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# ENDOMORPHISM CONGRUENCES

## Introduction

Considered in this paper the notion of endomorphism congruence has a close connection with a notion of result. To say most generally, results are fractions such that their numerators are elements of a certain linear space while their denominators are injective endomorphisms of that space. Generally, division by an endomorphism is lead out of range of elements. In connection with it we obtain the regular and singular results. Using the endomorphism congruence properties we can decide about regularity of certain results on the basis of the others confirming, if they are regular or singular.

The adequate examples in the Bittner operational calculus are given in this article.

## 1. Results and operators [1], [2]

Let $L(X, X)$ be the space of endomorphisms of a linear space $X$ (over a field $\Gamma$). Moreover, let $\pi(X)$ be a multiplicative and commutative semigroup of injective endomorphisms of $L(X, X)$.

When $X$ and $\pi(X)$ are given, we can introduce ordered pairs

$$\xi := [x, A], \ x \in X, \ A \in \pi(X)$$

and the quality relation

$$([x, A] = [y, B]) \overset{\text{def}}{\Longleftrightarrow} (Bx = Ay), \ x, y \in X, \ A, B \in \pi(X)$$

which is of equivalence type.

By this relation the entire set of considered pairs is divided into equivalence classes. These classes are called results. An arbitrary representative $\xi = [x, A]$ of such a class is also called a result. For such representative the fraction symbol $\xi = \frac{x}{A}$ is applied.

If in the set of results $\Xi(X, \pi(X))$ (denoted briefly by $\Xi(X)$) we introduce the operations

$$\frac{x}{A} + \frac{y}{B} := \frac{Bx + Ay}{AB}, \qquad \gamma\left(\frac{x}{A}\right) := \frac{\gamma x}{A},$$

where $x, y \in X$, $\gamma \in \Gamma$, $A, B \in \pi(X)$, then $\Xi(X)$ is a linear space over the field $\Gamma$. The elements of $X$ can be identified with the results, since the map

$$x \mapsto \frac{Ax}{A}, \; x \in X, \; A \in \pi(X)$$

is an isomorphism.

The elements $x \in X$ are called regular results and the elements $\xi \in \Xi(X) \setminus X$ are called singular results (cf [6]).

In $\Xi(X)$ we can also define the operation

$$B\left(\frac{x}{A}\right) := \frac{Bx}{A},$$

where $x \in X$, $A \in \pi(X)$, $B \in L(X, X)$, $AB = BA$.

With the given endomorphism $R \in L(X, X)$ commutative with the operations $A \in \pi(X)$, the linear operation given by the formula

$$\mu\frac{x}{B} := \frac{Rx}{AB}$$

is called an operator on the results space $\Xi(X)$ and denoted as $\mu = \frac{R}{A}$. The operator $\mu_0 = \frac{AR}{A}$ is identified with the endomorphism $R$.

The operator sum, the product of an operator by an element of $\Gamma$ and the superposition of operators are operators. The division by an operator, the numerator of which is an injection, defined as the product by the inverse of the operator, is also an operator.

## 2. Endomorphism congruences

Let $End(X)$ be a commutative algebra of endomorphisms of $L(X, X)$ (with the usual multiplication of endomorphisms).

It is said that two endomorphisms $A, B \in End(X)$ are congruent by the modulus $M \in \pi(X)$ when there exists an endomorphism $K \in End(X)$ such that

$$(A - B)x = KMx$$

for all $x \in X$. Then we denote

$$A \equiv B \pmod M.$$

This relation is a congruence. If $A \equiv B \pmod M$, then the elements

$$\frac{(A - B)x}{M}$$

are the regular results for all $x \in X$, i.e. the operator $\frac{A-B}{M}$ is an endomorphism of $X$.

The set

$$J_M := \{KM : K \in End(X)\}, \; M \in \pi(X)$$

is an ideal of the ring $End(X)$.

When $A, B \in End(X)$, we can write

$$A \equiv B \; (\mathrm{mod}\, J_M),$$

if $A - B \in J_M$. Therefore

$$A \equiv B \; (\mathrm{mod}\, M) \leftrightarrow A \equiv B \; (\mathrm{mod}\, J_M).$$

EXAMPLE 1. Let $X$ be a certain real linear space. Moreover, let $End(X)$ be an algebra of endomorphisms

$$Ax := a \cdot x, \qquad x \in X,$$

where $a$ is a given integer. The modulus $M \in \pi(X)$ is defined by the formula

$$Mx := m \cdot x, \qquad x \in X,$$

where $m$ is a given natural number.

Then

$$A \equiv B \; (\mathrm{mod}\, M)$$

if and only if

$$a \equiv b \; (\mathrm{mod}\, m)$$

in the classical congruence of integers sense.

EXAMPLE 2. Let $X := C^0(Q, R^1)$, where $Q \subset R^1$. The set $End(X)$ is defined as the algebra of endomorphisms

$$Ax := \{A(t)x(t)\}, \qquad x = \{x(t)\} \in X,$$

where $\{A(t)\} \in X$ is a given polynomial.

The modulus $M \in \pi(X)$ is defined as the endomorphism

$$Mx := \{M(t)x(t)\}, \qquad x = \{x(t)\} \in X,$$

where the given polynomial $\{M(t)\} \in X$ satisfies the condition $M(t) \neq 0$ for all $t \in Q$.

Then

$$A \equiv B \; (\mathrm{mod}\, M)$$

if and only if $\{M(t)\}$ is a divisor of the polynomial $\{A(t) - B(t)\}$.

Obviously the same modulus congruences can be added, substracted and multiplied by sides. It can be generalized by induction for any finite number of congruences. In particular, the both sides of a congruence can be multiplied by the same endomorphism and they can be also raised to the same power with a natural exponent.

Therefore, if $A_i, U \in End(X)$, $i = 0, 1, \ldots, n$ and

$$W(U) := A_0 + A_1 U + \ldots + A_n U^n,$$

then $W(U) \in End(X)$ and

$$A \equiv B \pmod{M} \quad \text{implies} \quad W(A) \equiv W(B) \pmod{M}. \tag{1}$$

## 3. Operational calculus

In accordance with notation used e.g. in [2], the Bittner operational calculus is the system

$$CO(L^0, L^1, S, T_q, s_q, q, Q),$$

where $L^0$ and $L^1$ are linear spaces over a field $\Gamma$.

The linear operation $S : L^1 \to L^0$ (denoted as $S \in L(L^1, L^0)$), called the (abstract) derivative, is a surjection. Moreover, $Q$ is a nonempty arbitrary set of indices $q$ for the operations $T_q \in L(L^0, L^1)$ such that $ST_q f = f$, $f \in L^0$, called integrals, and for the operations $s_q \in L(L^1, L^1)$ such that $s_q x = x - T_q Sx$, $x \in L^1$, called limit conditions. The kernel of $S$, i.e. the set $Ker\, S := \{c \in L^1 : Sc = 0\}$, is called the space of constants for the derivative $S$.

Assume that $L^1 \subset L^0$. Then

$$Ker\, S \subset L^1 \subset L^0$$

and the integrals $T_q$, $q \in Q$ are endomorphisms of $L^0$. The iterations of these operations can be also formed.

## 4. Examples of congruences in the operational calculus

A. It is not difficult to check that any integral $T_q$, $q \in Q$ is an injection of $L^0$. None of two endomorphisms $A, B, A \neq B$ of $L^0$ and $KerS$ are congruent by the modulus $T_q$, because

$$\xi = \frac{(A - B)c}{T_q}$$

is a singular result for all $c \in Ker\, S \setminus \{0\}$. In reality, if $d := (A - B)c$,where $c \in Ker\, S \setminus \{0\}$, then $d \in Ker\, S \setminus \{0\}$ and $T_q \xi = d$. Then with $\xi \in L^0$ it would be $s_q T_q \xi = s_q d = d$ and hence $d = 0$, what is impossible (cf [2]).

B. If $R$ is an endomorphism of $L^0$ and $L^1$, commutative with the derivative $S$ and the limit condition $s_q$, then $R$ is commutative with the integral $T_q$.

Moreover, if an abstract differential equation

$$Sx = Rx, \quad x \in L^1 \tag{2}$$

with the limit condition

$$s_q x = 0$$

has only zero solution, then $R$ is called the $q$-logarithm.

If $R$ is a $q$-logarithm, then there exists a semigroup $\pi(L^0)$ such that $I - T_q R \in \pi(L^0)$, where $I := id_{L^0}$ and the result

$$\xi = \frac{c}{I - T_q R}, \quad c \in Ker\, S$$

is well defined.

If the result $\xi$ is regular, then it is called an exponential element and denoted by the symbol $e^{Rt_q}c$ (see [2], cf [3, 6]).

The exponential element $x = e^{Rt_q}c$ is a solution of the equation (2) with the limit condition $s_q x = c$. Moreover, for every $n \in N$ we have

$$x = c + T_q Rc + \ldots + T_q^n R^n c + T_q^{n+1} R^{n+1} x.$$

The expression

$$w_n = c + T_q Rc + \ldots + T_q^n R^n c$$

is called the $n$-th Taylor polynomial for the exponential element $x = e^{Rt_q}c$ (in the point $q \in Q$).

Let

$$W(U) := I + U + \ldots + U^n,$$

where $n \in N$, $I := \mathrm{id}_{L^0}$, $U \in End(L^0)$.

Then

$$w_n = W(T_q R)c.$$

Since

$$I \equiv T_q R \ (\mathrm{mod}\,(I - T_q R)),$$

so, on the basis of (1), the result

$$\eta = \frac{w_n}{I - T_q R}$$

is regular if and only if $\xi$ is the regular result.

C. Assume that the abstract differential equation

$$S^2 x + Sx + x = 0, \quad x \in L^2 := \{x \in L^1 : Sx \in L^1\}$$

with the limit conditions

$$s_q x = 0, \quad s_q Sx = 0$$

has only zero solution.

Then there exists a semigroup $\pi(L^0)$ such that $I + T_q + T_q^2 \in \pi(L^0)$. By induction it can be proved that the congruence

$$(I + T_q)^{2n+1} \equiv -T_q^{n+2} \ (\mathrm{mod}\,(I + T_q + T_q^2))$$

holds for all non-negative integer $n$.

Due to the above mentioned the solution of the abstract integral equation

$$(I + T_q + T_q^2)\xi = (I + T_q)^{2n+1} f, \qquad \xi \in \Xi(L^0),$$

where $f \in L^0$ is a given element and $n$ is an arbitrary but fixed non-negative integer, is a regular result if and only if the solution of the equation

$$(I + T_q + T_q^2)\eta = -T_q^{n+2} f, \qquad \eta \in \Xi(L^0)$$

is a regular result.

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