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# Y-VALUED SOLUTIONS FOR SEMILINEAR GENERALIZED WAVE EQUATION

## 1. Introduction

Let $R^n$ be $n$-dimensional Euclidean space, $n \in \mathbb{N}$ and $\mathbb{N}$ the set of integers.

ASSUMPTION 1. Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$ be a bounded domain with boundary $\Gamma$ of class $C^{2m}$, $m \in \mathbb{N}$ and closure $\overline{\Omega}$. Let $t \in \langle 0, T\rangle$, $T \in \mathbb{R}^+ = \langle 0, \infty)$ and let functions $\phi, \psi : \Omega \to \mathbb{R}$ and $a_{pq} : \overline{\Omega} \times \langle 0, T\rangle \to \mathbb{R}$ be given, $p$, $q$ are multiindices

$$p = (p_1, p_2, \dots, p_m), \quad q = (q_1, q_2, \dots, q_m), \quad |p| = \sum_{i=1}^{m} p_i, |q| = \sum_{i=1}^{m} q_i.$$

Let $H^s(\Omega)$ and $H_0^s(\Omega)$ be Sobolev spaces with the norms $\|.\|_{H^s}, s \in \mathbb{R}$. Let $A_0(t, x, D)$, $t \in \langle 0, t\rangle$, be a family linear elliptic operators [5] of order $2m, m \in \mathbb{N}$, in the divergence form

$$A_0(t, x, D) = \sum_{|p|=|q|=0}^{m} (-1)^{|p|} D^p(a_{pq}(x, t) D^q). \tag{1}$$

ASSUMPTION 2. $a_{pq} \in C^{2m,1}(\overline{\Omega} \times \langle 0, T\rangle)$, $a_{pq} = a_{qp}$, $|p| \leq m$, $|q| \leq m$.

ASSUMPTION 3. The operators $A_0(t, x, D)$, $t \in \langle 0, T\rangle$, are uniformly strongly elliptic in $\Omega$, i.e. there is a constant $c > 0$ that

$$\sum_{|p|=|q|=0}^{m} (-1)^{|p|} a_{pq}(x, t) \xi^p \xi^q \geq c|\xi|^{2m},$$

for every $x \in \overline{\Omega}$, $t \in \langle 0, T\rangle$, $\xi \in \mathbb{R}^n$.

Then the operators $A_0(t, x, D)$ for $t \in \langle 0, T\rangle$ satisfy Gårding's inequality, i.e. there exist constants $C_1 \geq 0$ such that

$$a(v,v,t) \geq C_1\|v\|^2_{H^m(\Omega)} - C_2\|v\|^2_{L_2(\Omega)}$$

for any $v \in H_0^m(\Omega)$ and $t \in \langle 0,T\rangle$, where the bilinear form $a(v,w,t)$ being given by the formula

(2) $$a(v,w,t) = \sum_{|p|=|q|=0}^{m} \int_\Omega a_{pq}(x,t)D^p v(x)D^q w(x)dx, \qquad w,v \in H^m(\Omega).$$

If $C_2 \neq 0$ we can replace the operators $A_0(t,x,D)$ by the operators $A(t,x,D) = A_0(t,x,D) + \lambda\mathbb{I}$, where $\mathbb{I}$ — the identity operator, and $\lambda \geq C_2$. Then for any $v \in W_0^m(\Omega)$ and $t \in \langle 0,T\rangle$

(3) $$a(v,v,t) \geq C_1\|v\|^2_{H^m(\Omega)}.$$

In this paper existence of solutions of the following equation

(E) $$u_{tt}+A(t,x,D)u = f(t,x,u,u_t,Du,\ldots,D^{m-1}u), \quad x \in \Omega, \quad t \in \langle 0,T\rangle,$$

with the initial conditions

(IC) $$u(x,0) = \phi(x), \quad u_t(x) = \psi(x), \quad x \in \Omega,$$

and the boundary conditions

(BC) $$D^\beta u|_\Gamma = 0, \quad \text{for } |\beta| \leq m-1,\ t \in \langle 0,T\rangle,\ \beta = (\beta_1,\ldots,\beta_n)$$

has been investigated.

Now the problem (E), (IC), (BC) will be set in an abstract form and the theory of an evolution system $U(t,s)$, $0 \leq s \leq t \leq T$, on a certain Banach space will be applied in order to find solutions. This provides us with the existence and uniqueness of solutions in the sense of this Banach space [4], [5].

With the elliptic operators $A(t,x,D)$, $t \in \langle 0,T\rangle$, we associate linear operators $A(t)$, $t \in \langle 0,T\rangle$, in $L^2(\Omega)$. This is done as follows:

$$D(A(t)) = D = H^{2m}(\Omega) \cap H_0^m(\Omega) \quad \text{and} \quad A(t)u = A(t,x,D)u \quad \text{for } u \in D.$$

It is obvious that $D$ is dense in $L^2(\Omega)$.

LEMMA 1. *By Assumptions* 1–3 *and density of $D$ in $L^2(\Omega)$ we have*:

(i) *the operator $A(t)$, for every $t \in \langle 0,T\rangle$, can be extended to self-adjoint operator (proof* [6], *p.* 126);

(ii) *for any $\alpha \in (0,1)$, the operator $A^\alpha(t)$ is self-adjoint and $D_\alpha = D(A^\alpha(t))$ is also independent of $t$ (proof* [5], *p.* 109); *for $\alpha = 1/2$, in our case $D_{1/2} = D(A^{1/2}(t)) = H_0^m(\Omega)$*;

(iii) $a(t,v,w) = (A^{1/2}(t)v, A^{1/2}(t)w)$ *(proof* [5], *p.* 29).

We can set the problem (E), (IC), (BC) as the abstract initial value problem

$$u_{tt} + A(t)u = f_1(t, u, u_t), \tag{4}$$

$$u(0) = \phi, u_t(0) = \psi, \tag{5}$$

where $f_1(t, u, u_t)(x) = f(t, x, u, u_t, Du, \ldots, D^{m-1}u)$.

Next, the problem (4), (5) can be written in the form

$$\frac{dw}{dt} = \mathbb{A}(t)w + F(t, w), \tag{6}$$

$$w_0 = w(0) = \begin{pmatrix} \phi \\ \psi \end{pmatrix}, \tag{7}$$

where $\mathbb{A} = \begin{pmatrix} 0 & \mathbb{I} \\ -A(t) & 0 \end{pmatrix}$, $F(t, w) = \begin{pmatrix} 0 \\ f_1(t, w) \end{pmatrix}$, $w = \begin{pmatrix} u \\ u_t \end{pmatrix}$.

Let $\mathbb{H}_0 = H_0^m(\Omega) \times L_2(\Omega)$ and $D(\mathbb{A}(t)) = \mathbb{D} := (H^{2m}(\Omega) \cap H_0^m(\Omega)) \times H_0^m(\Omega)$, for $t \in \langle 0, T \rangle$. The space $\mathbb{H}_0$ is the Hilbert space with scalar product

$$(w_1, w_2)_{\mathbb{H}_0} = (A^{1/2}(t)v_1, A^{1/2}(t)v_2) + (z_1, z_2),$$

of $w_1 = \begin{pmatrix} v_1 \\ z_1 \end{pmatrix}$, $w_2 = \begin{pmatrix} v_2 \\ z_2 \end{pmatrix}$.

## 2. An evolution system

Let $X$ be a Banach space with the norm $\|.\|$. For every $t \in \langle 0, T \rangle$, let $A(t) : D(A(t)) \subset X \to X$ be a linear operator in $X$ and $f : \langle 0, T \rangle \times X \to X$ be a function.

Definition 1. A two parameter family $\{U(t, s)\}$ of bounded linear operators $U(t, s)$, $0 \le s \le t \le T$, on $X$, is called an evolution system if the following two conditions are satisfied:

(i) $U(s, s) = \mathbb{I}$, $U(t, r)U(r, s) = U(t, s)$ for $0 \le s \le r \le t \le T$,
(ii) $(t, s) \to U(t, s)$ is strongly continuous for $0 \le s \le t \le T$.

Definition 2. A family $\{A(t)\}$, $t \in \langle 0, T \rangle$, of infinitesimal generators of $C_0$ semigroups on $X$ is called stable if there are constants $M \ge 1$ and $\omega \in R$ such that resolvent set satisfies conditions:

$$\varrho(A(t)) \supset (0, \infty) \qquad \text{for } t \in \langle 0, T \rangle$$

and

$$\Big\| \bigcap_{j=1}^{k} R(\lambda; A(t_j)) \Big\| \le M(\lambda - \omega)^{-k}$$

for $\lambda > \omega$ and for every finite sequence $0 \le t_1 \le t_2 \le \ldots \le t_k \le T$, $k = 1, 2, \ldots$; $M$ and $\omega$ are called the stability constants.

R e m a r k 1. Any family $\{A(t)\}$, $t \in \langle 0,T\rangle$, of infinitesimal generators of $C_0$ semigroups of contractions is stable ([4], p. 131).

DEFINITION 3. Let $X$ and $Y$ be Banach spaces with norms $\|.\|$ and $\|.\|_Y$, respectively. $Y$ is densely and continuously imbedded in $X$, if $Y$ is dense subspace of $X$ and there is a constant $C > 0$ such that $\|w\| \leq C\|w\|_Y$ for $w \in Y$.

DEFINITION 4. Let $X$ be a Banach space and $S : D(S) \subset X \to X$ a linear operator in $X$. The subspace $Y$ of $X$ is an invariant subspace of $S$ if $S : D(S) \cap Y \to Y$.

DEFINITION 5. Let $A$ be a infinitesimal generator of $C_0$ semigroup $S(s)$, $s \in \mathbb{R}^+$. A subspace $Y$ of $X$ is called $A$-admissible, if it is an invariant subspace of $S(s)$, $s \in \mathbb{R}^+$, and the restriction of $S(s)$ to $Y$ is a $C_0$ semigroup in $Y$ (i.e., it is strongly continuous in the norm $\|.\|_Y$).

LEMMA 2. *Let, for each $t \in \langle 0,T\rangle$, $A(t)$ be the infinitesimal generator of $C_0$ semigroup $S_t(s)$, $s \in \mathbb{R}^+$, on $X$. The following conditions* (H1)–(H3) (*usually referred to as the "hyperbolic" case*):

(H1) *$\{A(t)\}$, $t \in \langle 0,T\rangle$, is a stable family with stability constants $M, \omega$,*

(H2) *$Y$ is $A(t)$-admissible, for $t \in \langle 0,T\rangle$, and the family $\{\widetilde{A}(t)\}$, $t \in \langle 0,T\rangle$, of the parts of $A(t)$ in $Y$, is a stable family in $Y$ with stability constants $\widetilde{M}$, $\widetilde{\omega}$,*

(H3) *for $t \in \langle 0,T\rangle$, $Y \subset D(A(t))$, $A(t)$ is a bounded operator from $Y$ into $X$ and $t \to A(t)$ is continuous in $B(Y,X)$ norm $\|.\|_{Y\to X}$; guarantee existence of a unique evolution system $U(t,s)$, $0 \leq s \leq t \leq T$, in $X$ satisfying*

(E1) $$\|U(t,s)\| \leq M \exp[\omega(t-s)] \qquad \text{for } 0 \leq s \leq t \leq T,$$

(E2) $$\frac{\partial^+}{\partial t} U(t,s)v|_{t=s} = A(s)v \qquad \text{for } v \in Y,\ 0 \leq s \leq T,$$

(E3) $$\frac{\partial}{\partial s} U(t,s)v = -U(t,s)A(s)v \qquad \text{for } v \in Y,\ 0 \leq s \leq t \leq T,$$

*where the right-hand derivative in* (E2) *and the derivative in* (E3) *are in the strong sense in $X$* (*proof* [4], *p.* 135).

LEMMA 3. *Let $\{A(t)\}$, $t \in \langle 0,T\rangle$, be a stable family of infinitesimal generators of $C_0$ semigroup on $X$. If $D(A(t)) = D$ is independent of $t$ and $A(t)v$ is continuously differentiable in $X$ for $v \in D$, then there exists a unique evolution system $U(t,s)$, $0 \leq s \leq t \leq T$, satisfying* (E1)–(E3) *and*

(E4) *$U(t,s)Y \subset Y$ for $0 \leq s \leq t \leq T$,* (E5) *$U(t,s)v$ is continuous in $Y$ for $0 \le s \le t \le T$ and $v \in Y$, where $Y$ is $D$ equipped with the norm $\|v\|_Y = \|v\| + \|A(0)v\|$, for $v \in Y = D$. (proof* [4], *p.* 145, *see also* [1], [3]).

THEOREM 1. *If Assumptions* 1–3 *are satisfied, then the family $\{\mathbf{A}(t)\}$ of operators $\mathbf{A}(t)$, $t \in \langle 0,T\rangle$, is generator of a unique evolution system on the space $\mathbf{H}_0$ having the properties* (E1)–(E5).

Proof. At first we will prove that $\mathbf{A}(t)$ is a dissipative operator for each $t \in \langle 0,T\rangle$. By Lemma 1, we have

$$[\mathbf{A}(t)w,w]_{\mathbf{H}_0} = \left[\begin{pmatrix} 0 & \mathbb{I} \\ -A(t) & 0 \end{pmatrix}\begin{pmatrix} v \\ z \end{pmatrix}, \begin{pmatrix} v \\ z \end{pmatrix}\right]_{\mathbf{H}_0} = \left[\begin{pmatrix} z \\ -A(t)v \end{pmatrix}, \begin{pmatrix} v \\ z \end{pmatrix}\right]_{\mathbf{H}_0} =$$
$$= (A^{1/2}(t)z, A^{1/2}(t)v) + (-A(t)v,z) = (z, A(t)v) - (A(t)v,z) =$$
$$= (A(t)v,z) - (A(t)v,z) = 0.$$

Thus $\mathbf{A}(t)$ is dissipative for each $t \in \langle 0,T\rangle$.

Now we will prove that for any $\lambda > 0$ the range of $(\lambda E - \mathbf{A}(t))$, for each $t \in \langle 0,T\rangle$ is all of $\mathbf{H}_0$, $E = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. We have to prove that equation $(\lambda E - \mathbf{A}(t))w = F$, $F = \begin{bmatrix} h \\ g \end{bmatrix}$, $w = \begin{bmatrix} v \\ z \end{bmatrix}$ has a solution $w \in (H^{2m}(\Omega) \cap H_0^m(\Omega)) \times H_0^m(\Omega)$ for any $(h,g) \in H_0^m(\Omega) \times L^2(\Omega)$. This equation is equivalent to the system of equations $(A(t) + \lambda^2\mathbb{I})v = g + \lambda h$, $z = \lambda v - h$. It is clear that $g + \lambda h \in L^2(\Omega)$. Due to (3) the bilinear form (2) is coercive for $t \in \langle 0,T\rangle$ with $C_2 = 0$. We may therefore apply the Lax–Milgram theorem and derive the existence of a unique weak solution $v \in H_0^m(\Omega)$ of boundary value problem $(A(t) + \lambda^2\mathbb{I})v = g + \lambda h$ for $\lambda > 0$ ([2], p. 43). The coefficients $a_{pq}(x,t)$ of $A(t,x,D)$ (Assumption 2) and the boundary $\Gamma$ (Assumption 1) are smooth enough that we can apply regularization theory ([2], p. 67). So we obtain $v \in H^{2m}(\Omega)$ and finally $v \in H^{2m}(\Omega) \cap H_0^m(\Omega)$. From the equation $z = \lambda v - h$ and the condition $h \in H_0^m(\Omega)$ we have $z \in H_0^m(\Omega)$. This means that $\mathbf{A}(t)$ is maximal operator for each $t \in \langle 0,T\rangle$. It is also clear that $D(\mathbf{A}(t)) = (H^{2m}(\Omega) \cap H_0^m(\Omega)) \times H_0^m(\Omega)$ is dense in $\mathbf{H}_0$.

All assumptions of Lummer–Phillips Theorem ([4], p. 14) are satisfied, so $\{\mathbf{A}(t)\}$, $t \in \langle 0,T\rangle$, is a family of the infinitesimal generators of $C_0$ semigroups of contractions on $\mathbf{H}_0$, i.e.,

$$\|T_t(s)\| \le 1 \quad \text{for } s \in \mathbb{R}^+ \quad \text{and} \quad t \in \langle 0,T\rangle.$$

By Remark 1, the family $\{\mathbf{A}(t)\}$, $t \in \langle 0,T\rangle$, is stable with stability constants $M = 1$ and $\omega = 0$. It is obvious that $D(\mathbf{A}(t))$ are independent of $t \in \langle 0,T\rangle$. Assumption 2 implies that $\mathbf{A}(t)w$ is continuously differentiable in $\mathbf{H}_0$ for

$w \in \mathbb{D}$. All assumptions of Lemma 3 are satisfied, then there exists a unique evolution system $U(t,s)$, $0 \le s \le t \le T$, on $\mathbb{H}_0$ satisfying (E1)–(E5).

## 3. Existence of solutions

At first we recall some definitions.

DEFINITION 1. A function $w : \langle 0,T\rangle \to \mathbb{H}_0$ is said to be a mild solution of the problem (6), (7) if $w \in C(\langle 0,T\rangle, \mathbb{H}_0)$ for any $w_0 \in \mathbb{H}_0$, and $w$ satisfies the following integral equation

$$w(t) = U(t,0)w_0 + \int_0^t U(t,s)F(s,w(s))\,ds, \qquad 0 \le t \le T.$$

DEFINITION 2. A function $w : \langle 0,T\rangle \to \mathbb{H}_0$ is said to be a $\mathbb{Y}$-valued solution of the problem (6), (7), if $w \in C(\langle 0,T\rangle;\mathbb{Y}) \cap C^1((0,T>;\mathbb{H}_0)$ and equation (6) is satisfied in $\mathbb{H}_0$.

The set $\mathbb{Y}$ is domain $D(\mathbb{A}(t)) = \mathbb{D} = (H^{2m}(\Omega) \cap H_0^m(\Omega)) \times H_0^m(\Omega)$ with the norm $\|w\|_{\mathbb{Y}} = \|w\|_{\mathbb{H}_0} + \|\mathbb{A}(0)\|_{\mathbb{H}_0}$ for $w \in \mathbb{D}$.

DEFINITION 3. A function $F : \langle 0,T\rangle \times \mathbb{H}_0 \to \mathbb{H}_0$ is said to be Lipschitz continuous in $w$, uniformly in $t \in \langle 0,T\rangle$, with constant $L > 0$, if $\|F(t,w_2) - F(t,w_1)\|_{\mathbb{H}_0} \le L\|w_2 - w_1\|_{\mathbb{H}_0}$ for every $t \in \langle 0,T\rangle$, $w_1, w_2 \in \mathbb{H}_0$.

DEFINITION 4. A function $F : \langle 0,\infty) \times \mathbb{H}_0 \to \mathbb{H}_0$ is said to be locally Lipschitz continuous in $w$, uniformly in $t$ on bounded intervals, if for every constants $r \ge 0$, $\tau \ge 0$ there exists a constant $L(r,\tau)$ such that

$$\|F(t,w_2) - F(t,w_1)\|_{\mathbb{H}_0} \le L(r,\tau)\|w_2 - w_1\|_{\mathbb{H}_0}$$

for every $t \in \langle 0,\tau\rangle$ and $w_1, w_2 \in \mathbb{H}_0$ with $\|w_1\|_{\mathbb{H}_0} \le r$, $\|w_2\|_{\mathbb{H}_0} \le r$.

THEOREM 2. *If Assumptions* 1–3 *are satisfied and the function* $F$ : $\langle 0,T\rangle \times \mathbb{H}_0 \to \mathbb{H}_0$ *is continuous in* $t \in \langle 0,T\rangle$ *and Lipschitz continuous in* $w$*, uniformly in* $t \in \langle 0,T\rangle$*, then for every* $w_0 \in \mathbb{H}_0 = H_0^m(\Omega) \times L_2(\Omega)$ *there exists a unique mild solution* $w \in C(\langle 0,T\rangle; H_0^m(\Omega) \times L_2(\Omega))$ *of the problem* (6), (7).

It has been proved in general case in [4], [5].

THEOREM 3. *If Assumptions* 1–3 *are satisfied and function* $F : \langle 0,\infty) \times \mathbb{H}_0 \to \mathbb{H}_0$ *is continuous in* $t$ *for* $t \ge 0$ *and locally Lipschitz continuous in* $w$*, uniformly in* $t$ *on bounded intervals, then for every* $w_0 \in \mathbb{H}_0$ *there exists a unique mild solution* $w \in C(\langle 0,t_{\max}), H_0^m(\Omega) \times L_2(\Omega))$ *of the problem (6), (7) with either* $t_{\max} = \infty$ *or* $t_{\max} < \infty$*. Moreover, if* $t_{\max} < \infty$ *then* $\lim_{t\to t_{\max}} \|w(t)\|_{\mathbb{H}_0} = \infty$.

Proof. The proof is similar to that of Theorem 1.4 ([4], p. 185), but in our case we have to put $M_2(t_0) = \max\{\|U(t,s)\|;\ 0 \le s \le t \le t_0 + 1\}$ and use integral equation

$$w(t) = U(t,t_0)w_0 + \int_{t_0}^{t} U(t,s)F(s,w(s))\,ds.$$

THEOREM 4. *If Assumptions 1–3 are satisfied and function $F : \langle 0,T\rangle \times \mathbb{Y} \to \mathbb{Y}$ is Lipschitz continuous in $\mathbb{Y}$, uniformly in $t \in \langle 0,T\rangle$ and for each $w \in \mathbb{Y}$ continuous from $\langle 0,T\rangle$ into $\mathbb{Y}$, then for $w_0 \in \mathbb{Y}$ the problem* (6), (7) *has a unique $\mathbb{Y}$-values solution on $\langle 0,T\rangle$, i.e.*

$$w \in C(\langle 0,T\rangle;(H^{2m}(\Omega)\cap H_0^m(\Omega))\times H_0^m(\Omega))\cap C^1((0,T>;H_0^m(\Omega)\times L_2(\Omega)).$$

Proof. In a standard way we can prove existence of the mild solution $\omega \in C(\langle 0,T\rangle;\mathbb{Y})$ which satisfies the integral equation

$$\omega(t) = U(t,0)w_0 + \int_0^t U(t,s)F(s,\omega(s))\,ds$$

in $\mathbb{Y}$ and a fortiori in $\mathbb{H}_0$ for a given $w_0 \in \mathbb{Y}$.

Let $g(s) = F(s,\omega(s))$, $s \in \langle 0,T\rangle$. Then, by the assumptions of our theorem, it follows that $g(s) \in \mathbb{Y}$ for $s \in \langle 0,T\rangle$ and $g \in C(\langle 0,T\rangle;\mathbb{Y})$. Theorem 5.2 ([4], p. 146) guarantees existence of a unique $\mathbb{Y}$-valued solution $w$ on $\langle 0,T\rangle$ for the linear problem

$$(8)\qquad \begin{cases} \dfrac{dw}{dt} + \mathbf{A}(t)w = g(t), \\ w(0) = w_0 \end{cases}$$

for $g \in C(\langle 0,T\rangle;\mathbb{Y})$ and $w_0 \in \mathbb{Y}$. This solution is then clearly also a mild solution of (8) and therefore

$$w(t) = U(t,0)w_0 + \int_0^t U(t,s)g(s)\,ds =$$

$$= U(t,0)w_0 + \int_0^t U(t,s)F(s,\omega(s))\,ds = \omega(t).$$

So $w = \omega$ and $\omega$ is a $\mathbb{Y}$-valued solution of problem (6), (7) on $\langle 0,T\rangle$.

THEOREM 5. *If Assumptions 1–3 are satisfied and the function $F : \langle 0,T\rangle \times \mathbb{Y} \to \mathbb{Y}$ is continuous in $t, T \in \mathbb{R}^+$, and locally Lipschitz continuous in $\mathbb{Y}$, uniformly in $t$ on $\langle 0,T\rangle$, then for every $w_0 \in \mathbb{Y}$ the problem* (6), (7) *has a unique $\mathbb{Y}$-valued solution*

$$w \in C(\langle 0, t_{\max}); (H^{2m}(\Omega) \cap$$
$$\cap H_0^m(\Omega)) \times H_0^m(\Omega)) \cap C^1((0, t_{\max}); H_0^m(\Omega) \times L^2(\Omega))$$

*on a maximal interval* $\langle 0, t_{\max})$ $t_{\max} \le T$. *Moreover, if* $t_{\max} < T$, *then*

$$\lim_{t \to t_{\max}} [\|w(t)\|_{\mathbb{H}_0} + \|\mathbf{A}(0)w(t)\|_{\mathbb{H}_0}] = \infty.$$

Proof. The proof of this theorem is similar to that of Theorem 4. It has been also used the results of Theorem 3.

Remark 2. Similar results, as in Theorems 2–4, have been obtained in the papers [1], [3], under a little bit weaker assumptions.

Due to Yamaguchi's Theorem ([7], Appendix) the following Lemma holds.

LEMMA 4. *Let* $f(t, x, a)$ *be defined on* $\langle 0, T\rangle \times \overline{\Omega} \times R^{N+1}$, $a = (a_0, a_1, \ldots$ $\ldots, a_N) \in R^{N+1}$, $N := \frac{(n+m-1)!}{n!(m-1)!}$. *Let* $f(x, t, a)$ *be of* $C^{sm+1}$*-class in* $(x, a) \in$ $\overline{\Omega} \times R^{N+1}$ *and* $D_x^k D_a^l f(x, t, a)$, $0 \le k + l \le sm + 1$, *be continuous in* $t$ *on* $t \in \langle 0, T\rangle$, $s \in \mathbb{N}$ *and* $s > [\frac{n-2}{2m}] + 1$. *Let* $B(Q) = \{a : a \in R^{N+1}; |a_i| \le Q, i = 0, 1, 2, \ldots, N\}$, *where* $Q$ *is some positive real number. Set*

$$h_Q(t) = \max_{0 \le k+l \le sm+1} \sup_{x \in \overline{\Omega}\, a \in B(Q)} |D_x^k D_a^l f(t, x, a)| \quad \text{and denote}$$

$$H_Q = \{u \in C(\langle 0, T\rangle, H^{sm}(\Omega)) \cap C^1(\langle 0, T\rangle, H^{(s-1)m}),$$
$$|D^\beta u(t, x)| \le Q, \quad |\beta| \le m - 1, \quad |u_t(t, x)| \le Q\}.$$

*Then the following assertions hold:*
(L1) *there exists a positive constant* $C_1$ *and a function* $h_Q : \langle 0, T\rangle \to \langle 0, T\rangle$ *defined above such that for any* $u \in H_Q$ *we have*

$$\|f(t, \ldots, u(t,.), u_t(t,.), Du(t,.), \ldots, D^{m-1}u(t,.))\|_{\mathbb{H}^{s-1}} \le$$
$$\le C_1 h_Q(t)[\|u(t,.)\|_{H^{sm}} + \|u_t(t,.)\|_{H^{(s-1)m}} + 1],$$

(L2) *there exists a positive constant* $C_2$ *such that for any* $u(t,.), \widetilde{u}(t,.) \in H_Q$ *satisfying the conditions*

$$[\|u(t,.)\|_{H^{sm}} + \|u_t(t,.)\|_{H^{(s-1)m}}] \le C_3,$$
$$[\|\widetilde{u}(t,.)\|_{H^{sm}} + \|\widetilde{u}_t(t,.)\|_{H^{(s-1)m}}] \le C_3,$$

*for some positive constant* $C_3$, *we have*

$$\|f(t,.,\widetilde{u}(t,.), \widetilde{u}_t(t,.), D\widetilde{u}(t,.), \ldots, D^m\widetilde{u}(t,.)) +$$

$$- f(t,.,u(t,.),u_t(t,.),Du(t,.),\dots,D^{m-1}u(t,.))\|_{\mathbb{H}^{s-1}} \leq$$

$$\leq C_2 h_Q(t)[\|\widetilde{u}(t,.) - u(t,.)\|_{H^{sm}} + \|\widetilde{u}_t(t,.) - u_t(t,.)\|_{H^{(s-1)m}}].$$

Remark 3. If we denote

$$F = \begin{pmatrix} 0 \\ f \end{pmatrix}, \quad w(t) = \begin{pmatrix} u(t,.) \\ u_t(t,.) \end{pmatrix}, \quad \widetilde{w}(t) = \begin{pmatrix} \widetilde{u}(t,.) \\ \widetilde{u}_t(t,.) \end{pmatrix}$$

we obtain from Lemma 4 the following conditions

$$(\widetilde{\mathrm{L}}1) \qquad \|F(t,w(t,.))\|_{\mathbb{H}_s} = \|f\|_{H^{(s-1)m}} \leq C_1 h_Q(t)[\|w(t,.)\|_{\mathbb{H}_s} + 1],$$

$$(\widetilde{\mathrm{L}}2) \qquad \|F(t,\widetilde{w}(t,.)) - F(t,w(t,.))\|_{\mathbb{H}_s} \leq C_2 h_Q(t)\|\widetilde{w}(t,.) - w(t,.)\|_{\mathbb{H}_s},$$

where $\mathbf{H}_s = H^{sm}(\Omega) \times H^{(s-1)m}(\Omega)$.

The condition $(\widetilde{\mathrm{L}}1)$ means that, if $w(t,.) \in \mathbb{H}_s$ then $F(t,w(t,.)) \in \mathbf{H}_s$ for $t \in \langle 0,T\rangle$. The condition $(\widetilde{\mathrm{L}}2)$ means that the function $F$ is locally Lipschitz continuous in $w$ with respect to norm $\mathbb{H}_s$.

THEOREM 6. *Let*

(i) $f \in C^{2m+1}$, *and a function* $f$ *satisfies assumptions of Lemma* 4 *with* $s = 2$,

(ii) *for any* $x \in \partial\Omega$, $t \in \langle 0,T\rangle$ $D_x^k D_a^l f(x,t,0) = 0$ *for* $k + l \leq m - 1$,

(iii) *Assumptions* 1–3 *are satisfied*,

*then for* $(\phi,\psi) \in (H^{2m}(\Omega) \cap H_0^m(\Omega)) \times H_0^m(\Omega)$ *there exists a unique local* $\mathbb{Y}$-*valued solution of the problem* $(E)$, $(IC)$, $(BC)$, *i.e.*

$$u \in C(\langle 0,t_{\max}); H^{2m}(\Omega) \cap$$
$$\cap H_0^m(\Omega)) \cap C^1(\langle 0,t_{\max}), H_0^m(\Omega)) \cap C^2(\langle 0,t_{\max}), L_2(\Omega))).$$

*If* $t_{\max} < T$, *then*

$$\lim_{t\to t_{\max}} [\|u(t,.)\|_{H_0^m(\Omega} +$$
$$+ \|A(0)u(t,.)\|_{L_2} + \|u_t(t,.)\|_{H_0^m(\Omega)} + \|u_t(t,.)\|_{L_2(\Omega)}] = \infty.$$

Proof. The norm $\|F(t,w)\|_{\mathbb{Y}}$ is equivalent to $\|F(t,w)\|_{H_2}$. Lemma 4 with $s = 2$ $(1 \leq n < 2m + 2)$ and condition (ii) guarantee that $F : \langle 0,T\rangle \times \mathbb{Y} \to \mathbb{Y}$ and it is locally Lipschitz continuous in $\mathbb{Y}$, uniformly in $t$ on $\langle 0,T\rangle$. So all assertions of Theorem 5 are satisfied and this implies the thesis of our theorem.

## References

[1] J. Bochenek: *An abstract semilinear first order differential equation in the hyperbolic case* (manuscript 1993).
[2] A. Friedman: *Partial Differential Equations*, New York, 1969.
[3] M. Kozak: *An abstract nonlinear temporally inhomogeneous equation*, Demonstratio Math., 23 (1990), 993–1003.
[4] A. Pazy: *Semigroups of linear operations and applications to partial equations*, Springer-Verlag, 1983.
[5] H. Tanabe: *Equations of Evolution*, Pitman, 1979.
[6] E. Zeidler: *Nonlinear Functional Analysis and its Applications* II.*A*, Springer-Verlag, 1990.
[7] M. Yamaguchi: *Existence and stability of global bounded classical solutions of initial boundary value problem for semilinear wave equations*, Funkcialaj Ekwacioj, 23 (1980), 289–308.

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