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PRODUCT FINAL DIFFERENTIAL STRUCTURES ON THE PLANE AND PRINCIPAL-DIRECTED CURVES

Product final differential structures $\mathcal{S}^1$ and $\mathcal{S}^2$ on $\mathbb{R}^2$ were defined in paper [2]. The differential spaces $\mathbb{R}\times_1\mathbb{R} = (\mathbb{R}^2, \mathcal{S}^1)$ and $\mathbb{R}\times_2\mathbb{R} = (\mathbb{R}^2, \mathcal{S}^2)$ have many common properties and they can be considered together as the differential space $\mathbb{R}\times_k\mathbb{R} = (\mathbb{R}^2, \mathcal{S}^k)$ where $k = 1$ or $k = 2$. In the above-mentioned paper it was proved that every regular curve in $\mathbb{R}\times_k\mathbb{R}$ is contained in a principal line, i.e. a straight line which is vertical or horizontal. This leads to a characterization of such curves as regular ones in $\mathbb{R}^2$ which are contained in principal lines. It is easily seen that every smooth curve in $\mathbb{R}\times_k\mathbb{R}$ is smooth in $\mathbb{R}^2$, but not conversely in general. In this paper we present a characterization of arbitrary smooth curves in $\mathbb{R}\times_k\mathbb{R}$ as some smooth ones in $\mathbb{R}^2$ (Theorem 2.20). It turns out that the characterization obtained does not depend on k (Corollary 2.21).

In Section 1 we first observe that every smooth curve in $\mathbb{R}\times_k\mathbb{R}$ is principal-directed in $\mathbb{R}^2$, however, there are smooth curves in $\mathbb{R}^2$ which are not smooth in $\mathbb{R}\times_k\mathbb{R}$ (Example 2.22). For this reason, we start with considerations of principal-directed curves in $\mathbb{R}^2$. Next, we distinguish and study certain classes of such curves, especially, the class of locally K-subordinate curves which is exactly that of all smooth ones in $\mathbb{R}\times_k\mathbb{R}$ (Theorem 2.20).

In Section 2 we introduce in different ways the classes of locally K-subordinate sets in $\mathbb{R}^2$ and of C^∞ subsets of $\mathbb{R}\times_k\mathbb{R}$. We prove that these classes are identical (Theorem 2.15) and show that they can be used for a characterization of smooth curves in $\mathbb{R}\times_k\mathbb{R}$. Moreover, it turns out that such curves are proper for a characterization of smooth maps from $\mathbb{R}\times_k\mathbb{R}$ to $\mathbb{R}\times_\ell\mathbb{R}$ where $k, \ell \in \{1,2\}$ (Propositions 2.24 and 2.25). By definitions, the class of principal-directed curves (locally K- subordinate sets) in $\mathbb{R}^2$ and its subclasses considered here do not depend on $\mathbb{R}\times_k\mathbb{R}$. Since the major part of this paper is devoted to the study of such classes, therefore this portion of our paper has respect to the classical differential geometry on the plane.

Clearly, one may generalize considerations from this paper to those for $\mathbb{R}^n$ where $n > 2$ (compare [2], Section 5). It seems that such generalizations can, to a considerable extent, be obtained as the corresponding combinatorial n-variants with respect to our case $n = 2$. However, we must be careful whether direct generalized properties can hold since the topological and differential structures of $\mathbb{R}^n$ ($n > 2$) are much more complicated than the corresponding ones of the plane.

1. Locally K-subordinate curves

In what follows, $k = 1, 2$ is fixed but arbitrary. First, we recall the definition of the differential structure $\mathcal{S}^k$ on $\mathbb{R}^2$ (see [2]). For any $a, b \in \mathbb{R}$ consider the maps $i_b : \mathbb{R} \to \mathbb{R}^2$ and $j_a : \mathbb{R} \to \mathbb{R}^2$ defined by $i_b(x) = (x, b)$ and $j_a(y) = (a, y)$. Let us denote by $\mathcal{F}^k$ the family of all real functions (real continuous functions) on $\mathbb{R}^2$ when $k = 1$ ($k = 2$). We set

$$\mathcal{S}^k = \{\alpha \in \mathcal{F}^k : \alpha \circ i_b \in C^\infty(\mathbb{R}) \wedge \alpha \circ j_a \in C^\infty(\mathbb{R})\ \forall a, b \in \mathbb{R}\}.$$

It is seen that $\mathcal{S}^k$ is a differential structure on $\mathbb{R}^2$ and the differential space $\mathbb{R} \times_k \mathbb{R}$ is defined to be the pair $(\mathbb{R}^2, \mathcal{S}^k)$. We shall regard $\mathbb{R} \times_k \mathbb{R}$ as a topological space under the Sikorski topology defined to be the weakest one on $\mathbb{R}^2$ in which all functions from $\mathcal{S}^k$ are continuous (see [3], §14).

By an interval of $\mathbb{R}$ we will mean a nonsingle (i.e. nonsingle-element) connected subspace of $\mathbb{R}$. Every curve in $\mathbb{R}^2$ is assumed to be a continuous map $c : I \to \mathbb{R}^2$ where $I = \mathrm{dom}(c)$ is an interval of $\mathbb{R}$. A map $c : I \to \mathbb{R} \times_k \mathbb{R}$ is called a *smooth curve* in $\mathbb{R} \times_k \mathbb{R}$ if it is a smooth map of differential spaces where $I = \mathrm{dom}(c)$ is an interval of $\mathbb{R}$ regarded as a differential space under the natural structure induced from $\mathbb{R}$. Since $\mathcal{S}^k$ contains all real smooth functions on $\mathbb{R}^2$, it follows that every smooth curve in $\mathbb{R} \times_k \mathbb{R}$ is smooth in $\mathbb{R}^2$ (in the usual sense).

Let $c = (\alpha, \beta) : I \to \mathbb{R}^2$ be a smooth curve, that is, $\alpha, \beta \in C^\infty(I)$, we define the k-th derivative of c at $s \in I$ to be the vector $(D^k c)(s) = [D^k(\alpha)(s), D^k(\beta)(s)]$. By $\dot{c} = [\dot{\alpha}, \dot{\beta}] = D^1 c$ will be denoted the *canonical vector field tangent* to c. We call c *regular* (*stationary*) at s if $\|\dot{c}(s)\| > 0$ ($\|\dot{c}(s)\| = 0$), where $\|\dot{c}(s)\| = (\dot{\alpha}(s)^2 + \dot{\beta}(s)^2)^{1/2}$. Denote by $\mathrm{dom}_R(c)$ ($\mathrm{dom}_S(c)$) the set of all regular (stationary) parameters of c. If $\mathrm{dom}_R(c) = \mathrm{dom}(c)$ ($\mathrm{dom}_S(c) = \mathrm{dom}(c)$), then c is called *regular* (*totally stationary*). We say that c is *completely stationary* at s or that s is a *singular parameter* of c if $(D^k c)(s) = [0, 0]$ for all $k \geq 1$. The set of all such parameters of c will be denoted by $\mathrm{dom}_{CS}(c)$. Obviously, $\mathrm{dom}_R(c)$ is an open subset of $\mathrm{dom}(c)$ but $\mathrm{dom}_S(c)$ and $\mathrm{dom}_{CS}(c)$ are closed subsets of $\mathrm{dom}(c)$. We call c

V-directed (*H-directed*) at s in case the vector $\dot{c}(s)$ is vertical (horizontal). Moreover, c is called *P-directed* at s if it is V-directed or H-directed at s. If $X \in \{V, H, P\}$, we denote by $\mathrm{dom}_X(c)$ the set of all parameters s of c such that c is X-directed at s. It is seen that $\mathrm{dom}_X(c)$ is a closed subset of $\mathrm{dom}(c)$. We say that c is *X-directed* in case $\mathrm{dom}_X(c) = \mathrm{dom}(c)$. Moreover, a P-directed curve will also be called *principal-directed.* These definitions immediately imply

1.1. LEMMA. *If c is a smooth curve in $\mathbb{R}^2$, then the following equalities hold*:

(a) $\mathrm{dom}_V(c) \cup \mathrm{dom}_H(c) = \mathrm{dom}_P(c)$;
(b) $\mathrm{dom}_V(c) \cap \mathrm{dom}_H(c) = \mathrm{dom}_S(c)$. ∎

Clearly, this lemma implies

1.2. COROLLARY. *Every regular P-directed curve in $\mathbb{R}^2$ is V-directed or H-directed.* ∎

If $X \in \{V, H\}$, then by an *X-principal line* we shall mean a straight line in $\mathbb{R}^2$ which is vertical if $X = V$ and horizontal if $X = H$. In turn, by a *(P-)principal line* we shall mean a straight line in $\mathbb{R}^2$ which is vertical or horizontal.

Let $X \in \{V, H, P\}$. A curve c in $\mathbb{R}^2$ is called *locally X-subordinate* at a parameter s if there are a neighbourhood U of s in $\mathrm{dom}(c)$ and an X-principal line L such that $c(U) \subseteq L$. The set of all such parameters of c will be denoted by $\mathrm{loc}_X(c)$. Obviously, $\mathrm{loc}_X(c)$ is an open subset of $\mathrm{dom}(c)$. Let the symbol int stand for the interior operation in $\mathrm{dom}(c)$. By an easy verification we get

1.3. LEMMA. *If c is a smooth curve in $\mathbb{R}^2$, then the following conditions hold*:

(a) $\mathrm{loc}_V(c) = \mathrm{int}\,\mathrm{dom}_V(c)$;
(b) $\mathrm{loc}_H(c) = \mathrm{int}\,\mathrm{dom}_H(c)$;
(c) $\mathrm{loc}_P(c) \subseteq \mathrm{int}\,\mathrm{dom}_P(c)$;
(d) $\mathrm{loc}_V(c) \cup \mathrm{loc}_H(c) = \mathrm{loc}_P(c)$;
(e) $\mathrm{loc}_V(c) \cap \mathrm{loc}_H(c) = \mathrm{int}\,\mathrm{dom}_S(c) = \mathrm{int}\,\mathrm{dom}_{CS}(c)$. ∎

We say that c is *locally X-subordinate* in case $\mathrm{loc}_X(c) = \mathrm{dom}(c)$. From Lemma 1.3 it follows immediately

1.4. COROLLARY. *If $X \in \{V, H, P\}$, then every smooth locally X-subordinate curve in $\mathbb{R}^2$ is X-directed. Conversely, if $X \in \{V, H\}$, then every X-directed curve in $\mathbb{R}^2$ is locally X-subordinate.* ∎

Throughout this paper in several constructions we use the following real smooth function ϑ on $\mathbb{R}$ defined by

$$\vartheta(t) = \begin{cases} 0 & \text{for } t \leq 0 \\ \exp(-1/t) & \text{for } t > 0. \end{cases} \tag{1.1}$$

The following example shows that a P-directed curve need not be locally P-subordinate, which means that the inclusion in condition (c) of Lemma 1.3 is essential.

1.5. EXAMPLE. Let $c : \mathbb{R} \to \mathbb{R}^2$ be a smooth curve defined by

$$c(t) = \begin{cases} (\vartheta(-t), 0) & \text{for } t < 0 \\ (0,0) & \text{for } t = 0 \\ (0, \vartheta(t)) & \text{for } t > 0. \end{cases}$$

Clearly, c is P-directed but not locally P-subordinate at 0. ∎

Let $X \in \{V, H, P\}$. A curve $c : I \to \mathbb{R}^2$ is called *globally X-subordinate* if there is an X-principal line L such that $c(I) \subseteq L$. It is easy to verify

1.6. PROPOSITION. *If $X \in \{V, H\}$ and c is a smooth curve in $\mathbb{R}^2$, then the following conditions are equivalent*:

(a) *c is X-directed*;
(b) *c is locally X-subordinate*;
(c) *c is globally X-subordinate.* ∎

Note that from Corollary 1.2 and Proposition 1.6 we get

1.7. COROLLARY. *If c is a P-directed curve in $\mathbb{R}^2$, then* $\mathrm{dom}_R(c) \subseteq \mathrm{loc}_P(c)$. *More precisely, c restricted to any connected component of* $\mathrm{dom}_R(c)$ *is globally P-subordinate, so every regular P-directed curve is globally P-subordinate.* ∎

Obviously, every globally P-subordinate curve is locally P-subordinate, but conversely this need not be satisfied.

1.8. EXAMPLE. Let $c : \mathbb{R} \to \mathbb{R}^2$ be a smooth curve defined by

$$c(t) = \begin{cases} ((\vartheta(-1-t), 0) & \text{for } t < -1 \\ (0,0) & \text{for } -1 \leq t \leq 0 \\ (0, \vartheta(t)) & \text{for } t > 0. \end{cases}$$

Clearly, c is locally P-subordinate. Moreover, observe that the image of c is contained in $(\mathbb{R} \times \{0\}) \cup (\{0\} \times \mathbb{R})$ but it is not contained in $\mathbb{R} \times \{0\}$ or $\{0\} \times \mathbb{R}$, separately. ∎

Let c be a smooth curve in $\mathbb{R}^2$. By a *nonsingular parameter* of c we shall mean any element of the set

$$\mathrm{dom}_{NS}(c) = \mathrm{dom}(c) \setminus \mathrm{dom}_{CS}(c).$$

We say that c is *nonsingular* (*almost regular*) if $\mathrm{dom}_{CS}(c) = \emptyset$ ($\mathrm{int}\,\mathrm{dom}_{CS}(c) = \emptyset$). Since $\mathrm{int}\,\mathrm{dom}_S(c) = \mathrm{int}\,\mathrm{dom}_{CS}(c)$ by Lemma 1.3, c is almost regular if and only if $\mathrm{int}\,\mathrm{dom}_S(c) = \emptyset$, i.e. $\mathrm{cl}\,\mathrm{dom}_R(c) = \mathrm{dom}(c)$. Lemma 1.3 and Proposition 1.6 imply

1.9. PROPOSITION. *Every nonsingular (almost regular) locally P-subordinate curve in $\mathbb{R}^2$ is globally P-subordinate.* ∎

1.10. LEMMA. *Every nonsingular P-directed curve in $\mathbb{R}^2$ is locally P-subordinate.*

Proof. Let c be a nonsingular P-directed curve in $\mathbb{R}^2$ and let $s \in \mathrm{dom}(c)$. First, if s is regular, there is a neighbourhood U of s such that the curve $c' = c|U$ is regular and P-directed. Then, by Corollary 1.7, c' is globally P-subordinate, which means that c is locally P-subordinate at s.

Suppose now that s is not regular, which means that $s \in \mathrm{dom}_S(c) \cap \mathrm{dom}_{NS}(c)$ because c is nonsingular. Without loss of generality, we can assume further that $s = 0$. Clearly, there is the least positive integer $k \geq 1$ such that $(D^k\dot{c})(0) \neq 0$. Thus, if $c = (\alpha, \beta)$, then $(D^k\dot{c})(0) = [(D^k\dot{\alpha})(0), (D^k\dot{\beta})(0)] \neq 0$. We can assume that $(D^k\dot{\alpha})(0) \neq 0$, so there is $\varepsilon > 0$ such that

$$(D^k\dot{\alpha})(t) \neq 0 \text{ for } t \in U = (-\varepsilon, \varepsilon).$$

Since $(D^i\dot{\alpha})(0) = 0$ for $i < k$, by the Taylor formula we have

$$\dot{\alpha}(t) = \frac{(D^k\dot{\alpha})(\theta t)}{k!}t^k \text{ for } t \in U$$

where $\theta = \theta(t) \in (0;1)$, whence $\dot{\alpha}(t) \neq 0$ for $t \in U\backslash\{0\}$. Therefore and since c is P-directed, we have $\dot{\beta}(t) = 0$ for $t \in$ U. Thus, $c|U$ is globally V-subordinate, so $0 \in \mathrm{loc}_P(c)$. ∎

Clearly, this Lemma and Proposition 1.9 imply

1.11. PROPOSITION. *Every nonsingular P-directed curve in $\mathbb{R}^2$ is globally P-subordinate.* ∎

Note that this proposition implies the following corollary being a generalization of Corollary 1.7.

1.12. COROLLARY. *If c is a P-directed curve in $\mathbb{R}^2$, then $\mathrm{dom}_{NS}(c) \subseteq \mathrm{loc}_P(c)$. More precisely, c restricted to any connected component of $\mathrm{dom}_{NS}(c)$ is globally P-subordinate.* ∎

Applying this Corollary and Proposition 1.6 one has

1.13. PROPOSITION. *If $X \in \{V, H, P\}$ and c is a smooth curve in $\mathbb{R}^2$, then the following conditions are equivalent:*

(a) $\mathrm{dom}(c) = \mathrm{dom}_X(c)$;
(b) $\mathrm{dom}_{NS}(c) \subseteq \mathrm{dom}_X(c)$;
(c) $\mathrm{dom}_R(c) \subseteq \mathrm{dom}_X(c)$;
(d) $\mathrm{dom}_{NS}(c) \subseteq \mathrm{loc}_X(c)$;
(e) $\mathrm{dom}_R(c) \subseteq \mathrm{loc}_X(c)$. ∎

Remark that since condition (a) of Proposition 1.13 means that c is X-directed, we can regard the other ones as characterizations of X-directed curves among smooth curves in $\mathbb{R}^2$.

For any $a, b \in \mathbb{R}$, we put $V_a = \{a\} \times \mathbb{R}$ and $H_b = \mathbb{R} \times \{b\}$. By a *principal cross* we shall mean a subset K of $\mathbb{R}^2$ of the form $\mathbb{K}_p = V_a \cup H_b$ where $p = (a, b)$ is called the *origin* of K. The principal cross $\mathbb{K} = \mathbb{K}_o$ with origin $o = (0,0)$ will also be called the *central principal* one. A curve c in $\mathbb{R}^2$ is called *locally K-subordinate* at a parameter s if there are a neighbourhood U of s in $\mathrm{dom}(c)$ and a principal cross K such that $c(U) \subseteq K$. The set of all such parameters will be denoted by $\mathrm{loc}_K(c)$. We say that c is *locally K-subordinate* provided that $\mathrm{loc}_K(c) = \mathrm{dom}(c)$. Clearly, every locally P-subordinate curve is locally K-subordinate, but not conversely in general. However, every smooth locally K-subordinate curve is P-directed. By an easy verification we get

1.14. PROPOSITION. *If $X \in \{P, K\}$ and if c is a P-directed curve in $\mathbb{R}^2$, then the following conditions are equivalent*:

(a) $\mathrm{dom}(c) = \mathrm{loc}_X(c)$;
(b) $\mathrm{dom}_S(c) \subseteq \mathrm{loc}_X(c)$;
(c) $\mathrm{dom}_{CS}(c) \subseteq \mathrm{loc}_X(c)$. ∎

One can see that this proposition can be false in the case when $X \in \{V, H\}$. For example, the horizontal curve $c = (\mathrm{id}_{\mathbb{R}}, 0) : \mathbb{R} \to \mathbb{R}^2$ satisfies $\mathrm{dom}_S(c) = \mathrm{dom}_{CS}(c) = \mathrm{loc}_V(c) = \emptyset$ but $\mathrm{dom}(c) = \mathbb{R} \neq \mathrm{loc}_V(c) = \emptyset$. However, we have

1.15. PROPOSITION. *If $X \in \{V, H\}$ and if c is a P-directed curve in $\mathbb{R}^2$, then the following conditions are equivalent*:

(a) $\mathrm{dom}(c) = \mathrm{loc}_X(c)$;
(b) $\mathrm{dom}_S(c) \subseteq \mathrm{loc}_X(c) \neq \emptyset$;
(c) $\mathrm{dom}_{CS}(c) \subseteq \mathrm{loc}_X(c) \neq \emptyset$.

Proof. Since the cases $X = V$ and $X = H$ are completely analogous, we can assume further that $X = V$. Obviously, it remains to prove the implication (c) $\Rightarrow$ (a), or equivalently, the statements (1) and (2) below.

(1) *If* $\emptyset = \mathrm{dom}_{CS}(c) \subseteq \mathrm{loc}_V(c) \neq \emptyset$, *then* $\mathrm{dom}(c) = \mathrm{loc}_V(c)$.

Indeed, we have $\mathrm{dom}_{CS}(c) \subseteq \mathrm{loc}_V(c) \subseteq \mathrm{loc}_P(c)$ and from Proposition 1.14 for $X = P$ it follows that $\mathrm{dom}(c) = \mathrm{loc}_P(c)$, and so, $\mathrm{loc}_V(c) \cup \mathrm{loc}_H(c) = \mathrm{dom}(c)$ by Lemma 1.3(d). Furthermore, from condition (e) of this lemma we get $\mathrm{loc}_V(c) \cap \mathrm{loc}_H(c) = \emptyset$. Therefore, since $\mathrm{loc}_V(c)$ and $\mathrm{loc}_H(c)$ are open in the connected space $\mathrm{dom}(c)$ and $\mathrm{loc}_V(c) \neq \emptyset$, we have $\mathrm{dom}(c) = \mathrm{loc}_V(c)$.

(2) *If* $\emptyset \neq \mathrm{dom}_{CS}(c) \subseteq \mathrm{loc}_V(c)$, *then* $\mathrm{dom}(c) = \mathrm{loc}_V(c)$.

Observe first that this statement is trivial in the case when $\mathrm{dom}_{CS}(c) = \mathrm{dom}(c)$. Therefore, we can assume further that $\mathrm{dom}_{CS}(c) \neq \mathrm{dom}(c)$, i.e. $\mathrm{dom}_{NS}(c) \neq \emptyset$. Let us take a parameter $s \in \mathrm{dom}_{NS}(c)$. Consider the sets $F^- = \{t \in \mathrm{dom}_{CS}(c) : t < s\}$ and $F^+ = \{t \in \mathrm{dom}_{CS}(c) : t > s\}$. Note that F^- and F^+ are closed disjoint subsets of $\mathrm{dom}(c)$ such that $F^- \cup F^+ = \mathrm{dom}_{CS}(c)$. Without loss of generality we can assume that $F^- \neq \emptyset$. Let us set $t^- = \max F^-$ and $t^+ = \min F^+$ if $F^+ \neq \emptyset$. Consider the interval I defined to be $(t^-;t^+)$ if $F^+ \neq \emptyset$ and $(t^-;+\infty) \cap \mathrm{dom}(c)$ if $F^+ = \emptyset$. Of course, $I \cap \mathrm{dom}_{CS}(c) = \emptyset$ and I is a neighbourhood of s in $\mathrm{dom}(c)$. Let $d = c|I$ and note that d is a P-directed curve such that $\mathrm{dom}_{CS}(d) = \emptyset$. Moreover, observe that $\mathrm{loc}_V(d) \neq \emptyset$ because $\mathrm{loc}_V(d) = \mathrm{loc}_V(c) \cap I$ and $t^- \in \mathrm{dom}_{CS}(c) \subseteq \mathrm{loc}_V(c)$, which means that d satisfies the assumption of statement (1). Therefore, by statement (1) we have $\mathrm{dom}(d) = \mathrm{loc}_V(d)$, whence $s \in \mathrm{loc}_V(d) \subseteq \mathrm{loc}_V(c)$ and since s can be an arbitrary point of $\mathrm{dom}_{NS}(c)$, we conclude that $\mathrm{dom}_{NS}(c) \subseteq \mathrm{loc}_V(c)$. Thus and since $\mathrm{dom}(c) = \mathrm{dom}_{CS}(c) \cup \mathrm{dom}_{NS}(c)$ and $\mathrm{dom}_{CS}(c) \subseteq \mathrm{loc}_V(c)$, it follows that $\mathrm{dom}(c) = \mathrm{loc}_V(c)$.

To sum up we have proved the statements (1) and (2) which are equivalent to the implication (c) $\Rightarrow$ (a). ∎

Let c be a P-directed curve in $\mathbb{R}^2$. Let us set

$$\mathrm{dom}_{PS}(c) = \mathrm{dom}(c) \setminus \mathrm{loc}_P(c)$$

and note that $\mathrm{dom}_{PS}(c)$ is a closed subset of $\mathrm{dom}(c)$. Moreover, from Corollary 1.12 it follows that $\mathrm{dom}_{PS}(c) \subseteq \mathrm{dom}_{CS}(c)$. Obviously, by Proposition 1.14 we get

1.16. COROLLARY. *If c is a P-directed curve in $\mathbb{R}^2$, then the following conditions are equivalent*:

(a) $\mathrm{dom}(c) = \mathrm{loc}_K(c)$;
(b) $\mathrm{dom}_S(c) \subseteq \mathrm{loc}_K(c)$;
(c) $\mathrm{dom}_{CS}(c) \subseteq \mathrm{loc}_K(c)$;
(d) $\mathrm{dom}_{PS}(c) \subseteq \mathrm{loc}_K(c)$. ∎

It is easy to verify

1.17. COROLLARY. *Under the same assumptions, if* $\mathrm{dom}_{PS}(c)$ *is discrete in* $\mathrm{dom}(c)$, *then* c *is locally* K*-subordinate.* ∎

2. A characterization of smooth curves in $\mathbb{R} \times_k \mathbb{R}$

We shall regard $\mathbb{R}^2$ as a real normed vector space under the coordinatewise operations and the norm $\|p\| = (x^2+y^2)^{1/2}$ for $p = (x, y)$. In particular, $\mathbb{R}^2$ will be regarded as a topological space under the Euclidean topology. For any $p \in \mathbb{R}^2$ denote by τ_p the *translation* of $\mathbb{R}^2$ via p, i.e. $\tau_p(x) = x + \mathrm{p}$. If $A \subseteq \mathbb{R}^2$, we set $A + p = \tau_p(A)$. Clearly, $\mathbb{K}_p = \mathbb{K} + p$ for $p \in \mathbb{R}^2$.

Let A be a subset of $\mathbb{R}^2$. We shall regard A as a differential space with structure $C^\infty(A)$ of all real smooth functions on A. Clearly, A is a differential space of class $\mathcal{D}_0$ (see [4], Theorem (2.1)). For any $x \in A$ denote by T_xA the tangent vector space of A at x. We associate with A the dimension function $\delta_A : A \to \mathbb{Z}^+$ defined by $\delta_A(x) = \dim T_x\mathrm{A}$. It is well known that δ_A is upper semicontinuous (see [1], Corollary 1). A point p of A is called *regular* (*singular*) if δ_A is continuous (discontinuous) at p, or equivalently, constant (nonconstant) locally at p. Moreover, it is also known that the set A^* ($\mathrm{sing}\, A$) of all regular (singular) points of A is an open (closed) and dense (boundary) subset of A (see [1], Corollary 3). We set

$$A^i = \{p \in A : \delta_A(p) = i\} \text{ for } i = 0, 1, 2.$$

Clearly, A^0, A^1 and A^2 are disjoint and $A^0 \cup A^1 \cup A^2 = A$. Since δ_A is upper semicontinuous, it follows that A^0 and $A^0 \cup A^1$ are open subsets of A and A^2 is a closed subset of A. Moreover, it is known that A^0 consists of all isolated points of A (see [1], Proposition 2), so A^0 is a discrete subset of A.

Let $p \in A \subseteq \mathbb{R}^2$. If $X \in \{V, H, P\}$, we say that A is *locally* X*-subordinate* at p in case there are a neighbourhood U of p in $\mathbb{R}^2$ and an X-principal line L such that $A \cap U \subseteq L$. Moreover, we say that A is *locally* K*-subordinate* at p if there is a neighbourhood U of p in $\mathbb{R}^2$ such that $A \cap U \subseteq \mathbb{K}_p$. For $X \in \{V, H, P, K\}$ we denote by $\mathrm{loc}_X A$ the set of all points p of A such that A is locally X-subordinate at p. Clearly, $\mathrm{loc}_X A$ is an open subset of A. We call A *locally* X*-subordinate* in case $\mathrm{loc}_X A = A$. If $X \in \{V, H, P\}$ and A is contained in an X-principal line, we call A *globally* X*-subordinate.* Similarly, if A is contained in a principal cross, we say that it is *globally* K*-subordinate.* Obviously, we have the following lemmas.

2.1. LEMMA. *For any subset* A *of* $\mathbb{R}^2$ *the following conditions hold*:

(a) $\mathrm{loc}_V A \cap \mathrm{loc}_H A = A^0$;

(b) $\mathrm{loc}_V A \cup \mathrm{loc}_H A = \mathrm{loc}_P A \subseteq A^0 \cup A^1$;

(c) $\mathrm{loc}_K A \backslash \mathrm{loc}_P A \subseteq A^2$. ∎

2.2. LEMMA. *If A is a locally K-subordinate subset of $\mathbb{R}^2$, then the following conditions hold*:

(a) $A^1 = \mathrm{loc}_P A \backslash A^0$;

(b) $A^2 = \mathrm{loc}_K A \backslash \mathrm{loc}_P A \subseteq \mathrm{sing} A$ *and A^2 is a discrete subset of A.* ∎

It is easy to verify

2.3. PROPOSITION. *Let A be a connected subset of $\mathbb{R}^2$.*

(1) *If $X \in \{V, H, P\}$ and A is locally X-subordinate, then A is globally X-subordinate*;

(2) *If A is locally K-subordinate, then* $\mathrm{loc}_P A = A^*$. ∎

By applying Lemma 2.2 we get

2.4. PROPOSITION. *If A is a nonsingle connected locally K-subordinate subset of $\mathbb{R}^2$, then the following conditions hold*:

(a) $A^0 = \emptyset$;

(b) $A^1 = \mathrm{loc}_P A = A^*$;

(c) $A^2 = \mathrm{sing} A$. ∎

From this Proposition and Lemma 2.2 we obviously get

2.5. COROLLARY. *If A is a connected locally K-subordinate subset of $\mathbb{R}^2$, then the following statements hold*:

(1) $\mathrm{sing} A$ *is a discrete closed subset of A*;

(2) *If A is closed in $\mathbb{R}^2$, then* $\mathrm{sing} A$ *is a discrete closed subset of $\mathbb{R}^2$*;

(3) *If A is compact, then* $\mathrm{sing} A$ *is finite.* ∎

The following example shows that if A is a disconnected and compact locally H-subordinate subset of $\mathbb{R}^2$, then the set $\mathrm{sing} A$ can be dense in itself and of the continuum power.

2.6. EXAMPLE. Consider the closed interval $I = [0; 1] \subseteq \mathbb{R}$. Let C be the Cantor set regarded as a subset of I, that is, C consists of all $x \in I$ which have the following representations

$$x = \sum_{i=1}^{\infty} \xi_i 3^{-i}$$

where $\xi_i = 0, 2$. It is known that C is dense in itself and compact boundary subset of I of the continuum power. Clearly,

$$I \backslash C = \bigcup_{n=1}^{\infty} (a_n; b_n)$$

where $\{(a_n; b_n) : n \in \mathbb{N}\}$ is a family of disjoint open intervals of $\mathbb{R}$. For any $n \in \mathbb{N}$ let us take a discrete countable subset P_n of $(a_n; b_n)$ such that $a_n, b_n \in \overline{P}_n$. Consider the set

$$A = \bigcup_{n=1}^{\infty} P_n \cup C.$$

We can regard that $A \subseteq \mathbb{R}^2$ via the identification $x \mapsto (x, 0)$. Clearly, A is a disconnected and compact globally H-subordinate subset of $\mathbb{R}^2$. Moreover, note that

$$A^0 = A^* = \bigcup_{n=1}^{\infty} P_n, \quad A^1 = \operatorname{sing} A = C \quad \text{and} \quad A^2 = \emptyset. \ \blacksquare$$

Let $X \in \{V, H, P, K\}$. Denote by $\operatorname{lso}(X)$ the class of all locally X-subordinate subsets of $\mathbb{R}^2$. It is easy to verify

2.7. Proposition. *Let $X \in \{V, H, P, K\}$. The class* $\operatorname{lso}(X)$ *has the following properties*:

(1) *If $A \in \operatorname{lso}(X)$ and $B \subseteq A$, then $B \in \operatorname{lso}(X)$*;

(2) *If $A \subseteq \mathbb{R}^2$ and for each $p \in A$ there is a neighbourhood U of p in $\mathbb{R}^2$ such that $A \cap U \in \operatorname{lso}(X)$, then $A \in \operatorname{lso}(X)$*;

(3) *If in addition $X \in \{V, H, K\}$, then $A, B \in \operatorname{lso}(X)$ involves $A \cup B \in \operatorname{lso}(X)$.* $\blacksquare$

The proposition above immediately implies

2.8. Corollary. *Let $X \in \{V, H, K\}$. If $\mathfrak{F}$ is a locally finite family of sets from the class $\operatorname{lso}(X)$, then $\bigcup \mathfrak{F} \in \operatorname{lso}(X)$.* $\blacksquare$

The following example shows that the union of a countable family of sets from the class $\operatorname{lso}(K)$ as well as the closure of a set of this class need not belong to $\operatorname{lso}(K)$. Analogous examples we can construct for $X \in \{V, H, P\}$.

2.9. Example. Let us set $\mathbb{K}_n = \mathbb{K}_{(2^{-n}, 2^{-n})}$ for $n \in \mathbb{N}$. Consider the families $\mathfrak{F}_1 = \{\mathbb{K}_n : n \in \mathbb{N}\}$ and $\mathfrak{F}_0 = \{\mathbb{K}\} \cup \mathfrak{F}_1$. Clearly, $\mathfrak{F}_1$ is a locally finite family of sets from $\operatorname{lso}(K)$, so $\bigcup \mathfrak{F}_1 \in \operatorname{lso}(K)$ by Corollary 2.8. On the other hand, $\mathfrak{F}_0$ is not such a family and $\bigcup \mathfrak{F}_0 \notin \operatorname{lso}(K)$ because $\bigcup \mathfrak{F}_0$ is not locally K-subordinate at o. Finally, note that $\bigcup \mathfrak{F}_0$ is the closure of $\bigcup \mathfrak{F}_1$. $\blacksquare$

If A is a subset of $\mathbb{R} \times_k \mathbb{R}$ ($A \subseteq \mathbb{R}^2$), then by $\mathcal{S}^k(A)$ will be denoted the differential structure on A induced from $\mathbb{R} \times_k \mathbb{R}$. We say that A is a C^∞ *subset* of $\mathbb{R} \times_k \mathbb{R}$ in case $\mathcal{S}^k(A) = C^\infty(A)$. Obviously, every vertical or horizontal line in $\mathbb{R} \times_k \mathbb{R}$ is such a subset. Let us denote by $\mathrm{sub}^\infty(\mathbb{R} \times_k \mathbb{R})$ the class of all C^∞ subsets of $\mathbb{R} \times_k \mathbb{R}$. We need the following lemmas.

2.10. LEMMA. (see [2], Lemma 2.1). *For every $p \in \mathbb{R}^2$ the translation τ_p is a diffeomorphism of $\mathbb{R} \times_k \mathbb{R}$, and so, the family $\mathrm{sub}^\infty(\mathbb{R} \times_k \mathbb{R})$ is invariant under any translation of $\mathbb{R}^2$.* ∎

Let us set $\mathbb{R}^2_o = \mathbb{R}^2 \setminus \{o\}$ where $o = (0,0)$.

2.11. LEMMA (see [2], Corollary 1.4). *Assume that $\alpha \in \mathcal{F}^k$. Then $\alpha \in \mathcal{S}^k$ if and only if $\alpha|\mathbb{R}^2_o \in \mathcal{S}^k(\mathbb{R}^2_o)$ and $\alpha|\mathbb{K} \in C^\infty(\mathbb{K})$.* ∎

By an easy verification we get

2.12. LEMMA. *The following properties hold*:

(1) *If $A \in \mathrm{sub}^\infty(\mathbb{R} \times_k \mathbb{R})$ and $B \subseteq A$, then $B \in \mathrm{sub}^\infty(\mathbb{R} \times_k \mathbb{R})$.*

(2) *If $A \subseteq \mathbb{R}^2$ and for each $p \in A$ there is an open neighbourhood U of p in $\mathbb{R}^2$ such that $A \cap U \in \mathrm{sub}^\infty(\mathbb{R} \times_k \mathbb{R})$, then $A \in \mathrm{sub}^\infty(\mathbb{R} \times_k \mathbb{R})$.* ∎

2.13. LEMMA. *For any $p \in \mathbb{R}^2$ we have $\mathbb{K}_p \in \mathrm{sub}^\infty(\mathbb{R} \times_k \mathbb{R})$.*

Proof. By Lemma 2.10 and since $\mathbb{K}_p = \mathbb{K} + p$ for $p \in \mathbb{R}^2$, it suffices to show that the central principal cross $\mathbb{K}$ is a C^∞ subset of $\mathbb{R} \times_k \mathbb{R}$. Since $\mathbb{K}$ equipped with the structure $\mathcal{S}^k(\mathbb{K})$ is a differential subspace of $\mathbb{R} \times_k \mathbb{R}$, we conclude that the differential structure $\mathcal{S}^k(\mathbb{K})$ is generated by the restrictions $\varphi|\mathbb{K}$ for $\varphi \in \mathcal{S}^k$. From the definition of $\mathcal{S}^k$ it follows that for any $\varphi \in \mathcal{S}^k$ we have $\varphi \circ i \in C^\infty(\mathbb{R})$ and $\varphi \circ j \in C^\infty(\mathbb{R})$ where $i(x) = (x, 0)$ and $j(y) = (0, y)$ for $x, y \in \mathbb{R}$. It is seen that if $\varphi' = \varphi \circ i$ and $\varphi'' = \varphi \circ j$, then the function $\widetilde{\varphi} : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$\widetilde{\varphi}(x, y) = \varphi'(x) + \varphi''(y) - \varphi(o)$$

belongs to $C^\infty(\mathbb{R}^2)$. Note that $\widetilde{\varphi}|\mathbb{K} = \varphi|\mathbb{K}$, which means that the differential structure on $\mathbb{K}$ is generated by restrictions of smooth functions on $\mathbb{R}^2$, and so, $\mathbb{K}$ is a C^∞ subset of $\mathbb{R} \times_k \mathbb{R}$. ∎

2.14. LEMMA. *Let $\{p_n\}$ be an infinite sequence of distinct points of $\mathbb{R}^2$ such that $\lim p_n = o$ and $p_n \notin \mathbb{K}$ for each $n \in \mathbb{N}$. If $\{t_n\}$ is an infinite sequence of real numbers such that $\lim t_n = 0$, then there is a function $\varphi \in \mathcal{S}^2$ such that $\varphi|\mathbb{K} = 0$ and $\varphi(p_n) = t_n$ for each $n \in \mathbb{N}$.*

Proof. One can see that there is a discrete sequence $\{U_n\}$ of open subsets of $\mathbb{R}^2_o$ such that $p_n \in U_n$ and $\overline{U}_n \cap \mathbb{K} = \emptyset$ for each $n \in \mathbb{N}$. Next, we can choose a sequence $\{\varphi_n\}$ of real smooth functions on $\mathbb{R}^2$ such that

$0 \le \varphi_n(q) \le 1$ for $q \in \mathbb{R}^2$, $\varphi_n(p_n) = 1$ and supp $\varphi_n \subseteq U_n$. Define the function $\varphi : \mathbb{R}^2 \to \mathbb{R}$ by

$$\varphi(q) = \sum_{n=1}^{\infty} t_n \varphi_n(q).$$

Clearly, φ is continuous such that $\varphi|\mathbb{K} = 0$ and $\varphi(p_n) = t_n$ for $n \in \mathbb{N}$. Moreover, $\varphi|\mathbb{R}^2_o \in C^\infty(\mathbb{R}^2_o)$, so $\varphi \in \mathcal{S}^2$ by Lemma 2.11. ∎

2.15. THEOREM. $\mathrm{sub}^\infty(\mathbb{R} \times_1 \mathbb{R}) = \mathrm{sub}^\infty(\mathbb{R} \times_2 \mathbb{R}) = \mathrm{lso}(K)$.

Proof. Clearly, the inclusion $\mathrm{lso}(K) \subseteq \mathrm{sub}^\infty(\mathbb{R} \times_k \mathbb{R})$ follows from Lemmas 2.12 and 2.13. To prove the converse inclusion, suppose to the contrary that $A \in \mathrm{sub}^\infty(\mathbb{R} \times_k \mathbb{R}) \setminus \mathrm{lso}(K)$. This means that there is $p \in A$ such that A is not locally K-subordinate at p. By Lemma 2.10 we can assume that $p = o$. It follows that for every open neighbourhood U of o in $\mathbb{R}^2$ we have $(A \cap U) \setminus \mathbb{K} \neq \emptyset$. This implies that there is a sequence $\{p_n\}$ of distinct points of A such that $\lim p_n = o$ and $p_n \notin \mathbb{K}$ for $n \in \mathbb{N}$. Let us set $t_n = \|p_n\|^{1/2}$ for $n \in \mathbb{N}$. By Lemma 2.14 there is a function $\varphi \in \mathcal{S}^2 \subseteq \mathcal{S}^k$ such that $\varphi(o) = 0$ and $\varphi(p_n) = \|p_n\|^{1/2}$.

We set $\varphi' = \varphi|A$ and note that $\varphi' \in \mathcal{S}^k(A)$. Since A is a C^∞ subset of $\mathbb{R} \times_k \mathbb{R}$, it follows that $\varphi' \in C^\infty(A)$. Then there are an open neighbourhood U of o in $\mathbb{R}^2$ and a function $\psi \in C^\infty(\mathbb{R}^2)$ such that $\psi|A \cap U = \varphi'|A \cap U$. Clearly, $\psi(o) = \varphi(o) = 0$ and since $\lim p_n = o$, we have $\psi(p_n) = \varphi(p_n) = \|p_n\|^{1/2}$ for sufficiently large n. Hence we get

$$\lim \frac{\psi(p_n) - \psi(o)}{\|p_n\|} = \lim \|p_n\|^{-1/2} = \infty,$$

which means that the function ψ is not differentiable at o, a contradiction. This completes the proof of our assertion. ∎

A family $\Sigma \subseteq \mathrm{lso}(K)$ is called a C^∞ *generator* of $\mathrm{lso}(K)$ in case the following condition holds:

If $\alpha \in \mathcal{F}^1$ *and* $\alpha|A \in C^\infty(A)$ *for each* $A \in \Sigma$, *then* $\alpha|B \in C^\infty(B)$ *for all* $B \in \mathrm{lso}(K)$.

It is seen that the families of all principal lines and of all principal crosses are C^∞ generators of $\mathrm{lso}(K)$. Moreover, if Σ is a C^∞ generator of $\mathrm{lso}(K)$, then so is the family $\Sigma(\mathfrak{B}) = \{A \cap U : A \in \Sigma,\ U \in \mathfrak{B}\}$ where $\mathfrak{B}$ is an arbitrary topological base of $\mathbb{R}^2$. We obviously have

2.16. LEMMA. *If* Σ *is a* C^∞ *generator of* $\mathrm{lso}(K)$, *then*

$$\mathcal{S}^k = \{\alpha \in \mathcal{F}^k : \alpha|A \in C^\infty(A)\ \forall A \in \Sigma\}. \ ∎$$

It is easy to verify

2.17. PROPOSITION. *Let M be a differential space. Let Σ be a C^∞ generator of* $\mathrm{lso}(K)$. *A map (continuous map) $f : \mathbb{R}^2 \to M$ is smooth from $\mathbb{R} \times_1 \mathbb{R}$ ($\mathbb{R} \times_2 \mathbb{R}$) to M if and only if $f|A : A \to M$ is smooth for each $A \in \Sigma$.* ∎

We say that a curve $c : I \to \mathbb{R}^2$ is *locally C^∞ subordinate to $\mathbb{R} \times_k \mathbb{R}$* if for each $s \in I$ there is an open neighbourhood U of s in I such that $c(U) \in \mathrm{sub}^\infty(\mathbb{R} \times_k \mathbb{R})$. From Theorem 2.15 we obviously get

2.18. COROLLARY. *A curve c in $\mathbb{R}^2$ is locally K-subordinate if and only if it is locally C^∞ subordinate to $\mathbb{R} \times_k \mathbb{R}$.* ∎

Clearly, we have

2.19. LEMMA. *Let c be a smooth curve in $\mathbb{R}^2$. If c is locally C^∞ subordinate to $\mathbb{R} \times_k \mathbb{R}$, then it is smooth in $\mathbb{R} \times_k \mathbb{R}$.* ∎

2.20. THEOREM. *If c is a smooth curve in $\mathbb{R}^2$, then the following conditions are equivalent:*

(a) *c is smooth in $\mathbb{R} \times_k \mathbb{R}$;*
(b) *c is locally K-subordinate;*
(c) *c is locally C^∞ subordinate to $\mathbb{R} \times_k \mathbb{R}$.*

Proof. From Corollary 2.18 and Lemma 2.19 it follows that the implications (b)⇒(c) and (c)⇒(a) are satisfied. Thus, it remains to prove the implication (a)⇒(b). Suppose to the contrary that there exists a smooth curve $c : I \to \mathbb{R} \times_k \mathbb{R}$ which is not locally K-subordinate. This means that there is a parameter $s \in I$ such that for each $\varepsilon > 0$ we have

$$c([s-\varepsilon; s+\varepsilon] \cap I) \cap (\mathbb{R}^2 \backslash \mathbb{K}_p) \neq \emptyset$$

where $p = c(s)$. By Lemma 2.10 and since the parameterization of c may be changed, one can assume that $s = 0$ and $p = o$. Then there is a sequence $\{t_n\}$ of parameters of c converging to 0 such that the sequence $\{c(t_n)\}$ consists of distinct points of $\mathbb{R}^2 \backslash \mathbb{K}$ and $\lim c(t_n) = o$. By Lemma 2.14 there is a function $\varphi \in \mathcal{S}^k$ such that $\varphi(c(t_n)) = (-1)^n \cdot t_n$ and $\varphi(c(0)) = 0$. Hence we get $(\varphi(c(t_n)) - \varphi(c(0)))/t_n = (-1)^n$, which means that c is not smooth curve in $\mathbb{R} \times_k \mathbb{R}$ at 0, a contradiction. ∎

Denote by $\mathrm{cur}(\mathbb{R} \times_k \mathbb{R})$ the class of all smooth curves in $\mathbb{R} \times_k \mathbb{R}$. In turn, by $\mathrm{cur}(K)$ ($\mathrm{cur}^\infty(K)$) we denote the class of all locally K-subordinate curves (smooth curves) in $\mathbb{R}^2$. Clearly, Theorem 2.20 implies

2.21. COROLLARY. $\mathrm{cur}(\mathbb{R} \times_1 \mathbb{R}) = \mathrm{cur}(\mathbb{R} \times_2 \mathbb{R}) = \mathrm{cur}^\infty(K)$. ∎

By a *principal K-graph* in $\mathbb{R}^2$ we shall mean a compact connected locally K-subordinate subset of $\mathbb{R}^2$. The simplest example of such a graph is given by a principal closed segment, i.e. a closed segment lying in a principal line. One can see that every principal K-graph is a union of a finite family of principal closed segments. From Theorem 2.20 it follows that if c is a smooth curve in $\mathbb{R} \times_k \mathbb{R}$, then for any $a, b \in \operatorname{dom}(c)$ such that $a \leq b$ the image $c([a;b])$ is a principal K-graph.

The following example shows that a P-directed curve in $\mathbb{R}^2$ need not be locally K-subordinate, i.e. smooth in $\mathbb{R} \times_k \mathbb{R}$.

2.22. Example. Clearly, one can construct functions $\alpha', \beta' \in C^\infty([0;1])$ satisfying the following conditions:

$$\alpha'(t) > 0 \textit{ and } \beta'(t) = 0 \textit{ if } \frac{1}{k+1} < t < \frac{1}{k} \textit{ where } k \textit{ is odd};$$
$$\alpha'(t) = 0 \textit{ and } \beta'(t) > 0 \textit{ if } \frac{1}{k+1} < t < \frac{1}{k} \textit{ where } k \textit{ is even};$$
$$\alpha'(t) = \beta'(t) = 0 \textit{ if } t = \frac{1}{k} \; (k \in \mathbb{N}) \textit{ or } t = 0.$$

Let us set

$$\alpha(s) = \int_0^s \alpha'(t)\,dt \quad \text{and} \quad \beta(s) = \int_0^s \beta'(t)\,dt \text{ for } 0 \leq s \leq 1$$

and note that the curve $c = (\alpha, \beta) : [0;1] \to \mathbb{R}^2$ is P-directed. It is seen that we have the following decomposition:

$$\operatorname{im} c = \bigcup_{k=1}^{\infty} c(I_k) \cup \{o\} \text{ where } I_k = \left[\frac{1}{k+1}; \frac{1}{k}\right].$$

Clearly, $c(I_k)$ is a vertical (horizontal) closed segment in $\mathbb{R}^2$ provided that k is even (odd). One can see that this decomposition is unique in the following sense. If S is an arbitrary nonsingle segment in $\mathbb{R}^2$ such that $S \subseteq \operatorname{im} c$, then there is a unique $k \in \mathbb{N}$ such that $S \subseteq c(I_k)$. In addition, $\operatorname{im} c = c([0;1])$ is not a principal K-graph, so c is not smooth in $\mathbb{R} \times_k \mathbb{R}$. ■

It is easy to verify

2.23. Lemma. *For any principal K-graph G in $\mathbb{R}^2$ there is a smooth curve $c : [0;1] \to \mathbb{R} \times_k \mathbb{R}$ such that $c([0;1]) = G$.* ■

Denote by $\operatorname{gr}(K)$ the class of all principal K-graphs in $\mathbb{R}^2$. Clearly, $\operatorname{gr}(K)$ is a C^∞ generator of $\operatorname{lso}(K)$. If f is a smooth map from $\mathbb{R} \times_k \mathbb{R}$ to $\mathbb{R} \times_\ell \mathbb{R}$ where $k, \ell \in \{1,2\}$, then by Lemma 2.23 and Theorem 2.20 we conclude that $A \in \operatorname{gr}(K)$ involves $f(A) \in \operatorname{gr}(K)$. Denote by $\mathcal{S}^{k,\ell}$ the family of all smooth maps from $\mathbb{R} \times_k \mathbb{R}$ to $\mathbb{R} \times_\ell \mathbb{R}$. Moreover, we adopt that $\mathcal{F}^{1,1} = \mathcal{F}^{1,2}$ ($\mathcal{F}^{2,2}$) denotes the family of all maps (continuous maps) of $\mathbb{R}^2$. If c is a curve in $\mathbb{R}^2$ and f is a map of $\mathbb{R}^2$, we set $f_{\#}(c) = f \circ c$. It is easy to verify

2.24. PROPOSITION. *Let* $k, \ell \in \{1,2\}$, $(k,\ell) \neq (2,1)$, *and* $f \in \mathcal{F}^{k,\ell}$. *Then the following conditions are equivalent*:

(a) $f \in \mathcal{S}^{k,\ell}$;
(b) $f_{\#}(\mathrm{cur}(\mathbb{R} \times_k \mathbb{R})) \subseteq \mathrm{cur}(\mathbb{R} \times_\ell \mathbb{R})$;
(c) $f_{\#}(\mathrm{cur}^\infty(K)) \subseteq \mathrm{cur}^\infty(K)$;
(d) *If* $A \in \mathrm{gr}(K)$, *then* $f|A : A \to f(A)$ *is a smooth map of* C^∞ *subsets of* $\mathbb{R} \times_k \mathbb{R}$ *and* $\mathbb{R} \times_\ell \mathbb{R}$, *respectively.* ∎

Clearly, this proposition implies equalities $\mathcal{S}^{1,2} = \mathcal{S}^{1,1}$ and $\mathcal{S}^{2,2} = \mathcal{S}^{1,1} \cap \mathcal{F}^{2,2}$. One can ask whether there is a corresponding characterization of smooth maps from $\mathbb{R} \times_2 \mathbb{R}$ to $\mathbb{R} \times_1 \mathbb{R}$. This problem has a solution for all cases $k, \ell \in \{1,2\}$ (Proposition 2.25). Let $\mathcal{C}^{k,\ell}$ denote the family of all continuous maps from $\mathbb{R} \times_k \mathbb{R}$ to $\mathbb{R} \times_\ell \mathbb{R}$ with respect to the corresponding Sikorski topologies. It is easy to verify

2.25. PROPOSITION. *Let* $k, \ell \in \{1,2\}$ *and* $f \in \mathcal{C}^{k,\ell}$. *Then the following conditions are equivalent*:

(a) $f \in \mathcal{S}^{k,\ell}$;
(b) $f_{\#}(\mathrm{cur}(\mathbb{R} \times_k \mathbb{R})) \subseteq \mathrm{cur}(\mathbb{R} \times_\ell \mathbb{R})$;
(c) $f_{\#}(\mathrm{cur}^\infty(K)) \subseteq \mathrm{cur}^\infty(K)$;
(d) *If* $A \in \mathrm{gr}(K)$, *then* $f|A : A \to f(A)$ *is a smooth map of* C^∞ *subsets of* $\mathbb{R} \times_k \mathbb{R}$ *and* $\mathbb{R} \times_\ell \mathbb{R}$, *respectively.* ∎

By the definitions $\mathcal{C}^{2,2} = \mathcal{F}^{2,2}$, so in the case when $(k,\ell) = (2,2)$ Propositions 2.24 and 2.25 coincide. However, for the remaining cases, the following question arises: what are functions belonging to $\mathcal{C}^{k,\ell}$. Since we do not know any full answer to this question, Proposition 2.25 is less useful than Proposition 2.24. One can observe that a reason of such a situation is also justified by the fact that there is still open the question: what is the kind of the Sikorski topology of $\mathbb{R} \times_1 \mathbb{R}$, called shortly the $\mathcal{S}^1$-topology, i.e. the weakest one on $\mathbb{R}^2$ for which all functions from $\mathcal{S}^1$ are continuous (see [2], Question 5.6). However, it is easily seen that if ϕ and ψ are functions from $\mathcal{S}^1$, then the function $(\phi, \psi) : \mathbb{R}^2 \to \mathbb{R}^2$ defined by the assignment $(x,y) \mapsto (\phi(x), \psi(y))$ belongs to $\mathcal{C}^{1,2}$ where ϕ and ψ can be taken from $\mathcal{S}^1 \backslash \mathcal{S}^2$ (see [2], Example 1.5). Moreover, since the $\mathcal{S}^1$-topology is stronger than the Euclidean one, it follows that $\mathcal{C}^{2,2} \subseteq \mathcal{C}^{1,2}$ and $\mathcal{C}^{2,1} \subseteq \mathcal{C}^{2,2}$. But it seems to be much more complicated to state anything about $\mathcal{C}^{1,1}$.

References

[1] A. Kowalczyk, J. Kubarski, *A local property of the subspaces of Euclidean differential spaces*, Demonstratio Math., 11 (1978), 875–885.
[2] B. Przybylski, *Product final differential structures on the plane*, Demonstratio Math., 24 (1991), 573–599.
[3] R. Sikorski, *Abstract covariant derivative*, Colloq. Math., 18 (1967), 251–272.
[4] P. Walczak, *A theorem on diffeomorphisms in the category of differential spaces*, Bull. Acad. Sci., Ser. Sci. Math., Astronom., Phys., 21 (1973), 325–329.

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