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A CHARACTERIZATION OF LOW DEGREE POLYNOMIALS

This paper concerns with the solution of a functional equation that characterizes low degree polynomials. The method used for solving the equation is simple and elementary. Here, an answer is also provided to a problem posed by Walter Rudin in the *American Mathematical Monthly* [11] in a general setting.

1. Introduction

Let R be the set of all real numbers. A function $A : R \to R$ is said to be an *additive function* on reals if

$$A(x+y) = A(x) + A(y)$$

for all real numbers x and y. There are many papers dealing with the various aspects of additive functions. A comprehensive review on additive functions can be found in [9].

It is well known that for quadratic polynomials the Mean Value Theorem takes the form

$$\frac{f(x)-f(y)}{x-y} = f'\left(\frac{x+y}{2}\right). \tag{1}$$

It was shown in [1] (and also [6]) that the solution of the functional differential equation (1) is of the form $f(x) = ax^2 + bx + c$, where a, b and c are arbutrary real constants. This has the following interpretation. Let $f(t)$ denote the position of a moving object at time t. If the mean velocity

$$\frac{f(y)-f(x)}{y-x}$$

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during every interval $[x, y]$ is equal to the velocity $f'(\frac{x+y}{2})$ at the arithmetic mean $\frac{x+y}{2}$ of the end points x and y of the interval $[x, y]$, then the trajectory of the object is a parabola or a line. The above functional equation (1) is a special case of the following equation

$$\frac{f(x) - f(y)}{x - y} = \phi(\eta(x, y)),$$

where $\eta(x, y)$ is an *apriori* known function of x and y. Note that this equation contains no derivative and no mean value. In a recent paper, Aczel and Kuczma [2] have determined the solution of the above functional equation assuming $\eta(x, y)$ to be either arithmetic, or harmonic or geometric mean of x and y. Several authors (see [1], [2], [6], [8]) have treated equations of this type. If $\eta(x, y) = x + y$, then the above equation characterizes quadratic polynomials. For a generalization of (1) to higher derivatives the reader may refer to [3], [7] and [8]. For a generalization of (1) that characterizes polynomials of degree at most n the reader may refer to [3], [4], and [7]. The motivation behind the study of the above functional equations for characterizing polynomials can be found in [1], [2], [3], [5], [6], [7], [8], [10], [12] and references therein.

In the *American Mathematical Monthly* [11] the following problem was proposed by Walter Rudin: *"Let s and t be given real numbers. Find all differentiable functions f on the real line which satisfy*

$$\text{(RE)} \qquad f'(sx + ty) = \frac{f(y) - f(x)}{y - x}$$

for all real x,y, with $x \neq y$." Note that any solution of (RE) is intrinsically differentiable. In fact, if f is a solution of (RE), then $f \in C^\infty(R)$. The equation given by Rudin is a special case of the following functional equation

$$\text{(2)} \qquad g(sx + ty) = \frac{f(y) - f(x)}{y - x}$$

for all $x, y \in R$ with $x \neq y$. The solution of (2) was given by Baker [13] in the following theorem. *If s,t are given real numbers, then the necessary and sufficient condition for f,g to satisfy $f(y) - f(x) = (y - x)g(sx + ty)$ for all x,y is as follows: If $s = t = 0$ or $s^2 \neq t^2$, then f is a polynomial of degree at most 1 and $g = f'$. If $s = t \neq 0$, then f is a polynomial of degree at most 2 and $g(x) = f'(x/2t)$. If $s = -t \neq 0$, then $f(x) = a + A(x)$ and $g(x) = A(x/t)/(x/t)$ for $x \neq 0$, where a is a real constant and A is an additive function.* Also, the solution of (2) (and also of (RE)) was obtained independently by the last two authors of this paper.

It is the purpose of this paper to present an elementary and simple technique for determining solution of the following functional equation:

$$\text{(FE)} \qquad \frac{f(x)-g(y)}{x-y} = h(sx+ty)$$

for all real x, y with $x \neq y$. Here s and t are *a priori* known real parameters. This equation generalizes (2) and characterizes polynomials of low degrees.

2. The solution of the functional equation

Now we proceed to find the general solution of (FE) with no regularity assumptions (differentiability, continuity, measurability, etc.) imposed on h, g and f.

THEOREM 1. *Let s and t be the real parameters. Functions $f, g, h : R \to R$ satisfy (FE) for all $x, y \in R, x \neq y$ if and only if*

$$f(x) = \begin{cases} ax+b & \text{if } s=0=t \\ ax+b & \text{if } s=0,\ t\neq 0 \\ \alpha t x^2 + ax + b & \text{if } s=t\neq 0 \\ \dfrac{A(tx)}{t} + b, & \text{if } s=-t\neq 0 \\ \beta x + b & \text{if } s^2 \neq t^2 \end{cases}$$

$$g(y) = \begin{cases} ay+b & \text{if } s=0=t \\ ay+b & \text{if } s=0,\ t\neq 0 \\ \alpha t y^2 + ay + b & \text{if } s=t\neq 0 \\ \dfrac{A(ty)}{t} + c, & \text{if } s=-t\neq 0 \\ \beta y + b & \text{if } s^2 \neq t^2 \end{cases}$$

$$h(y) = \begin{cases} \text{arbitrary with } h(0)=a & \text{if } s=0=t \\ a & \text{if } s=0,\ t\neq 0 \\ \alpha y + a & \text{if } s=t\neq 0 \\ \dfrac{A(y)}{y} + \dfrac{(c-b)t}{y}, & \text{if } s=-t\neq 0,\ y\neq 0 \\ \beta & \text{if } s^2 \neq t^2, \end{cases}$$

where $A : R \to R$ is an additive function and a, b, c, α, β are arbitrary real constants.

Proof. To prove the theorem, we consider several cases depending on parameters s and t.

Case 1. Suppose $s = 0 = t$. Then (FE) yields

$$\frac{f(x) - g(y)}{x - y} = h(0)$$

which is

$$f(x) - ax = g(y) - ay,$$

where $a := h(0)$. From the above, we obtain

$$f(x) = ax + b \quad \text{and} \quad g(y) = ay + b, \tag{3}$$

where b is an arbitrary constant. Letting (2) into (FE), we see that h is an arbitrary function with $a = h(0)$. Thus we obtain the solution as asserted in theorem for the case $s = 0 = t$.

Case 2. Suppose $s = 0$ and $t \neq 0$. Then from (FE), we get

$$\frac{f(x) - g(y)}{x - y} = h(ty). \tag{4}$$

Putting $y = 0$ in (4), we see that

$$f(x) = ax + b, \qquad x \neq 0 \tag{5}$$

where $a = h(0)$ and $b = g(0)$. Letting (5) into (4), we obtain

$$ax + b - g(y) = (x - y)h(ty) \tag{6}$$

for all $x \neq y$ and $x \neq 0$. Equating the coefficients of x and the constant terms in (6), we get

$$h(ty) = a \quad \text{and} \quad g(y) = h(ty)y + b = ay + b \tag{7}$$

for all $y \in R$. Letting $x = 0$ in (4) and using (7), we see that $f(0) = b$. Thus (5) holds for all x in $\mathbf{R}$. From (5) and (7), we get the solution of the (FE) for this case as asserted in Theorem 1.

Case 3. Suppose $s \neq 0 \neq t$. Letting $x = 0$ in (FE), we get

$$g(y) = yh(ty) + b \tag{8}$$

for all $y \neq 0$ (where $b := f(0)$). Similarly, letting $y = 0$ in (FE), we get

$$f(x) = xh(sx) + c \tag{9}$$

for all $x \neq 0$ (where $c := g(0)$). Inserting (8) and (9) into (FE) and simplifying, we obtain

$$xh(sx) - yh(ty) + c - b = (x - y)h(sx + ty) \tag{10}$$

for all real nonzero x and y with $x \neq y$.

Replacing x by $\frac{x}{s}$ and y by $\frac{y}{t}$ in (10), we get

$$\frac{x}{s}h(x) - \frac{y}{t}h(y) + c - b = \left(\frac{x}{s} - \frac{y}{t}\right)h(x+y) \tag{11}$$

for all real nonzero x and y with $tx \neq sy$.

Subcase 3.1. Suppose $s = t$. Hence (11) yields

$$xh(x) - yh(y) = (b-c)t + (x-y)h(x+y). \tag{12}$$

Interchanging x with y in (12), we get $b = c$ and (12) reduces to

$$xh(x) - yh(y) = (x-y)h(x+y) \tag{13}$$

for all real nonzero x and y with $x \neq y$. Replacing y with $-y$ in (13), we obtain

$$xh(x) + yh(-y) = (x+y)h(x-y) \tag{14}$$

for all real nonzero x and y with $x + y \neq 0$. Letting $y = -x$ in (13), we see that

$$xh(x) + xh(-x) = 2xh(0). \tag{15}$$

Subtracting (14) from (13) and using (15), we get

$$2yh(0) = (x+y)h(x-y) - (x-y)h(x+y) \tag{16}$$

for all real nonzero x, y with $x + y$ and $x - y \neq 0$. Writing

$$u = x + y \quad \text{and} \quad v = x - y \tag{17}$$

in (16), we see that

$$(u-v)h(0) = uh(v) - vh(u)$$

which is

$$v[h(u) - h(0)] = u[h(v) - h(0)], \tag{18}$$

for all real nonzero $u, v, u - v$ and $u + v$. Thus

$$h(u) = \alpha u + a \tag{19}$$

for all real nonzero u in R (where $a := h(0)$). Notice that (19) also holds for $u = 0$. Using (19) in (FE), we get

$$f(x) - g(y) = (x-y)(\alpha tx + \alpha ty + a)$$

for all $x \neq y$. Thus, we obtain the asserted solution

$$f(x) = g(x) = \alpha tx^2 + ax + b \quad \text{and} \quad h(y) = \alpha y + a, \tag{20}$$

where α, a and b are arbitrary constants.

Subcase 3.2. Suppose $s = -t$. Then (11) yields

$$xh(x) + yh(y) + (b-c)t = (x+y)h(x+y) \tag{21}$$

for all real nonzero x and y with $x \neq y$. Define

$$A(x) = \begin{cases} xh(x) + (b-c)t & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases} \tag{22}$$

Then by (22), (21) reduces to

$$A(x) + A(y) = A(x+y) \tag{23}$$

for all real nonzero x, y and $x+y$. Next we show that A in (23) is additive on the set of reals. In order for A to be additive it must satisfy

$$\begin{cases} A(x) + A(-x) = A(0) = 0 \\ \text{or} \\ xh(x) - xh(-x) + 2(b-c)t = 0. \end{cases} \tag{24}$$

Interchanging y with $-y$ in (21), we get

$$xh(x) - yh(-y) + (b-c)t = (x-y)h(x-y). \tag{25}$$

Subtracting (25) from (21), we get

$$yh(y) + yh(-y) = (x+y)h(x+y) - (x-y)h(x-y).$$

Thus, using (22), we get

$$A(y) - A(-y) = A(x+y) - A(x-y) \tag{26}$$

for all real nonzero x, y, $x+y$ and $x-y$. Replacing x by $-x$ in (26), we obtain

$$A(y) - A(-y) = A(-x+y) - A(-x-y). \tag{27}$$

From (26) and (27), we get

$$A(x+y) + A(-(x+y)) = A(x-y) + A(-(x-y)). \tag{28}$$

Letting $u = x+y$ and $v = x-y$ in (28), we see that

$$A(u) + A(-u) = A(v) + A(-v)$$

for all real nonzero $u, v, u-v$ and $u+v$. Thus

$$A(u) + A(-u) = \gamma \tag{29}$$

for all real nonzero u (where γ is a constant). Using (22), we see from (29) that

$$xh(x) - xh(-x) + 2(b-c)t = \gamma, \tag{30}$$

for all real nonzero x. From (FE) with $s = -t$, we get

$$f(x) - g(y) = (x-y)h(-(x-y)t). \tag{31}$$

Interchanging x with y, we get

$$f(y) - g(x) = -(x-y)h((x-y)t). \tag{32}$$

Adding (31) to (32) and using (30), we get

$$(33)\quad \begin{aligned} & f(x) - g(x) + f(y) - g(y) = \\ & = -(x-y)h((x-y)t) + (x-y)h(-(x-y)t) = -\frac{\gamma}{t} + 2(b-c). \end{aligned}$$

Using (8) and (22), we obtain

$$(34)\qquad A(tx) = t[g(x) - c] \quad (x \neq 0).$$

Similarly, using (9) and (22), we get

$$(35)\qquad A(-tx) = -t[f(x) - b] \quad (x \neq 0).$$

So from (34) and (35), we see that

$$f(x) - g(x) = -\frac{A(-tx) + A(tx)}{t} + b - c = -\frac{\gamma}{t} + b - c.$$

Hence from above, we get

$$(36)\qquad f(x) - g(x) + f(y) - g(y) = -2\frac{\gamma}{t} + 2(b-c).$$

Comparing (33) with (36), we get $\gamma = 0$. Thus (29) yields

$$A(x) + A(-x) = 0,$$

for all real nonzero x. Evidently the above also holds for $x = 0$. Hence A is an additive function on the set of reals. From (22), (8) and (9), we obtain

$$(37)\qquad \begin{aligned} & f(x) = \frac{A(tx)}{t} + b, \quad g(y) = \frac{A(ty)}{t} + c \quad \text{and} \\ & h(y) = \frac{A(y)}{y} + \frac{(c-b)t}{y}, \end{aligned}$$

where b and c are arbitrary constants.

Subcase 3.3. Suppose $s^2 \neq t^2$, that is $s \neq \pm t$. Interchanging x with y in (11), we get

$$(38)\qquad \frac{y}{s}h(y) - \frac{x}{t}h(x) + c - b = \left(\frac{y}{s} - \frac{x}{t}\right)h(x+y)$$

for all nonzero x and y with $ty \neq sx$. Subtracting (38) from (11) and using $s + t \neq 0$, we get

$$(39)\qquad xh(x) - yh(y) = (x-y)h(x+y),$$

which is same as (13). Thus

$$(40)\qquad h(x) = \alpha x + b,$$

where α and b are arbitrary constants. Letting (40) into (38) and simplifying the resulting expression, we get

$$\alpha xy\left(\frac{1}{s}-\frac{1}{t}\right)=b-c$$

for all nonzero x and y with $tx \neq sy$ and $sx \neq ty$. Since $s \neq t$, we see that $\alpha = 0$ and $b = c$. Hence (40) becomes

$$h(x) = b. \tag{41}$$

From (41), (8) and (9), we obtained the asserted form of f, g and h. This completes the proof of the theorem.

Remark 1. In case of the functional equation (2) (that is when $g = f$), Subcase 3.2 simplifies to a great extent. If $g = f$, then the left side of the (FE) for $s = -t$ is symmetric in x and y. Thus using this symmetry one can conclude that h is an even function. The evenness of h implies that A in (23) is additive.

Remark 2. In Subcase 3.1, $h(y)$ is undefined at $y = 0$.

Remark 3. It is well known that the functional equation $A(x + y) = A(x) + A(y)$ has nonmeasurable solutions in addition to the continuous solution of the form $A(x) = ax$, where a is an arbitrary real constant. Since, additive function appears in the solution of (FE) for Subcase $s = -t$, it follows that (FE) has non-measurable solutions. However, all measurable solutions of (FE) are continuous and polynomials of low degree.

The following theorem is obvious from the Theorem 1.

THEOREM 2. *Functions $\phi, f : R \to R$ satisfy functional equation* (2) *for all $x, y \in R$ with $x \neq y$ if and only if*

$$f(x) = \begin{cases} ax + b & \text{if } s = 0 = t \\ ax + c & \text{if } s = 0,\ t \neq 0 \\ \alpha t x^2 + ax + b & \text{if } s = t \neq 0 \\ \dfrac{A(tx)}{t} + b, & \text{if } s = -t \neq 0 \\ \beta x + b & \text{if } s^2 \neq t^2 \end{cases}$$

$$\phi(y) = \begin{cases} \text{arbitrary with } \phi(0) = a & \text{if } s = 0 = t \\ a & \text{if } s = 0,\ t \neq 0 \\ \alpha y + y & \text{if } s = t \neq 0 \\ \dfrac{A(y)}{y}, & \text{if } s = -t \neq 0,\ y \neq 0 \\ \beta & \text{if } s^2 \neq t^2, \end{cases}$$

where $A : R \to R$ is an additive function and a, b, c, α, β are arbitrary real constants.

The following corollary addresses the problem proposed by Walter Rudin in [11].

COROLLARY 3. *The function $f : R \to R$ satisfies the equation*

$$f'(sx + ty) = \frac{f(y) - f(x)}{y - x}$$

for all $x, y \in R$ with $x \neq y$ if and only if

$$f(x) = \begin{cases} ax^2 + bx + c & \text{if } s = \frac{1}{2} = t \\ bx + c & \text{otherwise,} \end{cases}$$

where a, b and c are arbitrary real constants.

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