DEMONSTRATIO MATHEMATICA
Vol. XXVIII No 1 1995

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VARIETIES OF ALGEBRAS HAVING A DISTRIBUTIVE LATTICE OF SUBVARIETIES

1. Introduction and preliminaries

The question of describing in term of identities the varieties having a distributive lattice of subvarieties was raised by L. Bokut in 1976 in [5, problem 19] and in [3] too. Since the survey [2] of V.A. Artamonov in 1978 many results concerning the topic have been obtained [1, 11, 7, 9, 8, 14, 12, 13].

In the paper we consider the absolutely free algebra $F = K\{X\}$ of infinite rank on a countable set X of free generators $x_1, x_2, \ldots$ over a fixed field K of characteristic zero. F_m is the subalgebra of rank m generated by $x_1, x_2, \ldots, x_m$. We denote by S_n and GL_m the symmetric group and the general linear group, acting on the set of symbols $\{1, 2, \ldots, n\}$ and on a m-dimensional vector space, respectively.

Let I be a T-ideal in F and $\mathfrak{M}$ a variety corresponding to I. We denote F/I by $F\{\mathfrak{M}\}$ and $F_m/F_m \cap I$ by $F_m(\mathfrak{M})$. The space $P_n(\mathfrak{M})$ of all multilinear polynomials of degree n from $F_n(\mathfrak{M})$ has a structure of a left S_n-module. $F_m(\mathfrak{M})$ is a left GL_m-module too.

The irreducible S_n- and GL_m-modules are described by Young diagrams. For a partition $\lambda = (\lambda_1, \ldots, \lambda_r)$ of n, $\lambda_1 \geq \ldots \geq \lambda_r \geq 0$, $\lambda_1 + \ldots + \lambda_r = n$ we denote by $M(\lambda)$ and $N_m(\lambda)$ the S_n- and GL_m-modules corresponding to λ. One can look into [4, 15, 6, 10] for details on the representation theory of the symmetric and general linear groups.

The subvarieties of $\mathfrak{M}$ form a lattice $\Lambda(\mathfrak{M})$ with respect to the intersection and the union of subvarieties. The question of distributivity has been treated by consideration of $P_n(\mathfrak{M})$. Because of [2] $\Lambda(\mathfrak{M})$ is distributive iff $P_n(\mathfrak{M})$ for all n is a sum of pairwise non-isomorphic irreducible S_n-modules $M(\lambda)$.

It is known [6] that the homogeneous component $F_m^{(n)}(\mathfrak{M})$ of $F_m(\mathfrak{M})$ and $P_n(\mathfrak{M})$ have the same module structure. If $P_n(\mathfrak{M}) = \sum_\lambda k(\lambda)M(\lambda)$, then $F_m^{(n)}(\mathfrak{M}) = \sum_\lambda k(\lambda)N_m(\lambda)$. Thus for convenience the investigations in the paper are on the GL_m-structure of $F_m^{(n)}$. For $n = 2,3$ we have:

$$F_m^{(2)} = N_m(2) + N_m(1,1),$$
$$F_m^{(3)} = 2N_m(3) + 4N_m(2,1) + 2N_m(1^3).$$

Generators of the modules $N_m(3)$ are x_1^3 and $x_1x_2^2$; the modules with diagrams [2,1] are generated by $f_1 = x_1x_2x_1 - x_2x_1x_1$, $f_2 = x_1(x_2x_1) - x_2x_1^2$, $f_3 = x_1^2x_2 - x_2x_1x_1$ and $f_4 = x_1(x_1x_2) - x_2x_1^2$; generators of $N_m(1^3)$ are $S_{21}(x_1,x_2,x_3) = \sum_{\sigma\in S_3}(-1)^\sigma x_{\sigma(1)}x_{\sigma(2)}x_{\sigma(3)}$ and $S_{12}(x_1,x_2,x_3) = \sum_{\sigma\in S_3}(-1)^\sigma x_{\sigma(1)}(x_{\sigma(2)}x_{\sigma(3)})$, where $(-1)^\sigma$ means the sign of the permutation σ.

So identities are needed, which "glue" the isomorphic modules, so that the sum in $F_m^{(3)}(\mathfrak{M})$ will be of non-isomorphic ones only.

For the modules $N_m(3)$ such an identity is

(1) $\qquad \alpha_1x^3 + \beta_1xx^2 = 0, \qquad \text{for } (\alpha_1,\beta_1) \neq (0,0),\ \alpha_1,\beta_1 \in K.$

It means that

(1^a) $\qquad xx^2 = 0 \quad \text{if } \alpha_1 = 0, \text{ or}$

(1^b) $\qquad x^3 - kxx^2 = 0, \quad \text{where } k = \beta_1/\alpha_1 \text{ if } \alpha_1 \neq 0.$

For $N_m(2,1)$ the following system has to be fulfilled:

$$\begin{aligned}\gamma_{11}f_1 + \gamma_{12}f_2 + \gamma_{13}f_3 + \gamma_{14}f_4 &= 0\\ \gamma_{21}f_1 + \gamma_{22}f_2 + \gamma_{23}f_3 + \gamma_{24}f_4 &= 0\\ \gamma_{31}f_1 + \gamma_{32}f_2 + \gamma_{33}f_3 + \gamma_{34}f_4 &= 0\end{aligned} \tag{2}$$

and rank $A = 3$, where

$$A = \begin{bmatrix}\gamma_{11} & \gamma_{12} & \gamma_{13} & \gamma_{14}\\ \gamma_{21} & \gamma_{22} & \gamma_{23} & \gamma_{24}\\ \gamma_{31} & \gamma_{32} & \gamma_{33} & \gamma_{34}\end{bmatrix}, \qquad \gamma_{ij} \in K.$$

It means that the following identities hold:

(2^a) $\quad f_i = 0 \quad \text{for } i = 1,\dots,4, \text{ or}$

(2^b) $\quad f_i - k_if_4 = 0 \quad$ for $i = 1,2,3$ and for $k_1,k_2,k_3 \in K$, if $f_4 = 0$ is not an identity.

For the modules $N_m(1^3)$ the needed identity is

(3) $\qquad \alpha_2S_{21}(x_1,x_2,x_3) + \beta_2S_{12}(x_1,x_2,x_3) = 0,$

where $(\alpha,\beta_2)\neq(0,0)$, $\alpha_2,\beta_2\in K$. It means that

(3^a) $S_{12}(x_1,x_2,x_3)=0$ if $\alpha_2=0$, or

(3^b) $S_{21}(x_1,x_2,x_3)-pS_{12}(x_1,x_2,x_3)=0,$ where $p=\beta_2/\alpha_2$ if $\alpha_2\neq 0$.

In the paper the following varieties are examined:

$\mathfrak{M}_1=[*,k_1,k_2,k_3,*]$ with identities (1^a), (2^b) and (3^a),
$\mathfrak{M}_2=[*,k_1,k_2,k_3,p]$ with identities (1^a), (2^b) and (3^b),
$\mathfrak{M}_3=[k,k_1,k_2,k_3,*]$ with identities (1^b), (2^b) and (3^a),
$\mathfrak{M}_4=[k,*,*,*,*]$ with identities (1^b), (2^a) and (3^a),
$\mathfrak{M}_5=[k,*,*,*,p]$ with identities (1^b), (2^a) and (3^b),
$\mathfrak{M}_6$ — a variety, for which $P_3(\mathfrak{M}_6)=M(3)$.

For a multihomogeneous polynomial $f(x_1,\ldots,x_r)$ of degree λ_i in x_i we denote by

$$\mathrm{lin}(f)=f(x_1\mathrm{I}y_{11},\ldots,y_{1\lambda_1};\ldots;x_r\mathrm{I}y_{r1},\ldots,y_{r\lambda_r})$$

the linearization of $f(x_1,\ldots,x_m)$ which equals the multilinear in y_{ij} for $i=1,\ldots,r$ component of

$$f(x_1+y_{11}+\ldots+y_{1\lambda_1},\ldots,x_r+y_{r1}+\ldots+y_{r\lambda_r}).$$

We point that $f=0$ and $\mathrm{lin}(f)=0$ are equivalent [6].

PROPOSITION 1.1. *Let M be an S_n-submodule of P_n and let Q be the set of the multilinear consequences of degree $n+1$ of the polynomial identities of M. Then Q is an S_{n+1}-module of P_{n+1} which is a homomorphic image of the S_{n+1}-module*

$$((M^{\downarrow}S_{n-1})\otimes_K(M(2)+M(1^2))^{\uparrow}S_{n+1}+2(M\otimes_K M(1))^{\uparrow}S_{n+1},\ \textit{i.e.}$$

a) *In the first summand S_{n-1} acts on the set $\{1,\ldots,n-1\}$ fixing n. S_2 acts on $\{n,n+1\}$, the tensor product is an $S_{n-1}\times S_2$-module, where the direct product $S_{n-1}\times S_2$ is canonically embedded in S_{n+1}.*

b) *The consequences $f(x_1,\ldots,x_n)$. x_{n+1} for $f\in M$ and $x_{n+1}\cdot f(x_1,\ldots$ $\ldots,x_n)$ generate two factor-modules $M\otimes_K M(1)^{\uparrow}S_{n+1}$, where $S_n\times S_1$ is canonically embedded in S_{n+1}.*

COROLLARY 1.2. *Let λ be a partition of n and let $M(\lambda)\subset P_n$. Then the S_{n+1}-module $M'(\lambda)$ of all multilinear consequences of $M(\lambda)$ in P_{n+1} equals $\sum\alpha_\mu M(\mu)$, where the non-negative integers λ_μ are bounded by the number of diagrams $[\mu]$ obtained by the following devices:*

a) *We remove a box from $[\lambda]$ and obtain a diagram $[\nu]$. Then we add two new boxes to $[\nu]$ and produce a diagram $[\mu]$ such that these two new boxes do not belong to one and the same column of $[\mu]$ if we consider the module*

$M(2)$ (or do not belong to one and the same row of $[\mu]$ if we consider the module $M(1^2)$),

b) *We add a new box to $[\lambda]$ and obtain $[\mu]$.*

2. Consequences of degree 4 as linear combinations of the generators of the modules $N_m(\lambda)$

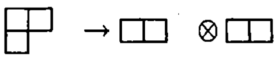
The symbols x, y, z, t will be used for the free generators of $K\{X\}$. The needed identities of degree 3 are written as:

$$
(2.1)\quad
\begin{aligned}
d_1 &= x^3 - kxx^2 = 0\\
d_1 &= xyx - yxx - k_1(x(xy) - yx^2) = 0\\
d_3 &= x(yx) - yx^2 - k_2(x(xy) - yx^2) = 0\\
d_4 &= x^2y - yxx - k_3(x(xy) - yx^2) = 0\\
d_5 &= S_{21}(x,y,z) - pS_{12}(x,y,z) = 0, \qquad k, k_1, k_2, k_3, p \in K.
\end{aligned}
$$

$A : N_1(4)$. Due to 1.2 we can have the following diagrams:

(A1) 

(A2) 

(A3) 

For (A 1) we have $d_1x = 0$ and $xd_1 = 0$. For (A 2) we linearize partially d_1 and substitute $u = x^2$ i.e. $d_1(x\mathrm{I}x^2) = 0$. For (A 3) we consider $d_1(y = x^2)$ for $i = 2, 3, 4$. So we get the system:

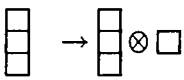
$$
(2.2)\quad
\begin{aligned}
&x^4 - kxx^2x = 0\\
&xx^3 - kx(xx^2) = 0\\
&x^4 + xx^2x + x^2x^2 - k(x^2x^2 + xx^3 + x(xx^2)) = 0\\
&xx^2x - x^4 - k_1(x(xx^2) - x^2x^2) = 0\\
&xx^3 - x^2x^2 - k_2(x(xx^2) - x^2x^2) = 0\\
&x^2x^2 - x^4 - k_3(x(xx^2) - x^2x^2) = 0.
\end{aligned}
$$

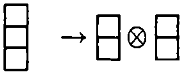
$B : N_4(1^4)$. The generators of the modules now are:

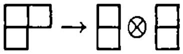
$$f_1 = S_{211}(x,y,z,t), \quad f_2 = S_{121}(x,y,z,t), \quad f_3 = S_{22}(x,y,z,t),$$
$$f_4 = S_{1(21)}(x,y,z,t), \; f_5 = S_{1(12)}(x,y,z,t).$$

The indices show the way of brackets in the standard polynomials, for example $S_{121} = \sum_{\sigma\in S_4}(-1)^\sigma x_{\sigma(1)}(x_{\sigma(2)}x_{\sigma(3)})x_{\sigma(4)}$.

According to Corollary 1.2 we have the following diagrams:

(B1) 

(B2) 

(B3) 

For (B 1) right and left multiplication of d_5 by t give

(b1) $$f_1 - pf_2 = 0,$$

(b2) $$f_4 - pf_5 = 0.$$

For (B 2) we consider $d_5(z = [z,t])$, in which three times a circling permutation of x, y, z, t is made alternating the signs. Then we transpose y and z in $d_5(z = [z,t])$ and again a circling of x, y, z, t is used. The sum of the six identities thus received leads to

(b3) $$f_1 - f_2 + f_3 - p(f_3 - f_4 + f_5) = 0.$$

For (B 3) in $d_2(x\mathrm{I}[z,t])$ we transpose x and y and then x and t. In the first circling permutation of x, y, z, t in $d_2(x\mathrm{I}[z,t])$ we transpose y and z and then x and y. In the second one the transpositions are of z and t and then of y and z. In the third one we transpose x and t and then z and t. So we come to 12 consequences, the sum of which is the identity

(b4) $$-f_1 + f_2 + 2f_3 - k_1(f_3 + 2f_4 + f_5) = 0.$$

Analogous transformations on d_3 and d_4 lead accordingly to:

(b5) $$-f_3 + f_4 + 2f_5 - k_2(f_3 + 2f_4 + f_5) = 0 \quad \text{and}$$

(b6) $$f_1 + 2f_2 + f_3 - k_3(f_3 + 2f_4 + f_5) = 0.$$

Identities (b1), ..., (b6) will be cited as (2.3) later on.

$C : N_2(3,1)$. The standard generators now are:

$a_1 = xyxx - yxxx$ and a_i $(1 = 2, \dots, 5)$ for brackets $(*(**)*)$, $(**)(**)$, $*(***)$ and $*(*(**))$;

$b_1 = x^2yx - yxxx$ and b_i $(i = 2, \dots, 5)$ for the respectvive brackets;

$c_1 = x^3y - yxxx$ and c_i $(i = 2, \dots, 5)$ for the respective brackets.

The system (2.5) in this case consists of identities

(c1) $2a_3 + b_1 - c_1 + c_2 - k_1(a_5 - b_3 + c_3 + 2c_4) = 0,$

(c2) $-a_1 + a_3 + b_1 + b_2 - k_1(-a_3 + c_3 + c_4 + c_5) = 0,$

(c3) $a_2 + a_3 + b_1 - k_1(b_5 + c_3 + c_4) = 0,$

(c4) $a_1 - k_1b_2 = 0,$

(c5) $-a_4 + b_4 - k_1(-a_5 + c_5) = 0,$

(c6) $a_1 - a_2 + b_2 - b_3 + c_3 - k(a_3 - a_4 + b_4 - b_5 + c_5) = 0,$

(c7) $2a_1 - a_3 - b_1 + 2b_2 - c_1 - c_2 + 2c_3 - k(2a_3 - a_5 - b_3 + 2b_4 - c_3 - c_4 + 2c_5) = 0,$

(c8) $a_1 + b_1 - 3c_1 - k(a_2 + b_2 - 3c_2) = 0,$

(c9) $a_4 + b_4 + c_4 - k(a_5 + b_5 + c_5) = 0,$

(c10) $a_3 - b_1 + c_1 - c_2 - p(a_5 - b_3 + c_3 - c_4) = 0$

and those of the system (2.4):

$$
(2.4)\qquad
\begin{aligned}
&2a_5 + b_3 - c_3 + c_4 - k_2(a_5 - b_3 + c_3 + 2c_4) = 0\\
&-a_3 + a_5 + b_3 + b_4 - k_2(-a_3 + c_3 + c_4 + c_5) = 0\\
&a_4 + a_5 + b_3 - k_2(b_5 + c_3 + c_4) = 0\\
&a_2 - k_2 b_2 = 0\\
&-a_5 + b_5 - k_2(-a_5 + c_5) = 0\\
&a_3 - b_1 + c_1 + 2c_2 - k_3(a_5 - b_3 + c_3 + 2c_4) = 0\\
&-a_1 + c_1 + c_2 + c_3 - k_3(-a_3 + c_3 + c_4 + c_5) = 0\\
&b_3 + c_1 + c_2 - k_3(b_5 + c_3 + c_4) = 0\\
&b_1 - k_3 b_2 = 0\\
&-a_4 + c_4 - k_3(-a_5 + c_5) = 0.
\end{aligned}
$$

$D : N_3(2, 1^2)$. The standard generators now are:

$$f_1 = \sum_{\sigma\in S_3} (-1)^\sigma x_{\sigma(1)}x_{\sigma(2)}x_{\sigma(3)}x_1, \qquad g_1 = \sum_{\sigma\in S_3} (-1)^\sigma x_{\sigma(1)}x_{\sigma(2)}x_1x_{\sigma(3)},$$

$$h_1 = \sum_{\sigma\in S_3} (-1)^\sigma x_{\sigma(1)}x_1x_{\sigma(2)}x_{\sigma(3)}.$$

For brackets $(*(**)*)$, $(**)(**)$, $*(***)$ and $*(*(**))$ the indices are $2,\dots,5$, respectively.

The system for $N_3(2, 1^2)$ is formed by identities

(d1) $-f_1 + f_3 + g_1 + g_2 - g_3 - h_2 - p(-f_3 + f_5 + g_3 + g_4 - g_5 - h_4) = 0,$

(d2) $f_1 - pf_2 = 0,$

(d3) $f_4 - g_4 + h_4 - p(f_5 - g_5 + h_5) = 0,$

(d4) $f_1 + f_3 - g_1 + g_2 + g_3 + 2h_1 + h_2 - p(f_3 + f_5 - g_3 + g_4 + g_5 + 2h_3 + h_4) = 0,$

(d5) $f_1 + 2f_2 + f_3 + g_1 - g_2 - g_3 + h_2 + 2h_3$
$\qquad -k(f_3 + 2f_4 + f_5 + g_3 - g_4 - g_5 + h_4 + 2h_5) = 0,$

(d6) $-f_1 + 3g_1 - k_1(f_2 + 3h_2) = 0,$

(d7) $2f_4 + g_4 - h_4 - k_1(f_5 + 2g_5 + h_5) = 0,$

(d8) $-f_1 + 2f_2 - g_1 - g_2 + h_2 - k_1(-f_3 + f_5 - g_3 - g_5 + 2h_5) = 0,$

(d9) $f_1 - 2f_3 - g_1 + g_2 - 2g_3 + 2h_1 + h_2$
$\qquad -k_1(-f_3 - f_5 + g_3 + 2g_4 - g_5 - 2h_3 + 2h_4) = 0,$

(d10) $f_1 - f_2 + f_3 - g_3 - k_1(g_3 - g_4 + h_4 - h_5) = 0$

and those of the following system (2.6):

$$\begin{aligned}
&-f_2 + 3g_2 - k_2(f_2 + 3h_2) = 0\\
&\ 2f_5 + g_5 - h_5 - k_2(f_5 + 2g_5 + h_5) = 0\\
&-f_3 + 2f_4 - g_3 - g_4 + h_4 - k_2(-f_3 + f_5 - g_3 - g_5 + 2h_5) = 0\\
&\ f_3 - 2f_5 - g_3 + g_4 - 2g_5 + 2h_3 + h_4\\
&-k_2(-f_3 - f_5 + g_3 + 2g_4 - g_5 - 2h_3 + 2h_4) = 0\\
&\ f_3 - f_4 + f_5 - g_5 - k_2(g_3 - g_4 + h_4 - h_5) = 0\\
&\ f_1 + 3h_1 - k_3(f_2 + 3h_2) = 0\\
&\ f_4 + 2g_4 + h_4 - k_3(f_5 + 2g_5 + h_5) = 0\\
&-f_1 + f_3 - g_1 - g_3 + 2h_3 - k_3(-f_3 + f_5 - g_3 - g_5 + 2h_5) = 0\\
&-f_1 - f_3 + g_1 + 2g_2 - g_3 - 2h_1 + 2h_2\\
&-k_3(-f_3 - f_5 + g_3 + 2g_4 - g_5 - 2h_3 + 2h_4) = 0\\
&\ g_1 - g_2 + h_2 - h_3 - k_3(g_3 - g_4 + h_4 - h_5) = 0.
\end{aligned} \tag{2.6}$$

Briefly the system of the consequences is denoted by (2.7).

$E : N_2(2^2)$. A standard generator in this case is

$$f_1 = \sum_{\sigma,\tau\in S_2} (-1)^{\sigma+\tau} x_{\sigma(1)} x_{\tau(1)} x_{\sigma(2)} x_{\tau(2)}.$$

For brackets $(*(**)*)$, $(**)(**)$, $*(***)$ and $*(*(**))$ the indices of the generators will be $2, \ldots, 5$.

Another generator is

$$f_6 = \sum_{\sigma,\tau\in S_2} (-1)^{\sigma+\tau} x_{\sigma(1)} x_{\sigma(2)} x_{\tau(1)} x_{\tau(2)}$$

and for the corresponding way of brackets $f_7, \ldots, f_{10}$.

The system (2.9) in this case is formed by identities (e1), …, (e6) and those of the system (2.8):

(e1) $2f_1 - f_2 + 2f_3 - f_6 - f_8 - f_9 - k(2f_3 - f_4 + 2f_5 - f_7 - f_8 - f_{10}) = 0,$

(e2) $-f_2 + f_6 + f_8 + f_9 - p(-f_4 + f_7 + f_8 + f_{10}) = 0,$

(e3) $f_6 - k_1 f_2 = 0,$

(e4) $f_4 - f_7 - k_1 f_5 = 0,$

(e5) $f_2 - f_6 + 2f_8 - f_9 - k_1(2f_4 - 2f_7 + f_8 + f_{10}) = 0,$

(e6) $-2f_1 - f_2 + f_6 - f_9 - k_1(-2f_3 + 2f_5 + 2f_5 + f_8 - f_{10}) = 0,$

$$
(2.8)\qquad
\begin{aligned}
&f_9 - k_2 f_2 = 0\\
&f_5 - f_{10} - k_2 f_5 = 0\\
&f_4 - f_7 - f_8 + 2f_{10} - k_2(2f_4 - 2f_7 + f_8 + f_{10}) = 0\\
&-2f_3 - f_4 - f_7 + f_8 - k_2(-2f_3 + 2f_5 + f_8 - f_{10}) = 0\\
&f_1 - k_3 f_2 = 0\\
&f_4 - k_3 f_5 = 0\\
&2f_2 + f_6 + f_8 - 2f_9 - k_3(2f_4 - 2f_7 + f_8 + f_{10}) = 0\\
&-2f_1 + 2f_3 + f_6 - f_8 - k_3(-2f_3 + 2f_5 + f_8 - f_{10}) = 0.
\end{aligned}
$$

3. Description of $P_4(\mathfrak{M}_i)$ for $i = 1, \ldots, 5$ and $P_n(\mathfrak{M}_6)$

Having already obtained the homogeneous linear systems for the standard generators of every module $N_m(\lambda)$, we determine the rank of the matrix of the corresponding system in any of the considered cases. If the system has a trivial solution only, there is no module with the corresponding diagram in $P_4(\mathfrak{M}_i)$. If the rank is not maximal, we define the multiplicities $k(\lambda)$ in the decomposition of $P_4(\mathfrak{M}_i)$ into a sum of irreducible modules i.e. in

$$
(3.1)\qquad
\begin{aligned}
P_4(\mathfrak{M}_i) = k(4)M(4) + k(1^4)M(1^4) + k(3,1)M(3,1)\\
+ k(2,1^2)M(2,1^2) + k(2^2)M(2^2)
\end{aligned}
$$

THEOREM 3.1. *For $\mathfrak{M}_1 = [*, k_1, k_2, k_3, *]$ in* (3.1)

a) $k(4) \leq 1$ *if,* $[*, k_1, 2, k_1 - 1, *]$ *and a generator of the module $M(4)$ is the complete linearization of x^2x^2,*

b) $k(1^4) \leq 1$ *if* $[*, k_1, 0, 1 - k_1, *]$ *and the module $M(1^4)$ is generated by* $S_{1(21)}(x_1, x_2, x_3, x_4)$,

c) $k(3,1) = 0$,

d) $k(2,1^2) \leq 1$ *if* $[*, 0, 1, -1, *]$, *where a generator of $M(2,1^2)$ is the linearization of* $\sum_{\sigma \in S_3}(-1)^\sigma x_{\sigma(1)}(x_{\sigma(2)}(x_{\sigma(3)}x_1))$,

e) $k(2^2) = 0$.

f) *Otherwise the variety $\mathfrak{M}_1$ is nilpotent of index* 4.

Proof. We consider the corresponding GL_m-modules.

(4): The system (2.2) leads to the following matrix of the coefficients of the generators x^2x^2, xx^3 and x^4:

$$
A = \begin{bmatrix} 1 & 1 & 0 \\ k_1 & 0 & -1 \\ k_2 - 1 & 1 & 0 \\ k_3 + 1 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} k_1 & 0 & -1 \\ 1 & 1 & 0 \\ k_2 - 2 & 0 & 0 \\ k_3 - k_1 + 1 & 0 & 0 \end{bmatrix}.
$$

Rank $A = 2$ gives the conditions on k_2 and k_3 in a).

(1^4): In this case the first three identities of (2.3) are changed, namely $f_2 = 0, f_5 = 0, f_3 - f_4 = 0$ and the matrix of the coefficients of f_1, f_3 and f_4 is the following:

$$A = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 2-k_1 & -2k_1 \\ 0 & -1-k_2 & 1-2k_2 \\ 1 & 1-k_3 & -2k_3 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 0 \\ -1 & 2-k_1 & 2-3k_1 \\ 0 & -1-k_2 & -3k_2 \\ 0 & 3-k_1-k_3 & 3-3k_1-3k_3 \end{bmatrix}$$

(3.1): Corresponding to (c6), ..., (c10) of (2.5) are now the identities:

$$a_5 + b_5 + c_5 = 0$$
$$a_5 - b_3 + c_3 - c_4 = 0$$
$$a_3 - a_4 + b_4 - b_5 + c_5 = 0$$
$$2a_3 - a_5 - b_3 + 2b_4 - c_3 - c_4 + 2c_5 = 0$$
$$a_3 + b_2 - 3c_2 = 0$$

The system has a trivial solution only.

($2, 1^2$): Corresponding to (d1), ..., (d5) of (2.7) are:

$$-f_3 + f_5 + g_3 + g_4 - g_5 - h_4 = 0$$
$$f_2 = 0$$
$$f_5 - g_5 + h_5 = 0$$
$$f_3 + f_5 - g_3 + g_4 + g_5 + 2h_3 + h_4 = 0$$
$$f_3 + 2f_4 + f_5 + g_3 - g_4 - g_5 + h_4 + 2h_5 = 0.$$

Easily we get the following matrix $A_{17\times 12}$:

$$\begin{bmatrix}
0 & -1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & -1 & -1 \\
0 & 1 & 0 & 2 & 0 & -1 & 1 & 0 & 0 & 2 & 1 & 1 \\
0 & 1 & 2 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 1 & 1 \\
-1 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & -3k_1 & 0 & 0 & 0 \\
0 & 0 & -2 & 3k_1 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 3k_1 \\
-1 & k_1 & 0 & 0 & -1 & k_1 & 0 & 0 & 1-k_2 & 0 & 0 & -k_1 \\
1 & k_1-2 & 0 & 2k_1 & -1 & -2-k_1 & -2k_1 & 2 & 1+k_2 & 2k_1 & -2k_1 & k_1 \\
1 & 1 & 0 & 0 & 0 & -1-k_1 & k_1 & 0 & 0 & 0 & -k_1 & k_1 \\
0 & 0 & 0 & k_2-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & k_2 \\
0 & k_2-1 & 2 & 0 & 0 & k_2-1 & -1 & 0 & 0 & 0 & 1 & -k_2 \\
0 & 1+k_2 & 0 & 2k_2-4 & 0 & -1-k_2 & 1-2k_2 & 0 & 0 & 2+2k_2 & 1-2k_2 & k_2-2 \\
0 & 1 & -1 & 0 & 0 & -k_2 & k_2 & 0 & 0 & 0 & -k_2 & k_2-1 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & -3k_3 & 0 & 0 & 0 \\
0 & 0 & 1 & -3k_3 & 0 & 0 & 2 & 0 & 0 & 0 & 1 & -3k_3 \\
-1 & 1+k_3 & 0 & 0 & -1 & k_3-1 & 0 & 0 & 0 & 2 & 0 & -k_3 \\
-1 & k_3-1 & 0 & 2k_3 & 1 & -1-k_3 & -2k_3 & -2 & 2+2k_2 & 2k_3 & -2k_3 & k_3 \\
0 & 0 & 0 & 0 & 1 & -k_3 & k_3 & 0 & 1-k_2 & -1 & -k_3 & k_3
\end{bmatrix}$$

Investigations on its rank give the result.

(2^2). Now the first two identities in (2.9) are

$$2f_3 - f_4 + 2f_5 - f_7 - f_8 - f_{10} = 0 \quad \text{and} \quad - f_4 + f_7 + f_8 + f_{10} = 0$$

The system has a trivial solution only.

Proof of condition f). The partial linearization of $x^4 = 0$ and the identities $a_1 = b_1 = c_1 = 0$ lead to $yxxx = xyxx = x^2yx = x^3y = 0$. New partial linearizations in x and $f_1 = f_6 = 0$ (generators for $N_2(2^2)$) lead to

$$x^2yy = xyyx = yxxy = y^2xx = xyxy = yxyx = 0.$$

In a similar way the partial linearizations of these identities in y and $f_1 = g_1 = h_1 = 0$ (generators of $N_3(2,1^2)$) lead to $zxxy = xyxz = x^2yz = xyzx = yxzx = yzxx = 0$.

New linearizations and $f_1 = 0$ (a generator of $N_4(1^4)$) give $zytx = 0$. Another way of brackets is treated anagously.

THEOREM 3.2. *For* $\mathfrak{M}_2 = [*, k_1, k_2, k_3, p]$ *the conditions on the multiplicities of the modules in* (3.1) *are the following*:

a) $k(4) \leq 1$ *if* $[*, k_1, 2, k_1 - 1, p]$. *The module* $M(4)$ *is generated by the complete linearization of* x^2x^2,

b) $k(1^4) \leq 1$ *if* $[*, k_1, 2 - k_1, k_1, 0]$, $[*, k_1, 1 - k_1, k_3, 1]$ *or*

$$[*, k_1, k_2, k_3, p \neq 0, 1 : k_2(p^2 - 2p - 1) = pk_1 + k_1 - 2,$$
$$k_1(p^2 + 3p + 1) = 2 + p - k_3 - pk_3].$$

The module $M(1^4)$ *is generated by* $S_{1(12)}(x_1, x_2, x_3, x_4)$,

c) $k(3,1) = 0$,

d) $1 \leq k(2,1^2) \leq 2$ *if* $[*, -2, -1, -1, -1]$. *Module generators are the linearizations of* $\sum_{\sigma \in S_3} (-1)^\sigma (x_{\sigma(1)} x_{\sigma(2)})(x_1 x_{\sigma(3)})$ *and* $\sum_{\sigma \in S_3} (-1)^\sigma x_{\sigma(1)} \times (x_{\sigma(2)} x_1 x_{\sigma(3)})$,

e) $k(2^2) \leq 1$ *if* $[*, 0, 1, -1, p]$. $M(2^2)$ *is generated by the complete linealization of* $\sum_{\sigma,\tau \in S_2} (-1)^{\sigma+\tau} x_{\sigma(1)} (x_{\tau(1)} (x_{\sigma(2)} x_{\tau(2)}))$. *The same unequality holds for both the cases* $[*, 2, -1, 1, 1]$ *and* $[*, -2, -1, -1, 5]$ *and the module* $M(2^2)$ *is generated by the complete linearization of* $\sum_{\sigma,\tau \in S_2} (-1)^{\sigma+\tau} x_{\sigma(1)} x_{\sigma(2)} \times (x_{\tau(1)} x_{\tau(2)})$.

f) *Otherwise* $\mathfrak{M}_2$ *is a nilpotent variety of index* 4.

THEOREM 3.3. *For* $\mathfrak{M}_3 = [k, k_1, k_2, k_3, *]$ *we have in* (3.1):

a) $k(4) \leq 1$ *if* $[0, k_1, k_1, -k_1, *]$, $[1, k_1, k_2 \neq 1, k_3, *]$ *or*

$$[k \neq 0; 1, k_1, k_2, k_3, * : k_3(k-1) = k_2 - k + kk_1, \ 2k_3 = 2k + k_1 - (3+k)k_2],$$
$$k(4) = 1 \quad \textit{if} \ [1, k_1 \neq 0, k_2, k_3, *] \ \textit{or} \ [1, k_1, k_2, k_3 \neq -1, *].$$

$M(4)$ *is generated by the complete linearization of*

$x(xx^2)$, $1 \leq k(4) \leq 2$ *if* $[1,0,1,-1,*]$. *The two generators are complete linearizations of* xx^2x *and* $x(xx^2)$,

b) $k(1^4) \leq 1$ *if* $[k, k_1, 0, 1-k_1, *]$. *The module* $M(1^4)$ *is generated by* $S_{1(12)}(x_1, x_2, x_3, x_4)$,

c) $k(3,1) \leq 1$ *if* $[0,0,0,0,*]$ *or* $[1,0,0,1,*]$. *In the first case* $M(3,1)$ *is generated by the complete linearization of* $\sum_{\sigma \in S_2}(-1)^\sigma x_{\sigma(1)}(x_1(x_1 x_{\sigma(2)}))$ *or by that of* $\sum_{\sigma \in S_2}(-1)^\sigma x_{\sigma(1)} x_1 x_1 x_{\sigma(2)}$ *in the second one,*

d) $k(2,1^2) = 0$,

e) $k(2^2) \leq 1$ *if* $[k \neq 1, 0, 1, -1, *]$. *The module* $M(2^2)$ *is generated by the complete linearization of* $\sum_{\sigma,\tau \in S_2}(-1)^{\sigma+\tau}(x_{\sigma(1)} x_{\tau(1)})(x_{\sigma(2)} x_{\tau(2)})$,

$k(2^2) = 2$ *if* $[1,0,1,-1,*]$. *Generators are the complete linearizations of* $\sum_{\sigma,\tau \in S_2}(-1)^{\sigma+\tau}(x_{\sigma(1)} x_{\tau(1)})(x_{\sigma(2)} x_{\tau(2)})$ *and of* $\sum_{\sigma,\tau \in S_2}(-1)^{\sigma+\tau} x_{\sigma(1)} \times (x_{\tau(1)} x_{\sigma(2)} x_{\tau(2)})$.

f) *Otherwise the variety* $\mathfrak{M}_3$ *is nilpotent of index* 4.

THEOREM 3.4. *If in* $\mathfrak{M}_4$ *and* $\mathfrak{M}_5$ $k \neq 1$ *then* $P_4(\mathfrak{M}_4) = \{0\}$ *and* $P_4(\mathfrak{M}_5) = \{0\}$. *For* $k = 1$ $P_4(\mathfrak{M}_4) = M(4)$ *and* $P_4(\mathfrak{M}_5) = M(4)$.

THEOREM 3.5. $P_n(\mathfrak{M}_6) = M(n)$ *for* $n \geq 3$.

Proof. Because of $f_i=0$ $(i=1,\ldots,4)$ and $S_{12}=S_{21}=0$ $x_1 \ldots x_k x_{k+1} \ldots x_4 - x_1 \ldots x_{k+1} x_k \ldots x_4 \in P_4(\mathfrak{M}_6) \cap T(\mathfrak{M}_6)$ for any brackets save $(**)(**)$. In the last case we refer to the corresponding to (2.2), (2.3), (2.5), (2.7) and (2.9) systems and get $P_4(\mathfrak{M}_6) = M(4)$. Using induction on n we see that

$$x_1 \ldots (x_k x_{k+1}) \ldots x_n - x_1 \ldots (x_{k+1} x_k) \ldots x_n \in P_n \cap T(\mathfrak{M}_6).$$

If $k > 1$ then

$$\begin{aligned}
&(x_1(\ldots x_k))(x_{k+1} x_{k+2} \ldots x_n) - (x_1(\ldots x_{k+1}))(x_k x_{k+2} \ldots x_n) \\
&= x_{k+1}(x_1(\ldots x_k) x_{k+2} \ldots x_n) - x_k(x_1(\ldots x_{k+1}) x_{k+2} \ldots x_n) \\
&= x_{k+1}(x_1(\ldots x_{k+2}) x_k \ldots x_n) - x_k(x_1(\ldots x_{k+2}) x_{k+1} \ldots x_n) \\
&= (x_1(\ldots x_{k+2}))(x_{k+1} x_k \ldots x_n) - (x_1(\ldots x_{k+2}))(x_k x_{k+1} \ldots x_n) = 0.
\end{aligned}$$

For $k = 1$

$$\begin{aligned}
&x_1(x_2, x_3 \ldots x_n) - x_2(x_1 x_3 \ldots x_n) \\
&= (x_2 x_3)(x_1 x_4 \ldots x_n) - (x_1 x_3)(x_2 x_4 \ldots x_n) \\
&= (x_2(x_1 x_4))(x_3 x_5 \ldots x_n) - (x_1(x_2 x_4))(x_3 x_5 \ldots x_n) = 0.
\end{aligned}$$

All the equalities are modulo $P_{n-1} \cap T(\mathfrak{M}_6)$. So $P_n(\mathfrak{M}_6) = M(n)$.

Acknowledgment: I would like to thank V. Drensky for suggesting this problem to me and the useful conversations during the preparation of the paper.

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Received November 3, 1992.