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ASYMPTOTIC BEHAVIOUR OF SOLUTIONS OF A DELAYED DIFFERENTIAL EQUATION

1. Introduction

In this paper we consider the asymptotic expansion of solutions of delayed differential equations

$$g(t)\dot{y}(t) = -ay(t) + \sum_{i+j=2}^{N} c_{ij}(t)y^i(t)y^j(t-r), \tag{1}$$

where $N \geq 2$ is an integer, $a > 0$, $r > 0$ are constants, $g(t) : R_+^0 \to R_+$, $c_{ij}(t) : R_+^0 \to R$ are continuous functions (further conditions will be given latter). The purpose of this paper is to prove that for each real parameter C and function $\psi \in B_o = \{\psi \in C^0[-r,0], \|\psi\| \leq 1,\ \psi(0) = 0\}$ which describe the power of the set of solutions, there is a solution $y(t) = y(t,C,\psi)$ of 1 which may be at $t \to \infty$ represented by asymptotic series (symbol $\approx$ denotes the asymptotic expansions)

$$y(t,C,\psi) \approx \sum_{k=1}^{\infty} f_k(t)\varphi^k(t,C) \tag{2}$$

where $\varphi(t,C)$ is the solution of equation $g(t)\dot{y}(t) = -ay(t)$, given by the formula $\varphi(t,C) = C\exp\int_0^t \frac{-a}{g(u)}\,du$, $f_1(t) \equiv 1$ and the functions $f_k(t)$ for $k = 2,\ldots,n$ are particular solutions of some system of auxiliary differential equations. To prove our results we will use Ważewski's topological method in the form, proposed by K. Rybakowski [5], which may be used for differential equations with retarded arguments. The first Lyapunoff's method is often used to construct the solutions of ordinary differential equations in the form of power-like series. Such a way is not possible here. First lefthand ends of existence intervals of partial sums can tend to infinity and, secondly, if it does not happen, the partial sums need not to converge uniformly. The modification of the first Lyapunoff's method were used in [6], [1].

Delayed differential equations appear in many technical problems. The form of equation (1) include some equations which have been recently considered. For example the logistic equation with recruitment delays

$$\dot{x}(t) = x(t-r)(A - Bx(t))$$

which were considered by Gopalsamy [2], with regard to the applications on ecology. After substitution $x(t) = \frac{A}{B} + y(t)$ have the form of equation (1), where $g(t) = 1$, $a = A$, $c_{11} = -B$ and $c_{ij} = 0$ for $i \neq 1$, $j \neq 1$, $N = 2$. Moreover also one branch of the equation partially solved with respect to derivative in Diblik's work [1] have (after solving with respect to derivatives) the form of the equation (1), in which are not terms with retarded arguments.

2. Preliminaries

To describe simply coefficients of power series raised to a power, it is suitable to denote: α, β — are sequences of nonnegative integers with finite sumation.

Let $\alpha = \{\alpha_k\}_{k=1}^{\infty}$, then we denote

$$|\alpha| = \sum_{k=1}^{\infty} \alpha_k,\ V(\alpha) = \sum_{k=1}^{\infty} k\alpha_k,\ \alpha! = \prod_{k=1}^{\infty} \alpha_k!,\ \max(\alpha) = \max\{k \mid \alpha_k \neq 0\}.$$

Let $\mathbf{a} = \{a_k\}_{k=1}^{\infty}$ be any sequence (of numbers or functions). We define

$$\mathbf{a}^{\alpha} = \prod_{k=1}^{\infty} a_k^{\alpha_k}, \qquad \text{where } a_k^0 = 1 \text{ for every } a_k.$$

Then it is possible to prove

$$\Big(\sum_{k=1}^{\infty} a_k \mathbf{x}^k\Big)^n = \sum_{k=n}^{\infty} \mathbf{x}^k \sum_{n}^{k} \frac{n!}{\alpha!}\mathbf{a}^{\alpha},$$

where $\sum_n^k$ denotes the sumation over all sequences such that $|\alpha| = n$, $V(\alpha) = k$. As we work with the product of the power series raised to a power, we denote $\sum_{i,j}^k$ is the sumation over all couples (α, β) such that $V(\alpha) + V(\beta) = k$, $|\alpha| = i$, $|\beta| = j$.

Throughout this paper $g(t)$, $G(t)$ denote functions such that

C1. $g(t) \in C^0[0,\infty)$, $g(t) > 0$ for $t \geq t_0$ and $g(t) = O(1)$ as $t \to \infty$.

C2. $G(t) = o(g(t))$ as $t \to \infty$, where $G(t) = \left(\int_0^t g^{-1}(u)\,du\right)^{-1}$

C3. there is a constant $\lambda > 0$ such that

$$\frac{g(t) - g(t-r)}{g(t-r)} = o(G^{\lambda}(t)) \quad \text{as } t \to \infty.$$

This condition enables us to consider relative large class of functions: $g(t)$ may be constant, a periodical function (r is a period) or there is a positive $\lim_{t\to\infty} g(t)$ and if $\lim_{t\to\infty} g(t) = 0$ in addition then the function $g(t)$ must satisfy

$$\int_0^t g(u)\,du = o(g^k(t)) \quad \text{as } t \to \infty,\ k > 0 \text{ is a constant.}$$

LEMMA 1. *Let functions $g(t)$, $G(t)$ satisfy the conditions* **C1, C2, C3.** *Then*:

1. $G(t) \sim G(t-K)$ *as* $t \to \infty$ *where* K *is any constant*
2. $g(t)(g^{-1}(t-ir) - g^{-1}(t-ir+r)) = o(G^\lambda(t))$ *as* $t \to \infty$.

Proof.

$$\lim_{t\to\infty} \frac{G(t)}{G(t-K)} = \lim_{t\to\infty} \frac{\int_0^t g^{-1}(u)\,du - \int_{t-K}^t g^{-1}(u)\,du}{\int_0^t g^{-1}(u)\,du} =$$

$$= 1 - \lim_{t\to\infty} G(t) \int_{t-K}^{t} g^{-1}(u)\,du = 1,$$

therefore the function $G(t)$ is a decraesing function and we obtain

$$\lim_{t\to\infty} G(t) \int_{t-K}^{t} g^{-1}(u)\,du \le \lim_{t\to\infty} \int_{t-K}^{t} G(u)g^{-1}(u)\,du \le Ko(1) = 0.$$

Moreover for $t \to \infty$ using **C3** we get

$$g(t) = g(t-ir)\prod_{j=1}^{i}(1 + o(G^\lambda(t-jr))) = g(t-ir)\prod_{j=1}^{i}(1 + o(G^\lambda(t)(1+o(1)))$$

$$= g(t-ir)\sum_{j=1}^{i}\binom{i}{j}1^j(o(G^\lambda(t)(1+o(1))))^{i-j} = g(t-ir)(1+o(G^\lambda(t)).$$

Thus

$$\frac{g(t)-g(t-ir)}{g(t-ir)} = o(G^\lambda(t)).$$

Eventually we get

$$g(t)(g^{-1}(t-ir) - g^{-1}(t-ir+r)) =$$

$$\frac{g(t)-g(t-ir)}{g(t-ir)} - \frac{g(t)-g(t-ir+r)}{g(t-ir+r)} = o(G^\lambda(t)).$$

LEMMA 2. *Let the coefficients of equation*

$$g(t)\dot{y}(t) = Ky(t) + E(t)f(t) \tag{3}$$

satisfy:

1. *$K > 0$ is a constant,*
2. *the functions $g(t)$, $G(t) = (\int_0^t g^{-1}(u)\,du)^{-1}$ fulfill* **C1**, **C2**, **C3**,
3. *$E(t) \equiv \exp \sum_{i=1}^n \int_{t-ir}^{t-ir+r} \frac{K_i}{g(u)}\,du$, where $K_i > 0$ are constants,*
4. *the function $f(t)$ has the asymptotic form $f(t) = G^\gamma(t)b(t) + O(G^{\gamma+\varepsilon_1}(t))$, as $t \to \infty$ where $\varepsilon_1 > 0$, γ are constants, $b(t) \in C^1[t_0, \infty)$ and moreover: $b(t) = o(G^\tau(t))$ as $t \to \infty$, for all $\tau > 0$, $g(t)\dot{b}(t) = o(G^\delta(t))$ as $t \to \infty$, where $\delta > 0$ is a constant.*

Then there exists the solution $Y(t)$ of (3) such that, the following asymptotic relations hold

$$Y(t) = E(t)G^\gamma(t)\Big(-\frac{b(t)}{K} + O(G^\varepsilon(t))\Big) \qquad \dot{Y}(t) = O(g^{-1}(t)G^{\gamma+\varepsilon}(t)),$$

where $0 < \varepsilon < \min(\lambda, \varepsilon_1, \delta, 1)$ is a constant.

Proof. After the subtitution $y(t) = x(t)E(t)$ the equation (3) has form:

$$g(t)\dot{x}(t) = \Big(K + \sum_{i=1}^n K_i g(t)(g^{-1}(t-ir) - g^{-1}(t-ir+r))\Big)x(t) + f(t). \tag{4}$$

We define the domain $\Omega = \{(x,t) | t > t_0,\ u(x,t) < 0\}$, where $u(x,t) = (ax + G^\gamma(t)b(t))^2 - G^{2(\gamma+\varepsilon)}(t)$. The assumptions of Picard-Lindelöf's theorem are locally satisfied in the domain Ω, therefore throught each $(x,t) \in \Omega$ goes a unique solution of (4). Using the assumptions 1, 2, 3, 4 we compute the trajectory derivative $\dot{u}(x,t)$ along the solution $x(t)$ of (3) on the bound $\partial\Omega$:

$$\dot{u}(x,t) = \frac{2}{g(t)}\{KG^{2(\gamma+\varepsilon)}(t) - G^{2(\gamma+\varepsilon+1)}(t) + G^{2(\gamma+\varepsilon+\lambda)}(t)o(1) \pm$$
$$\pm\, G^{2\gamma+\varepsilon}[G^\lambda(t)b(t)o(1) + KG^{\varepsilon_1}(t)O(1) - \gamma b(t)G(t) + G^\delta(t)o(1)]\}.$$

For sufficiently large t the construction of the number ε implies

$$\operatorname{sign} \dot{u}(x,t) = \operatorname{sign} \frac{2a}{g(t)} G^{2(\gamma+\varepsilon)}(t) = 1.$$

Then according to Ważewski's principle [4, p. 282] there is at least one solution $x(t)$ of (4) such that $x(t) \in \Omega$. The asymptotic form of the solution $x(t)$ and also $y(t) = E(t)x(t)$ is obtained from the construction of the domain Ω.

3. Main results

Let the formal solution of equation (1) be expressed in the form (2), where $\varphi(t,C)$ is the general solution of the equation $g(t)\dot{y}(t) = -ay(t)$, consequently $\varphi(t,C) \equiv C \exp \int_{t_0}^t \frac{-a}{g(s)}\,ds$, where C is a constant and $f_1(t) = 1$, $f_k(t)$ for $k \geq 2$ are unknown functions for the time being. After substituing

$y(t, C)$ in the equation (1) and comparing coefficients of the same powers $\varphi^k(t, C)$ we obtain an auxiliary system of linear differential equations:

$$(5_k) \qquad g(t)\dot{f}_k(t) = a(k-1)f_k(t) + \sum_{i+j=2}^{N} c_{ij}(t) \sum_{i,j}^{k} \frac{i!j!}{\alpha!\beta!} \mathbf{f}^{\alpha}(t)\mathbf{h}^{\beta}(t)$$

where

$$\mathbf{f}(t) = \{f_k(t)\}_{k=1}^{\infty}, \quad \mathbf{h}(t) = \{h_k(t)\}_{k=1}^{\infty} = \left\{ f_k(t-r) \exp \int_{t-r}^{t} \frac{ak}{g(s)}\, ds \right\}_{k=1}^{\infty}.$$

As $V(\alpha) + V(\beta) = k$ and $|\alpha| + |\beta| \geq 2$ yields $\alpha_l = 0$ and $\beta_l = 0$ for $l \geq k$, the auxiliary system (5_k) is recurrent. Therefore we may define recurrently two sequences of functions:

$$\mathbf{p}(t) = \{p_k(t)\}_{k=1}^{\infty}, \quad \mathbf{q}(t) = \{q_k(t)\}_{k=1}^{\infty} = \left\{ p_k(t-r) \exp \int_{t-r}^{t} \frac{ak}{g(s)}\, ds \right\}_{k=1}^{\infty},$$

$$p_1(t) = 1,$$

$$p_k(t) = \frac{1}{a(k-1)} \sum_{i+j=2}^{N} c_{ij}(t) \sum_{i,j}^{k} \frac{i!j!}{\alpha!\beta!} \mathbf{p}^{\alpha}(t)\mathbf{q}^{\beta}(t).$$

If $|\beta| \neq 0$, then the expresion $\exp \int_{t-r}^{t} \frac{ak}{g(s)}\, ds$ is included in $\mathbf{q}^{\beta}(t)$ and also in $p_k(t)$. Now using Lemma 2 we describe the asymptotic behaviour of particular solutions of the system 5_k.

THEOREM 1. *Let the functions $p_k(t)$ have the asymptotic form*

$$p_k(t) = E_k(t) G^{\gamma_k}(t)(b_k(t) + O(G^{\varepsilon_k}(t)))$$

as $t \to \infty$ where $\varepsilon_k > 0$, γ_k are constants, $b_k(t) \in C^1[t_k, \infty)$, $b_k(t) = o(g^{\tau}(t))$ as $t \to \infty$ for any positive τ, $g(t)\dot{b}_k(t) = o(g^{\lambda_k}(t))$, as $t \to \infty$, $\lambda_k > 0$ is a constant.

$$E_k(t) = \exp \sum_{i=1}^{n_k} K_k^i \int_{t-ir}^{t-ir+r} \frac{ds}{g(s)}.$$

Assume further there is a sequence $\{\nu_k\}_{k=1}^{\infty}$ such that

$$\nu_k \in (\gamma_k, \gamma_k + \min(\lambda, \delta_k, 1, \varepsilon_k - \Delta_k^*)),$$

where $\Delta_k^* = \max(\Delta_1, \ldots, \Delta_{k-1})$, $\Delta_1 = 0$, $\Delta_l = \gamma_l + \varepsilon_l - \nu_l$ for $l = 2, \ldots, k-1$.

Then the coeficients $f_k(t)$ of the series (2), *which are the solutions of the auxiliary system* (5_k), *i. e.*

(6_k) $\quad f_k(t) =$

$$= \int_t^\infty \frac{-1}{g(s)} \sum_{i+j=2}^{N} c_{ij}(s) \sum_{i,j}^{k} \frac{i!j!}{\alpha!\beta!} \mathbf{f}^\alpha(s)\mathbf{h}^\beta(s) \exp\left\{ - \int_t^s \frac{a(k-1)}{g(u)}\, du \right\} ds$$

can be expressed in the asymptotic form

$$(7_k) \qquad \begin{aligned} f_k(t) &= E_k(t)G^{\gamma_k}(t)\Big(- \frac{b_k(t)}{a(k-1)} + O(g^{\nu_k}(t))\Big) \\ \dot{f}_k(t) &= \frac{1}{g(t)} E_k(t) O(G^{\nu_k}(t)). \end{aligned}$$

Proof. The formulas (6_k) are obtained by integrating the system (5_k). The convergence of (6_k) is evident. It remains to show the asymptotic estimate (7_k). This will be done by induction.

For $k = 2$ the coefficients of the equation (5_2) satisfy the requirements of Lemma 2, thus the solution (6_2) has the form (7_2).

In spite of $\mathbf{f}(t)$ being substituted instead of $\mathbf{p}(t)$ and $\mathbf{h}(t)$ being substituted instead of $\mathbf{q}(t)$, in the recurrent definition of $p_k(t)$, the asymptotic form

$$p_k^*(t) = g^{\gamma_k}(t)(b_{1k}(t) + O(b_{0k}^*(t)g^{\varepsilon_k - \Delta_k^*}(t)))G_k(t)$$

has the same asymptotic properties like $p_k(t)$. Therefore the equation (5_k) satisfies the assumptions of Lemma 2, then (6_k) takes the form (7_k) and the theorem is proved.

Remark. The necessary condition for satisfying assumptions of Theorem 1 is $\lim_{t\to\infty} c_{ij}(t)\exp(-\tau G^{-1}(t)) = 0$. This is satisfied for example if functions $c_{ij}(t)$ have the same asymptotic behaviour $p_k(t)$.

We shall denote

$$y_n(t) = \sum_{k=1}^{n} f_k(t)\varphi^k(t, C) \quad \text{and} \quad \sum_n(t) = \sum_{i+j=2}^{N} c_{ij}(t) \sum_{i,j}^{k} \frac{i!j!}{\alpha!\beta!} \mathbf{f}^\alpha(t)\mathbf{h}^\beta(t).$$

THEOREM 2. *Let the assumptions of Theorem* 1 *hold and suppose that*

$$\lim_{t\to\infty} f_{n+1}^{-1}(t)\exp(-\tau G^{-1}(t)) = 0,$$

where $\tau < 1$ *is a constant. Then for every* $C \neq 0$ *and* $\psi \in C^0[-r, 0]$, $\|\psi\| \leq 1$, $\psi(0) = 0$ *there exists a solution* $y_C(t)$ *of equation (1) such that*

$$(8) \qquad |y_C(t) - y_n(t)| \leq \delta |f_{n+1}(t)\varphi^{n+1}(t, C)|$$

for $t \in [t_C, \infty)$ *where coefficients* $f_k(t)$ *are the solutions* (6_k) *of the system* (5_k), $\delta > 1$ *is a constant,* t_C *is a function of the parametr* C *and of* δ, n.

P r o o f. The existence of solution $y_C(t)$ which satisfies the inequality (8) will be proved by Ważewski's principle for retarded functional differential equations $\dot{y} = f(t, y_t)$, where y_t denotes an element of $C^0 = C^0[-r, 0]$ defined as $y_t(\theta) = y(t+\theta)$, $\theta \in [-r, 0]$ for any continuous mapping y from an interval $[-r+t, t]$ into R. For this method see [5]. The function $f(t, y_t) : R \times C^0[-r, 0] \to R$, defined by a formula

$$f(t,\phi) = \frac{1}{g(t)}\Big(- a\phi(0) + \sum_{i+j=2}^{N} c_{ij}(t)\phi^i(0)\phi^j(-r)\Big)$$

is continuous and Lipschitzian in ψ in each compact set in Ω^ε, where

$$\Omega^\varepsilon = \{(t,\psi) | t > t_0 - r;\ \|\varphi - y_{nt}\| < A(t)\},$$
$$\|\phi\| = \sup_{-r\le\theta\le 0} |\phi(\theta)| \quad \text{and} \quad A(t) = (\varepsilon+1) \max_{t-r\le\theta\le t} (f_{n+1}(\theta)\varphi^{n+1}(\theta, C))$$

for a positive constant ε. Thus for any $(t,\phi) \in \Omega^\varepsilon$ there exists the unique solution of the equation $y = f(t, y_t)$ [3,p. 42].

We shall prove that $\omega = \{(y,t) \mid l(y,t) < 0,\ t > t_C\}$, where $l(y,t) = (y - y_n(t))^2 - (\delta f_{n+1}(t)\varphi^{n+1}(t,C))^2$ is the regular polyfacial set with respect to the equation $\dot{y} = f(t, y_t)$, where $f(t,\phi)$ is defined as above. Then for all $K \in (-1, 1)$

$$\begin{aligned}\dot{l}(y,t) = \frac{2}{g(t)}\Big(&(\pm\delta\varphi^{n+1}(t,C)|f_{n+1}(t)|)\Big[- a(y_n(t) \pm \delta\varphi^{n+1}(t,C)|f_{n+1}(t)|)+\\ &+ \sum_{i+j=2}^{N} c_{ij}(t)(y_n(t) \pm \delta|f_{n+1}(t)|\varphi^{n+1}(t,C))^i \times (y_n(t-r)+\\ &+ K\delta|f_{n+1}(t-r)|\varphi^{n+1}(t-r,C))^j + ay_n(t) - \sum_{k=1}^{n}\varphi^k(t,C)\sum_k(t)\Big] -\\ &- (\delta\varphi^{n+1}(t,C))^2[-a(n+1)f_{n+1}^2(t) + g(t)\dot{f}_{n+1}(t)f_{n+1}(t)]\Big).\end{aligned}$$

Using binomial theorem for i, j-power in sumation $\sum_{i+j=2}^{N}$ we obtain

$$\begin{aligned}\dot{l}(y,t) = \frac{2}{g(t)}\Big(&(\delta\varphi^{n+1}(t,C))^2[-a(n+1)f_{n+1}^2(t) - g(t)\dot{f}_{n+1}(t)f_{n+1}(t)]\pm\\ &\pm \delta\varphi^{n+1}(t,C)\Big[- \sum_{k=1}^{n}\varphi^k(t,C)\sum_k(t) + \sum_{i+j=2}^{N} c_{ij}(t)(y_n^i(t)y_n^j(t-r)+\\ +y_n^i(t)\varphi^{n+1}(t,C)V_1(t) + &y_n^j(t-r)\varphi^{n+1}(t,C)V_2(t) + \varphi^{2n+2}(t,C)V_1(t)V_2(t))\Big]\Big),\end{aligned}$$

where

$$V_1(t)=\sum_{l=0}^{j-1}\binom{j}{l}(-1)^{j-l}y_n^l(t-r)\Big(K\delta|f_{n+1}(t-r)|\exp\int_{t-r}^{t}\frac{a(n+1)}{g(s)}\,ds\Big)^{j-l}\times$$
$$\times(\varphi^{n+1}(t,C))^{j-l-1},$$
$$V_2(t)=\sum_{l=0}^{i-1}\binom{i}{l}(-1)^{i-l}y_n^l(t)(\delta|f_{n+1}(t)|)^{i-l}(\varphi^{n+1}(t,C))^{i-l-1}.$$

Therefore

$$\{(\alpha,\beta)\mid V(\alpha)+V(\beta)\le n+1,\ |\alpha|+|\beta|\ge 2,\ \max(\alpha)\le n,\ \max(\beta)\le n\}=$$
$$\{(\alpha,\beta)|V(\alpha)+V(\beta)\le n+1,|\alpha|+|\beta|\ge 2\},$$

we obtain

$$\sum_{i+j=2}^{N}c_{ij}(t)y_n^i(t)y_n^j(t-r)=\sum_{k=1}^{n=1}\varphi^k(t,C)\sum_k(t)+$$
$$+\sum_{k=n+2}^{nN}\varphi^k(t,C)\sum_{i+j=2}^{N}c_{ij}(t)\sum_{i_nj_n}^{k}\frac{i!j!}{\alpha!\beta!}\mathbf{f}^\alpha(t)\mathbf{h}^\beta(t),$$

where $\sum_{i_nj_n}^k$ denotes the sumation ower all (α,β) such that $V(\alpha)+V(\beta)=k$, $|\alpha|=i$, $|\beta|=j$, $\max(\alpha)\le n$, $\max(\beta)\le n$.

Eventualy we get

$$\dot l(y,t)=\frac{2}{g(t)}\varphi^{2n+2}(t,C)\times$$
$$\times\Big[(anf_{n+1}^2(t)-g(t)\dot f_{n+1}(t)f_{n+1}(t))\delta^2\pm\delta f_{n+1}(t)\sum_{n+1}(t)\Big]\pm$$
$$\pm\,\delta\varphi^{2n+3}(t,C)\Big[\sum_{k=n+2}^{nN}\varphi^{k-n-2}(t,C)\sum_{i+j=2}^{N}c_{ij}(t)\sum_{i_nj_n}^{k}\frac{i!j!}{\alpha!\beta!}\mathbf{f}^\alpha(t)\mathbf{h}^\beta(t)\times$$
$$\times\sum_{i+j=2}^{N}c_{ij}(t)(+y_n^i(t)V_1(t)+y_n^j(t-r)V_2(t)+\varphi^{n+1}(t,C)V_1(t)V_2(t))\Big].$$

For sufficiently large $t>t_C$ and $\delta>1$ we deduce that

$$\operatorname{sign}\dot l(y,t)=\operatorname{sign}(anf_{n+1}^2(t)-g(t)\dot f_{n+1}(t)f_{n+1}(t)).$$

As $\lim_{t\to\infty}g(t)\frac{\dot f_{n+1}(t)}{f_{n+1}(t)}=\lim_{t\to\infty}(G^{\nu_{n+1}-\lambda_{n+1}}(t)=0$ we obtain

$$\operatorname{sign}\dot l(y,t)=\operatorname{sign}anf_{n+1}^2(t)=1.$$

Consequently ω is the polyfacial set regular with respect to the equation (1), $W = \partial\omega$, $Z = \{(y, t_C) \mid l(y, t_C) \le 0\}$.

We define $p : B = \overline{Z} \cap (Z \cup W) = Z \to C$ as:

$$p(z) = (y - y_n(t_C))(1 - |\psi|)\frac{(f_{n+1}(t)\varphi^{n+1}(t, C))_{t_C}}{r|f_{n+1}(t_C)\varphi^{n+1}(t_C, C)|} + (y_n(t))_{t_C} \text{ for } z = (y, t_C).$$

The mapping $p(z)$ is evidently continuos and for every $z \in B$ $p(z)$ satisfies:

$$(t_C + \theta, p(z)(\theta)) \in \omega \quad \text{for } \theta \in [-r, 0).$$

Moreover it holds: $Z \cap W$ is a retract of W but $Z \cap W$ is not a retract of Z. Then all assumptions of Ważewski principle for retarded functional differential equations are satisfied and thus there exists at least one solution $y_C(t)$ of (1) such that $y_C(t) \in \omega$ for $t > t_C$. The asymptotic form of the solution $y_C(t)$ is obtained from the construction of the domain ω and proof is complete.

Remark. As the relation $h_k(t) = f_k(t - r)\exp\int_{t-r}^{t} \frac{ak}{g(s)}\, ds$ is used in the definition of the sequences $\mathbf{f}(t)$ and $\mathbf{h}(t)$ and the function $h_k(t)$ is used in the definition of $f_{k+1}(t)$ the lefthand end of the existence interval of the function $f_{k+1}(t)$ is greater by r then the lefthand end of the existence interval of $f_k(t)$. If lefthand ends of the existence intervals of the functions $c_{ij}(t)$ are finite then lefthand ends of the existence intervals of the functions $f_k(t)$ must tend to infinity.

COROLLARY. *If all asumptions of Theorem 2 are satisfied for every n, then there exists the asymptotic expansion of the solution $y_C(t)$ in the form*

$$y_C(t) \approx \sum_{n=1}^{\infty} f_n(t)\varphi^n(t, C),$$

where the coefficients $f_n(t)$ are the solutions (5_n).

Proof. As

$$\lim_{t\to\infty} \frac{f_{n+1}(t)\varphi^{n+1}(t, C)}{f_n(t)\varphi^n(t, C)} =$$

$$= \lim_{t\to\infty} G^{\gamma_{n+1}-\gamma_n}(t)\frac{b_{1n+1}(t) + O(g^{\nu_{n+1}-\gamma_{n+1}}(t))}{b_{1n}(t) + O(g^{\nu_n-\gamma_n}(t))}\varphi(t, C) = 0$$

the assertion is proved.

EXAMPLE. We consider the equation:

$$\frac{1}{t}\dot{y}(t) = -2y(t) + y^2(t-1) + t\sin t\, y^2(t)y(t-1).$$

In this case we have $a = 2$, $r = 1$, $g(t) = \frac{1}{t}$, $\lambda = 2$, $\psi(t) = -1$. Then the auxiliary system (4_k) have a form:

$$\frac{1}{t}\dot{f}_n(t) = 2(n-1)f_n(t) + \frac{1}{(n-2)!}\exp(a_n t + b_n)(1 + O(t^{-0.9})).$$

Using Lemma 2 we obtain:

$$f_n(t) = \frac{1}{2(n-1)!}\exp(a_n t + b_n)(1 + O(t^{-0.9})),$$

where $a_n = n^2 + 2n - 2$ and $b_n = -\frac{1}{6}(2n^3 + 3n^2 - 11n + 6)$. Then according the Theorem 2 and corollary we obtain

$$y_C(t) \approx \sum_{n=1}^{\infty} \frac{C}{2(n-1)!}\exp\left(a_n t + b_n - \frac{t^2}{2}\right).$$

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