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A NOTE ON OPIAL TYPE INEQUALITIES INVOLVING PARTIAL SUMS

1. Introduction

In [8] Z. Opial found an interesting and useful integral inequality involving a function and its derivative. Further, a large number of papers deal with various extensions and generalizations of the Opial inequality (see [1], [5]–[10]). The main purpose of the present note is to establish two new Opial type inequalities involving partial sums. Our results are based on the Hardy inequality involving partial sums (see [2]–[4]) and the discrete analogue of the Opial inequality given by Wong in [10].

2. Result

First, we recall the known inequalities.

LEMMA 1 (see [2]–[4]). *Let* $\lambda_n > 0$, $a_n \geq 0$, $n = 1,2,\ldots$, *and* $\Lambda_n = \lambda_1 + \ldots + \lambda_n$, $A_n = \lambda_1 a_1 + \ldots + \lambda_n a_n$. *Then*

$$\sum_{n=1}^{m} \lambda_n \left(\frac{A_n}{\Lambda_n}\right)^p \leq \left(\frac{p}{p-1}\right)^p \sum_{n=1}^{m} \lambda_n a_n^p, \qquad p > 1.$$

LEMMA 2 (see [10]). *For nondecreasing sequence of nonnegative numbers* $\{u_n\}_1^\infty$ *we have*

$$\sum_{n=1}^{m} u_n^p (u_n - u_{n-1}) \leq \frac{(m+1)^p}{p+1} \sum_{n=1}^{m} (u_n - n_{n-1})^{p+1}, \qquad p \geq 1,$$

with $u_0 = 0$.

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Our result is given in the following theorem.

THEOREM. *Let $p \geq 1$, $q \geq 1$ and λ_n, a_n, Λ_n, A_n be as defined in Lemma 1. Then*

$$\sum_{n=1}^{m} \frac{A_n^p(A_n^q - A_{n-1}^q)}{\Lambda_n^{p+q-1}} \leq q(\frac{p+q}{p+q-1})^{p+q-1} \sum_{n=1}^{m} \lambda_n a_n^{p+q}, \tag{1}$$

$$\sum_{n=1}^{m} A_n^p(A_n^q - A_{n-1}^q) \leq \frac{q(m+1)^{p+q-1}}{p+q} \sum_{n=1}^{m} (\lambda_n a_n)^{p+q}, \tag{2}$$

where any number with suffix zero is zero.

Proof. Since $A_{n-1} \leq A_n$, we have

$$A_n^p(A_n^q - A_{n-1}^q) = A_n^p\Big(\sum_{k=0}^{q-1} A_n^{q-1-k} A_{n-1}^k\Big)(A_n - A_{n-1}) \leq \tag{3}$$

$$\leq q A_n^{p+q-1}(A_n - A_{n-1})$$

which implies, by using the Hölder inequality with indices $p+q$, $\frac{p+q}{p+q-1}$ and Lemma 1, the inequality

$$\sum_{n-1}^{m} \frac{A_n^p(A_n^q - A_{n-1}^q)}{\Lambda^{p+q-1}} \leq$$

$$\leq q \sum_{n=1}^{m} \left(\frac{A_n}{\Lambda_n}\right)^{p+q-1} (\lambda_n a_n) =$$

$$= q \sum_{n=1}^{m} \alpha_n^{\frac{1}{p+q}} a_n \lambda_n^{\frac{p+q-1}{p+q}} \left(\frac{A_n}{\Lambda_n}\right)^{p+q-1} \leq$$

$$\leq q\Big(\sum_{n=1}^{m} \lambda_n a_n^{p+q}\Big)^{\frac{1}{p+q}} \left[\sum_{n=1}^{m} \lambda_n (\frac{A_n}{\Lambda_n})^{p+q}\right]^{\frac{p+q-1}{p+q}} \leq$$

$$\leq q\Big(\sum_{n=1}^{m} \lambda_n a_n^{p+q}\Big)^{\frac{1}{p+q}} \left[\left(\frac{p+q}{p+q-1}\right)^{p+q} \sum_{n=1}^{m} \lambda_n a_n^{p+q}\right]^{\frac{p+q-1}{p+q}} =$$

$$= q\left(\frac{p+q}{p+q-1}\right)^{p+q-1} \sum_{n=1}^{m} \lambda_n a_n^{p+q}$$

proving (1).

From (3), using Lemma 2, we have

$$\sum_{n=1}^{m} A_n^p(A_n^q - A_{n-1}^q) \leq q\sum_{n=1}^{m} A_n^{p+q-1}(A_n - A_{n-1}) \leq$$

$$\leq \frac{q(m+1)^{p+q-1}}{p+q}\sum_{n=1}^{m}(A_n - A_{n-1})^{p+q} =$$

$$= \frac{q(m+1)^{p+q-1}}{p+q}\sum_{n=1}^{m}(\lambda_n a_n)^{p+q}$$

which proves (2).

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