{n-1}(1-\frac{t}{s+1})\}_{s=n}^{\infty}$ are bounded, so we have

$$\lim_{s\to\infty}(s+1)^{-r}Q_{s+1}\Big[\prod_{t=-m+1+n+r}^{n-1}\Big(1-\frac{t}{s+1}\Big)\Big]=0, \tag{36}$$

and (35) can be rewritten in the form

$$\sum_{j=n}^{\infty}(j+m-1-n-r)^{(m-r-1)}u_j =$$
$$= n^{-r}\prod_{t=-m+1+n+r}^{n-1}\Big(1-\frac{t}{n}\Big)Q_n+\sum_{j=n}^{\infty}\Delta\Big[j^{-r}\prod_{t=-m+1+n+r}^{n-1}\Big(1-\frac{t}{j}\Big)\Big]Q_{j+1},$$

And again, because the series on the left-hand side is convergent, so that on the right-hand side of (35) is convergent, too.

Applying the formula for the difference of a product, we can rewrite (35) in the form

$$\begin{aligned}\sum_{j=n}^{\infty}(j+m-1-n-r)^{(m-1-r)}u_j &= n^{-r}\prod_{t=-m+1+n+r}^{n-1}\Big(1-\frac{t}{n}\Big)Q_n+\\ &+\sum_{j=n}^{\infty}Q_{j+1}\prod_{t=-m+1+n+r}^{n-1}\Big(1-\frac{t}{j+1}\Big)\Delta j^{-r}+\\ &+\sum_{j=n}^{\infty}j^{-r}\Delta\Big[\prod_{t=-m+1+n+r}^{n-1}\Big(1-\frac{t}{j}\Big)\Big]Q_{j+1}.\end{aligned} \tag{37}$$

One can observe that

$$\begin{aligned}\sum_{j=n}^{\infty}\Big[\Delta\prod_{t=-m+1+n+r}^{n-1}\Big(1-\frac{t}{j}\Big)\Big] &= \lim_{s\to\infty}\sum_{j=n}^{s}\Delta\Big[\prod_{t=-m+1+n+r}^{n-1}\Big(1-\frac{t}{j}\Big)\Big]=\\ &= \lim_{s\to\infty}\Big[\prod_{t=-m+1+n+r}^{n-1}(1-\frac{t}{s+1})-\prod_{t=-m+1+n+r}^{n-1}\Big(1-\frac{t}{n}\Big)\Big]=\\ &= 1-\prod_{t=-m+1+n+r}^{n-1}(1-\frac{t}{n}).\end{aligned}$$

Hence, the series $\sum_{j=n}^{\infty}\Delta[\prod_{t=-m+1+n+r}^{n-1}(1-\frac{t}{j})]$ is convergent. Its terms being positive, it is absolutely convergent. The sequence $\{j^{-1}Q_{j+1}\}$ is bounded, so the series

$$\sum_{j=n}^{\infty} j^{-r} Q_{j+1} \Delta \Big[\prod_{t=-m+1+n+r}^{n-1} \Big(1 - \frac{t}{j}\Big)\Big]$$

is absolutely convergent.

From equality (37) it follows that the series

$$\sum_{j=n}^{\infty} Q_{j+1} \prod_{t=-m+1+n+r}^{n-1} \Big(1 - \frac{t}{j+1}\Big) \Delta j^{-r}$$

is convergent, too. Further, applying (37), we have

$$\Big|\sum_{j=n}^{\infty} (j+m-1-n-r)^{(m-1-r)} u_j\Big| \le n^{-r} \prod_{t=-m+1+n+r}^{n-1} \Big(1 - \frac{t}{n}\Big) |Q_n| +$$
$$+ \sum_{j=n}^{\infty} |Q_{j+1}| \Big| \prod_{t=-m+1+n+r}^{n-1} \Big(1 - \frac{t}{j+1}\Big)\Big| |\Delta j^{-r}| +$$
$$+ \sum_{j=n}^{\infty} j^{-r} \Big|\Delta \Big[\prod_{t=-m+1+n+r}^{n-1} \Big(1 - \frac{t}{j}\Big)\Big]\Big| |Q_{j+1}|.$$

Since the sequence $\{|Q_n|\}_{n=n_0}^{\infty}$ is bounded from above (see (33)) by a nonincreasing (see (8)) sequence $\{\Phi_n \varrho_n\}$, we have $|Q_j| \le \varrho_n \Phi_n$ and $|Q_{j+1}| \le \varrho_{j+1}\Phi_{j+1} \le \varrho_n \Phi_n$ for $j \ge n$. The sequence $\{\prod_{t=-m+1+n+r}^{n-1}(1 - \frac{t}{j}\}_{j=n}^{\infty}$ is nondecreasing positive and bounded from above by 1. Furthermore $\{n^{-r}\}_{n=n_0}^{\infty}$ is a positive nonincreasing sequence for $0 \le r \le m-1$, so we get

$$\Big|\sum_{j=n}^{\infty} (j+m-1-n-r)^{(m-1-r)} u_j\Big| \le$$
$$\le n^{-r} \Phi_n \varrho_n \Big[\prod_{t=-m+1+n+r}^{n-1} \Big(1 - \frac{t}{n}\Big)\Big] +$$
$$+ \sum_{j=n}^{\infty} |\Delta j^{-r}| \Phi_n \varrho_n \Big[\prod_{t=-m+1+n+r}^{n-1} \Big(1 - \frac{t}{j+1}\Big)\Big] +$$
$$+ \sum_{j=n}^{\infty} j^{-r} \Phi_n \varrho_n \Delta \Big[\prod_{t=-m+1+n+r}^{n-1} \Big(1 - \frac{t}{j}\Big)\Big] \le$$
$$\le n^{-r} \Phi_n \varrho_n \Big[\prod_{t=-m+1+n+r}^{n-1} \Big(1 - \frac{t}{n}\Big)\Big] + \Phi_n \varrho_n \Big[- \sum_{j=n}^{\infty} \Delta j^{-r} \Big] +$$

$$+ n^{-r}\Phi_n\varrho_n \sum_{j=n}^{\infty} \Delta\Big[\prod_{t=-m+1+n+r}^{n-1}\Big(1-\frac{t}{j}\Big)\Big] =$$

$$= n^{-r}\Phi_n\varrho_n\Big\{\prod_{t=-m+1+n+r}^{n-1}(1-\frac{t}{n})+1+1+$$

$$-\prod_{t=-m+1+n+r}^{n-1}\Big(1-\frac{t}{n}\Big)\Big\} = 2n^{-r}\Phi_n\varrho_n.$$

Finally, we obtain

$$|\Delta^r w_n| \leq \frac{1}{(m-1-r)!}\Big|\sum_{j=n}^{\infty}(j+m-1-n-r)^{(m-1-r)}u_j\Big| \leq \frac{2n^{-r}\Phi_n\varrho_n}{(m-1-r)!}$$

for $0 \leq r \leq m-1$. Now, in order to prove (12), we state that

$$\frac{|\Delta^r w_n|}{\Phi_n n^{-r}} \leq \frac{2\varrho_n}{(m-r-1)!},$$

because the sequence Φ_n is positive and $n > 0$. By the assumption (11) of our Lemma, we get $\lim_{n\to\infty}\varrho_n = 0$.

Hence

$$\frac{|\Delta^r w_n|}{\Phi_n n^{-r}} \to 0, \quad \text{as } n\to\infty,$$

so $\Delta^r w_n = o(\Phi_n n^{-r})$ for $0 \leq r \leq m-1$. This completes the proof of Lemma.

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INSTITUTE OF MATHEMATICS,
TECHNICAL UNIVERSITY OF POZNAŃ,
Piotrowo 3a
60-963 POZNAŃ, POLAND

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