DEMONSTRATIO MATHEMATICA
Vol. XXVII No 3–4 1994

Elżbieta Zając

SUBDIRECTLY IRREDUCIBLE LEFT NORMAL BANDOIDS, II

Dedicated to Professor Tadeusz Traczyk

Introduction

This paper is a sequel to the author's works [1] and [2]. It is concerned with the characterisation of finite subdirectly irreducible left normal bandoids. In [1] a general structure theorem for left normal bandoids was given. In [2] a family of subdirectly irreducibles was constructed. In this and in the next paper we show that this family consists of all finite subdirectly irreducible left normal bandoids.

The notation and terminology of [1] and [2] will be used without explanation or apology in this paper. Our numbering here begins with Section 7. References in Sections 1 through 6 are to the relevant parts of [1] and [2].

Recall that by Lemma 5.1, every nontrivial principal congruence of a finite bandoid $\underline{B}$ contains a principal congruence generated by pairs of elements a, b of B such that $a < b$ or a and b satisfy the condition 5.1(ii). Hence we conclude that the monolith of a finite subdirectly irreducible left normal bandoid $\underline{B}$ is a principal congruence $\Theta(a, b)$ on $\underline{B}$ with $a < b$ or a and b satisfying 5.1(ii). If $\Theta(a, b)$ with $a < b$ is the monolith of a subdirectly irreducible left normal bandoid, then this bandoid is called to be *subdirectly irreducible of the first type*. If $\Theta(a, b)$ with a and b satisfying 5.1(ii) is the monolith of $\underline{B}$, then $\underline{B}$ is called to be *subdirectly irreducible of the second type*. Note that the subdirectly irreducible left normal bandoids constructed in Section 5 are of the first type, and these constructed in Section 6 are of

This paper has been presented at the Conference on Universal Algebra and its Applications, organized by the Institute of Mathematics of Warsaw University of Technology held at Jachranka, Poland, 8–13 June 1993.

the second type. In this paper we give a necessary condition for a finite left normal bandoid to be subdirectly irreducible of the first type. A necessary condition for a finite bandoid to be subdirectly irreducible of the second type will be given in the last paper of this series.

In this paper our aim is to prove the following theorem:

THEOREM 7.0. *If $\underline{B} = (B, \cdot)$ is a finite subdirectly irreducible left normal bandoid with a monolith $\Theta(a, b)$ with $b < a$, then*

$$\underline{B} = \underline{B}(L, R)^{\rho}$$

for some finite distributive lattice $\underline{L}$ with exactly one coatom, and some relation $R \subseteq \leq_L$ satisfying the following condition

(i) for each join — irreducible element t in $L \setminus \{0\}$ there exist elements $t = x_1, x_2, \ldots, x_n, z_1, \ldots, z_{n-1}, z_n = 1$ of L such that for all $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, n-1$

$$(x_i, z_i) \in R \text{ and } x_1 \leq x_{j+1} \leq_L z_j.$$

7. The proof of the theorem

First we prove some lemmas which are necessary in the proof of the theorem.

Let $\underline{B} = (B, \cdot)$ be a finite left normal bandoid.

7.1. LEMMA. *Let $x, y \in B$ and x, y lie in the same orbit or x, y satisfy the condition 5.1(ii). Then the principal congruence $\Theta(x, y)$ on $\underline{B}$ is the equivalence relation on B generated by the set $\{(\alpha x, \alpha y) : \alpha \in L(B)\}$, i.e. $(z, t) \in \Theta(x, y)$ iff*

(*) *there exist elements $z_1, z_2, \ldots, z_n \in B$ such that $z_1 = z, z_n = t$ and for every $i < n$*

$$z_i = z_{i+1} \text{ or } \{z_i, z_{i+1}\} = \{\alpha x, \alpha y\} \text{ for some } \alpha \in L(B).$$

Proof. First note that the relation R defined by

$$(z, t) \in R \text{ iff } (z, t) \text{ satisfy } (*)$$

is exactly the equivalence relation $E(\{(\alpha x, \alpha y) : \alpha \in L(B)\})$ generates by the set $\{(\alpha x, \alpha y) : \alpha \in L(B)\}$. Indeed, the relation R is contained in $E(\{(\alpha x, \alpha y) : \alpha \in L(B)\})$ since $E(\{(\alpha x, \alpha y) : \alpha \in L(B)\})$ is an equivalence relation containing the set $\{(\alpha x, \alpha y) : \alpha \in L(B)\}$. It obviously contains $\{(\alpha x, \alpha y) : \alpha \in L(B)\}$. Moreover for every $z, t, u \in B$:

$(z, z) \in R$ by the definition of R, via $z_1 = z$,

$(z, t) \in R$ via $z_1, z_2, \ldots, z_n$ implies $(t, z) \in R$ via $z_n, \ldots, z_1$,

$(z, t) \in R$ via $z_1, z_2, \ldots, z_n$ and $(t, u) \in R$ via $t_1, t_2, \ldots, t_m$ imply that $(z, u) \in R$ via $z_1, z_2, \ldots, z_n, t_1, t_2, \ldots, t_m$.

Therefore $R = E(\{(\alpha x, \alpha y) : \alpha \in L(B)\})$.

Now we will prove that $R = \Theta(x,y)$. Note that it suffices to show that for every $c \in B$ and $\alpha \in L(B)$ the following condition hold:

$$(7.1.1) \qquad (c\alpha x, c\alpha y) = (\beta x, \beta y) \text{ for some } \beta \in L(B),$$

$$(7.1.2) \qquad ((\alpha x)c, (\alpha y)c) = (\gamma x, \gamma y) \text{ for some } \gamma \in L(B) \text{ or } (\alpha x)c = (\alpha y)c,$$

Indeed if $(z,t) \in R$ via $z_1, z_2, \ldots, z_n$ then by (7.1.1), $(cz, ct) \in R$ via $cz_1, cz_2, \ldots, cz_n$ and by (7.1.2) $(zc, tc) \in R$ via $z_1c, z_2c, \ldots, z_nc$. This completes the proof of the fact that R is a congruence on $\underline{B}$. Since obviously $R \subseteq \Theta(x,y)$ and $(x,y) \in R$, it follows that $\Theta(x,y) = R$.

To prove (7.1.1) note that for every $c \in B$ and $\alpha \in L(B)$, $c\alpha x = (L(c) \circ \alpha)x$, $c\alpha y = (L(c) \circ \alpha)y$ and $L(c) \circ \alpha \in L(B)$.

To prove (7.1.2) let us assume first that $\alpha = L(y_1) \circ \ldots \circ L(y_n)$ for some $y_1, \ldots, y_n \in B$. Then by Proposition 1.2.12 and Remark 1.2.5 we obtain: $(\alpha x)c = (\alpha x)y_1 c = (y_1 c)(\alpha x) = (L(y_1 c) \circ \alpha)x$. Analogously, $(\alpha y)c = (L(y_1 c) \circ \alpha)y$. Since $L(y_1 c) \circ \alpha \in L(B)$, we have that $((\alpha x)c, (\alpha y)c) = (\gamma x, \gamma y)$ for some $\gamma \in L(B)$.

Now let $\alpha = \mathrm{id}_B$, the identity mapping on B. We consider two cases: x, y are in a common orbit $vT(B)$ or x, y satisfy 5.1(ii). If $x, y \in vT(B)$, then $(\alpha x)c = xc = (vx)c = (vx)(vc) = (vc)(vx) = (vc)x = L(vc)x$. The third and fifth equalities hold by Proposition 1.2.12 and the fourth equality holds by Remark 1.2.5. Analogously we show that $(\alpha y)c = L(vc)y$. So (7.1.2) holds in this case.

Now let x, y satisfy 5.1(ii). If $xc < x$, then since $xT(B) \setminus yT(B) = \{x\}$, we have that $xc \leq y$ and moreover, since $(\{x,y\}, \cdot)$ is a left zero semigroup, $y = yx$. Hence, using (B5) (B6) and Corollary 1.2.16 we obtain $(\alpha y)c = yc = (yx)c = (yc)(xc) = (y(xc))c = (xc)c = xc(\alpha x)c$.

If $xc = x$ then $yc = y$. Indeed, if $yc < y$ then analogously as in the case $xc < x$, we show that $xc = yc$ and as a consequence of this we obtain $yc = x$ what implies that $x \leq y$, and contradicts the fact that $xT(B) \setminus yT(B) = \{x\}$. So we have $((\alpha x)c, (\alpha y)c = (xc, yc) = (x,y) = (\mathrm{id}_B x, \mathrm{id}_B y)$. Therefore (7.1.2) holds in this case as well.

For a set X, a relation $R \subseteq X^2$ and a subset U of X, the symbol $R_{|U}$ denotes the relation $R \cap U^2$ on X.

7.2. LEMMA. *Let x, y, c be elements of B and x, y lie in the same orbit or x, y satisfy 5.1(ii). Then*

$$\Theta(x,y)_{|cT(B)} = \omega_{cT(B)}.$$

Proof. Let $(z,t) \in \Theta(x,y)_{|cT(B)}$. Then $(z,t) \in \Theta(x,y), z = cz$ and $t = ct$. By Lemma 7.1 there exist elements $z_1, z_2, \ldots, z_n$ of B such that $z = z_1, t = z_n$ and for every $i = 1,2,\ldots,n-1 : z_i = z_{i+1}$ or $\{z_i, z_{i+1}\} = \{\alpha x, \alpha y\}$ for some $\alpha \in L(B)$. By Proposition 1.2.14 and definition of $L(B)$, if $cx = cy$ then $c\alpha x = x\alpha(cx) = c\alpha(cy) = c\alpha y$. Hence $cz_i = cz_{i+1}$ for every $i = 1,2,\ldots,n-1$. Consequently $z = cz = cz_1 = cz_n = t$. Therefore $\Theta(x,y)_{|cT(B)} \subseteq \omega_{cT(B)}$. Obviously $\omega_{cT(B)} \subseteq \Theta(x,y)_{|cT(B)}$ since $\Theta(x,y)$ is a congruence on $\underline{B}$. ∎

Recall that in this section we consider only finite subdirectly irreducible left normal bandoids of the first type.

Let $\underline{B} = (B,\cdot)$ be a finite subdirectly irreducible left normal bandoid. Let a, b be elements of B such that $b < a$ and $\Theta(a,b)$ is the monolith of $\underline{B}$. Then the following lemmas hold.

7.3. LEMMA. *The element a is maximal in $(B,\leq)$.*

Proof. Suppose on the contrary that $x \in B$ and $x > a$. Then $ax = aa$ and consequently, by Lemma 7.2, $\Theta(a,x)_{|aT(B)} = \omega_{aT(B)}$. Since $a, b \in aT(B)$ and $a \neq b$ it follows that $(a,b) \notin \Theta(a,x)$. Since $\Theta(a,x) \neq \omega_B$, the last statement gives a contradiction with the fact that $\Theta(a,b)$ is the monolith of $\underline{B}$.

7.4. LEMMA. *The element b is the only predecessor of a in $(B,\leq)$.*

Proof. First we prove that b is a predecessor of a in $(B,\leq)$. Suppose on the contrary that there is c in B such that $b < c < a$. Then $\Theta(b,c) = \Theta(cb,ca) \subseteq \Theta(b,a)$. On the other hand $\Theta(a,b) \subseteq \Theta(b,c)$ since $\Theta(a,b)$ is the monolith of $\underline{B}$ and $\Theta(b,c) \neq \omega_B$. Therefore $\Theta(b,c) = \Theta(a,b)$. Consequently $\Theta(b,c)$ is the monolith of $\underline{B}$. Hence by Lemma 7.3, c is maximal in $(B,\leq)$, a contradiction to the fact that $c < a$. So b is a predecessor of a.

It remains to show that there are no other predecessors of a. Suppose that d is a predecessor of a and $d \neq b$. Then $db \neq d$. Indeed, $b, d \in aT(B)$ and consequently $db = bd$, so $db = d$ implies that $d \leq b$, hence we conclude that d is not a predecessor of a, a contradiction.

Note that $\Theta(db,d) = \Theta(db,da) \subseteq \Theta(b,a) = \Theta(a,b)$. Since $\Theta(a,b)$ is the monolith of $\underline{B}$ and $\Theta(db,d) \neq \omega_B$ if follows that $\Theta(db,d) = \Theta(a,b)$. Consequently $\Theta(db,d)$ is the monolith of $\underline{B}$ and by Lemma 7.3, d is maximal in $(B,\leq)$, contradicting $d < a$.

7.5. LEMMA. *For every $x \in B$ the mapping $L(a) : (xT(B),\cdot) \to (aT(B),\cdot); y \to ay$ is a semilattice monomorphism.*

Proof. Let $x \in B$. By Proposition 1.2.7 it suffices to show that the mapping is one to one. Let $y, z \in xT(B)$ and $y \neq z$. Suppose on the contrary that $ay = az$. Then by Lemma 7.2, $\Theta(y,z)_{|aT(B)} = \omega_{aT(B)}$ and consequently $(a,b) \in \Theta(y,z)$, contradicting the fact that $\Theta(a,b)$ is the monolith of $\underline{B}$.

7.6. LEMMA. *Let $x_1, x_2, \ldots, x_n \in B$ and $i \in \{1,2,\ldots,n\}$. Then the following holds:*

$$x_1, \ldots, x_i a x_{i+1} \ldots x_n = x_1 x_2 \ldots x_n.$$

Proof. By Proposition 1.2.14, $ax_1, \ldots x_i a x_{i+1} \ldots x_n = ax_1 x_2 \ldots x_n$. Hence, by Lemma 7.5 we obtain: $x_1 \ldots x_i a x_{i+1} \ldots x_n = x_1 x_2 \ldots x_n$. ∎

7.7. LEMMA. *Let x, y satisfy the condition 5.1(ii). Then*

$$ax \neq ay.$$

Proof. Suppose that $ax = ay$. Then by Lemma 7.2, $\Theta(x,y)_{|aT(B)} = \omega_{aT(B)}$. Since $a, b \in aT(B)$ and $a \neq b$, we conclude that $(a,b) \in \omega(x,y)$. By the assumption that x, y satisfy 5.1(ii) it follows that $x \neq y$ and consequently $\Theta(x,y) \neq \omega_B$. Therefore $(a,b) \in \Theta(x,y)$ gives a contradiction to the fact that $\Theta(a,b)$ is the monolith of B. So $ax \neq ay$. ∎

7.8. LEMMA. *Let $(x,y) \in B^2 \setminus \{(b,a)\}$ and $x < y$. Then $ay = a$ implies $ax \neq b$.*

Proof. First assume that $y = a$. Then $ax = yx = x$. Since $(x,y) \neq (b,a)$ it follows that $ax \neq b$.

Now let $y \neq a$. Suppose on the contrary that $ax = b$. Using the fact that $x < y$ and Proposition 1.2.14 we obtain:

$$ax = ayx = ayax. \tag{7.8.1}$$

By Lemma 7.5 the left multiplication $L(a) : (y(T(B),\cdot) \to (aT(B),\cdot)$ is a monomorphism. So, by (7.8.1), $x = yax$. Since $ax = b$ it follows that $yb = x$. Moreover by Corollary 7.6 we have $ya = y$. Therefore $\Theta(y,x) = \Theta(ya,yb) \subseteq \Theta(a,b)$. Since $\Theta(y,x) \neq \omega_B$ and $\Theta(a,b)$ is the monolith of $\underline{B}$, it follows that $\Theta(y,x) = \Theta(a,b)$, i.e. $\Theta(y,x)$ is the monolith of $\underline{B}$, it follows that $\Theta(y,x) = \Theta(a,b)$, i.e. $\Theta(y,x)$ is the monolith of $\underline{B}$. So by Lemma 7.5

(7.8.2) the left multiplication $L(y) : (aT(B),\cdot) \to (yT(B),\cdot)$ is a monomorphism.

Note that the assumptions $y \neq a$ and $ay = a$ imply that $y \notin aT(B)$. Therefore $y \in yT(B) \setminus aT(B)$ and so $yT(B) \setminus aT(B) \neq \emptyset$.

Let t be a minimal element in $(yT(B) \setminus aT(B), \leq)$. We want to show that the elements t and at satisfy the condition 5.1(ii).

By Corollary 7.6 we have that $tat = tt = t$ and by Corollary 1.2.16, $(at)t = at$. So

$$(7.8.3) \qquad (\{t, at\}, \cdot) \text{ is a left zero semigroup.}$$

Now we show that $tT(B)\setminus(at)T(B) = \{t\}$. Obviously $t \in tT(B)\setminus(at)T(B)$. Indeed, $t \in (at)T(B)$ implies that $(at)t = t$, whence by Corollary 1.2.16, $at = t$ and consequently $t \in aT(B)$, a contradiction. We will show that t is the unique element of $tT(B) \setminus (at)T(B)$. Suppose on the contrary that $t' \in tT(B)\setminus(at)T(B)$ and $t' \neq t$. Then $t' < t$. Since t is minimal in $(yT(B)\setminus aT(B), \leq)$, it follows that $t' \in aT(B)$ and consequently $t' = at'$. Using the fact that $t' < t$ and (B3) we obtain: $at' = att' = (at)(tt') = (at)(t')$. So $t' = (at)t'$, i.e. $t' \in (at)T(B)$, contradicting $t' \in tT(B)\setminus(at)T(B)$. Therefore we have

$$(7.8.4) \qquad tT(B) \setminus (at)T(B) = \{t\}.$$

To prove that $t, (at)$ satisfy 5.1(ii) it remains to show that $(at)T(B)\setminus tT(B) = \{at\}$. Note that $at \in (at)T(B) \setminus tT(B) = \{at\}$. Indeed, $at \in tT(B)$ implies that $at = tat$, whence by Corollary 7.6 and (B.1), $at = tt = t$ and consequently $t \in aT(B)$, contradicting the fact that $t \in yT(B) \setminus aT(B)$. Therefore $at \in (at)T(B)\setminus tT(B)$. Suppose on the contrary that $t'' \neq at$ and $t'' \in (at)T(B)\setminus tT(B)$. Then $t'' < at$. So, by (7.8.2), $yt'' < yat$. By Corollary 7.6 $yat = yt$ and consequently, since $t \in yT(B), yat = t$. Therefore $yt'' < t$. Hence, because t is minimal in $(yT(B)\setminus aT(B), \leq)$, we get that $yt'' \in aT(B)$. Note that $t'' \in aT(B)$ as well and by (B2), $yyt'' = yt''$. Therefore, by (7.8.2) we obtain $yt'' = t''$. Hence $t'' \in yT(B)$, contradicting $t'' \in (at)T(B)\setminus tT(B)$. So we have

$$(7.8.5) \qquad (at)T(B) \setminus tT(B) = \{at\}.$$

From (7.8.3)–(7.8.5) it follows that the elements t and at satisfy the condition 5.1(ii). Consequently, by Lemma 7.7, $at \neq aat$, contradicting the axiom (B2). The last contradiction shows that $ax \neq b$. This completes the proof.

7.9. LEMMA. *Let $(x, y) \in B^2 \setminus \{(b, a)\}$ and $x < y$. Then for every $\alpha \in L(B)$*

$$\alpha y = a \text{ implies } ax \neq b.$$

Proof. Let $\alpha \in L(B)$ and $\alpha y = a$. If $\alpha = \mathrm{id}_B$ then obviously $\alpha x \neq b$ since $(x, y) \neq (b, a)$. Let $\alpha \neq \mathrm{id}_B$. Then $\alpha = L(z_1) \circ L(z_2) \circ \ldots \circ L(z_n)$ for some $z_1, z_2, \ldots, z_n \in B$. By Corollary 1.2.8, $\alpha x \leq \alpha y$. So $\alpha y, \alpha x \in aT(B)$. Consequently

$$(7.9.1) \qquad \alpha x = a\alpha x = az_1 z_2 \ldots z_n x$$

and

(7.9.2) $$\alpha y = a\alpha y = az_1 z_2 \dots z_n y.$$

Let $z_{i_1}, z_{i_2}, \dots, z_{i_k}$ be a subsequence of the sequence $z_1, z_2, \dots, z_n$ obtained by dropping all elements equal to a. By Proposition 1.2.14, from (7.9.1) and (7.9.2) it follows that

(7.9.3) $$\alpha x = az_{i_1} z_{i_2} \dots z_{i_k} x$$

and

(7.9.4) $$\alpha y = az_{i_1} z_{i_2} \dots z_{i_k} y.$$

Note that $z_{i_1} z_{i_2} \dots z_{i_k} \neq a$. Indeed, by Remark 1.2.10, $z_{i_1} z_{i_2} \dots z_{i_k} = a$ implies that $a \leq z_{i_1}$, and since $a \neq z_{i_1}$ we have $a < z_{i_1}$, contradicting the fact that, by Lemma 7.3, a is maximal in $(B, \leq)$. Therefore $(z_{i_1} z_{i_2} \dots z_{i_k} x, z_{i_1} z_{i_2} \dots z_{i_k} y) \neq (b, a)$. Moreover, by Remark 1.2.11 we have that $z_{i_1} z_{i_2} \dots z_{i_k} x \leq z_{i_1} z_{i_2} \dots z_{i_k} y$. If $z_{i_1} z_{i_2} \dots z_{i_k} x = z_{i_1} z_{i_2} \dots z_{i_k} y$, then by (7.9.3) and (7.9.4), $\alpha x = \alpha y$ and consequently $\alpha x \neq b$ since $\alpha y = a$. If $z_{i_1} z_{i_2} \dots z_{i_k} x < z_{i_1} z_{i_2} \dots z_{i_k} y$, then by Lemma 7.8, since $az_{i_1} z_{i_2} \dots z_{i_k} y = \alpha y = a$, we obtain that $az_{i_1} z_{i_2} \dots z_{i_k} x \neq b$ and consequently, by (7.9.3), $\alpha x \neq b$. ■

7.10. LEMMA. *Let $(x, y) \in B^2 \setminus \{(b, a)\}$ and $x < y$. Then there exists $\alpha \in L(B)$ such that $\alpha y = a$ and $\alpha x < b$.*

Proof. Clearly, if $y = a$ then for $\alpha := \mathrm{id}_B$ we have that $\alpha y = a$ and, by Lemma 7.4, $\alpha x < b$.

Let $y \neq a$. Suppose on the contrary that for every $\alpha \in L(B), \alpha y = a$ implies $\alpha x < b$. Then, by Lemma 7.4, for every $\alpha \in L(B), \alpha y = a$ implies $\alpha x = a$ or $\alpha x = b$. Consequently, by Corollary 7.9 for every $\alpha \in L(B), \alpha y = a$ implies $\alpha x = a$. On the other hand, by Corollary 1.2.8 for every $\alpha \in L(B)$: $\alpha x \leq \alpha y$. Hence by Lemma 7.3, for every $\alpha \in L(B), \alpha x = a$ implies $\alpha y = a$. So we have

(7.10.1) $$\alpha x = a \text{ iff } \alpha y = a$$

for every $\alpha \in L(B)$. We will show that $(a, b) \notin \Theta(x, y)$. This will give a contradiction with the fact that $\Theta(a, b)$ is the monolith of $\underline{B}$. Suppose on the contrary that $(a, b) \in \Theta(x, y)$. Then by Lemma 7.1 there exist elements, say $z_1, z_2, \dots, z_n$ such that $a = z_1, b = z_n$ and for each $i = 1, 2, \dots, n, z_i = z_{i+1}$ or for some $\alpha \in L(B), \{z_i, z_{i+1}\} = \{\alpha x, \alpha y\}$. Hence, by (7.10.1), it follows that $a = z_1 = z_2 = \dots = z_n = b$, a contradiction to $b < a$. Therefore $(a, b) \in \Theta(x, y)$, contradicting the fact that $\Theta(a, b)$ is the monolith of $\underline{B}$. This contradiction completes the proof. ■

7.11. LEMMA. *Let $x, y \in aT(B)$ and x be a predecessor of y in $(B, \leq)$. Let α be an element of $L(B)$, such that $\alpha y = a$ and $\alpha x < b$.*

Then αx is the supremum of the set $\{v \in aT(B) | vy = x\}$ in $(B, \leq)$.

Proof. Let us denote $Z := \{v \in aT(B) | vy = x\}$. First we show that αx is an upper bound of Z in $(B, \leq)$.

Let $v \in Z$. By Proposition 1.2.14, $v\alpha y = v\alpha(vy)$ and consequently, since $vy = x$, we obtain $v\alpha y = v\alpha x$. Because $\alpha y = a, v = va = v\alpha y = v\alpha x$. Since αx and a lie in the same orbit $aT(B)$, we have that $v\alpha x = (\alpha x)v$. Therefore $v = (\alpha x)v$, i.e. $v \leq \alpha x$ for all $v \in Z$. Now it suffices to show that $\alpha x \in Z$. By Remark 1.2.5, since $x \leq y$, it follows that $xy = yx = x$. Therefore $x \in Z$ and consequently, as was shown in the first part of the proof $x \leq \alpha x$. Hence by Remark 1.2.11, $xy \leq (\alpha x)y$, i.e. $x \leq (\alpha x)y$. On the other hand $(\alpha x)y \leq y$ since by Remark 1.2.5 and Corollary 1.2.16, $y(\alpha x)y = ((\alpha x)y)y = (\alpha x)y$. Since x is a predecessor of y, we have the following

$$(\alpha x)y = y \text{ or } (\alpha x)y = x. \tag{7.11.1}$$

Suppose that $(\alpha x)y = y$. Then $\alpha(\alpha x)y) = \alpha y$. Corollary 1.2.8 and Proposition 1.2.14 it follows that $\alpha((\alpha x)y) = (\alpha(\alpha x))(\alpha y) = (\alpha x)(\alpha y)$. Consequently $(\alpha x)(\alpha y) = \alpha y$. Since $\alpha y = a$ and, by Corollary 1.2.8, $\alpha x \leq \alpha y$, we obtain that $\alpha x = (\alpha x)(\alpha y) = (\alpha x)a = a$, contradicting $\alpha x < b$. Therefore

$$(\alpha x)y \neq y. \tag{7.11.2}$$

From (7.11.1) and (7.11.2) it follows that $(\alpha x)y = x$ and in consequence $\alpha x \in Z$, what completes the proof. ∎

7.12. Remark. Let x, y, α satisfy the hypothesis of Lemma 7.11. Then

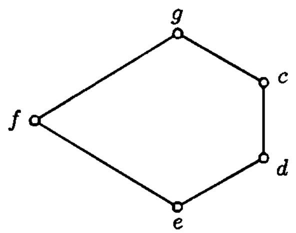
$$y \leq \alpha x.$$

Proof. This follows immediately from (7.11.2). ∎

Since B is finite and all orbits of $\underline{B}$ are semilattices with unit, it follows that each pair (x, y) of elements of an orbit has a join $x + y$, the supremum of $\{x, y\}$ in $(B, \leq)$. So for any x in B, the algebra $(xT(B), +, \cdot)$ is a lattice.

7.13. LEMMA. *The lattice* $(aT(B), +, \cdot)$ *is modular.*

Proof. Suppose on the contrary that $(aT(B), +, \cdot)$ contains as a subalgebra a copy of the lattice N_5, say the lattice pictured below.



Additionaly let us assume that d is a predecessor of c. By Lemma 7.10 we may assume that α is an element of $L(B)$ such that $\alpha x = a$ and $\alpha d < b$. By Lemma 7.11, $\alpha d = \Sigma(v \in aT(B) | vc = d)$ and consequently, since $dc = d$ we obtain

$$\alpha d \geq d. \tag{7.13.1}$$

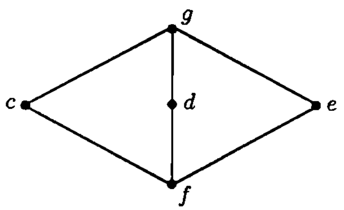
Note that $fc = fd$. Hence by Proposition 1.2.14 and Remark 1.2.5, it follows that $f = fa = f\alpha c = f\alpha(fc) = f\alpha(fd) = f(\alpha d) = (\alpha d)f$ and consequently

$$\alpha d \geq f. \tag{7.13.2}$$

By (7.13.1) and (7.13.2) we obtain that $f + d \leq \alpha d$, i.e. $g \leq \alpha d$. Hence $c \leq \alpha d$. But by Remark 7.12 $c \leq \alpha d$, a contradiction. ∎

7.14. LEMMA. *The lattice* $(aT(B), +, \cdot)$ *is distributive.*

Proof. By Lemma 7.13 the lattice $(aT(B), +, \cdot)$ is modular. So it remains to show that it does not contain as a subalgebra a copy of the lattice M_3. Suppose on the contrary that the lattice pictured below is a subalgebra of $(aT(B), +, \cdot)$.



Let z be an element of B such that f is a predecessor of z and $z \leq e$. By Lemma 7.10 there exist an element of $L(B)$, say α, such that $\alpha z = a$ and $\alpha f < b$. By Lemma 7.11, $\alpha f \geq c + d$ since $cz = f$ and $de = f$. Therefore $\alpha f \geq g$ and consequently $\alpha f \geq z$, contradicting the fact that by Remark 7.12, $z \not\leq \alpha f$. So the proof is complete. ∎

Recall that for a lattice $\underline{L}$ the symbol $JI(L)$ denotes the set of all join — irreducible elements of $\underline{L}$. The principal ideal of a lattice, generated by x is denoted by (x).

7.15. LEMMA. *Let* $t \in JI(aT(B))$ *and* s *be a predecessor of* t *in* $(B, \leq)$. *Then for every* $z \in B, zs \neq zt$ *implies that the left multiplication* $L(z) : ((t], \cdot) \to (zT(B), \cdot)$ *is a monomorphism.*

Proof. Let $zs \neq zt$. By Proposition 1.2.7 it suffices to show that the left multiplication $L(z) : ((t], \cdot) \to (zT(B), \cdot)$ is one to one.

Let $u, v \in (t]$ and $u \neq v$. Suppose on the contrary that $zu = zv$. Since $u \neq v$, we have that $uv \neq v$ or $vu \neq u$. Without loss of generality we may assume that $uv \neq v$. By Proposition 1.2.14 and Remark 1.2.5, $vuv = vu = uv$, whence $uv \leq v$. Therefore $uv < v$. Let w be a predecessor of v such that

$$uv \leq w. \tag{7.15.1}$$

By Lemma 7.10 we may assume that $\alpha v = a$ and $\alpha w < b$. By Corollary 1.2.8 and (7.15.1) it follows that

$$\alpha(uv) \leq \alpha w. \tag{7.15.2}$$

By Remark 7.12 we have

$$v \not\leq \alpha w. \tag{7.15.3}$$

From (7.15.2) and (7.15.3) it follows that $v \leq \alpha(uv)$ and consequently $t \not\leq \alpha(uv)$, i.e.

$$\alpha(uv)t \neq t. \tag{7.15.4}$$

Since $\alpha w < b < a$, from (7.15.2) it follows that $\alpha(uv) \in aT(B)$. Hence by Remark 1.2.5 we conclude that $t\alpha(uv) = \alpha(uv)t$. So, by Remark 1.2.10, $\alpha(uv)t \leq t$ and in consequence, by (7.15.4) we obtain $\alpha(uv)t < t$. Since s is a predecessor of t and $t \in JI(aT(B))$, the last inequality implies

$$\alpha(uv)t \leq s. \tag{7.15.5}$$

Note that

$$\begin{aligned}
zt &= zta && \text{since } t \in aT(B)\\
&= zt\alpha v && \text{since } \alpha v = a\\
&= zt\alpha(zv) && \text{since by assumption } zu = zv \text{ and consequently, by}\\
& && \text{Proposition 1.2.7, } zuv = (zu)(zv) = (zv)(zv) = zv\\
&= zt\alpha(uv) && \text{by Proposition 1.2.14}\\
&= z(\alpha(uv)t) && \text{by Remark 1.2.5.}
\end{aligned}$$

Hence by (7.15.5) and Remark 1.2.11 we have $zt \leq zs$. On the other hand, by Remark 1.2.11, since $s < t$, we have $zs \leq zt$. So $zt = zs$, contradicting the assumption $zt \neq zs$. This contradiction shows that the mapping $L(z) : ((t], \cdot) \to (zT(B), \cdot); x \mapsto zt$ is one to one. ∎

For an element y of a finite subdirectly irreducible left normal bandoid $\underline{B}$ with monolith $\Theta(a, b)$ where $b < a$, let us consider the set $Z^y := \{z \in aT(B) | yz = y\}$. Note that the set Z^y is nonempty since by Corollary 7.6, $a \in Z^y$. Since B is finite there exist the meet $\prod Z^y$ of all elements of the semilattice $(aT(B), \cdot)$ lying in Z^y. Moreover, by Proposition 1.2.7, $y(\prod Z^y) = \prod\{yz | z \in Z^y\} = y$. Obviously $\prod Z^y \in aT(B)$. So $\prod Z^y \in Z^y$. Let us define $y^a := \prod Z^y$.

7.16. LEMMA. *For every $y \in B$ the left multiplication $L(y) : ((y^a], \cdot) \to (yT(B), \cdot)$ is an isomorphism.*

Proof. By Proposition 1.2.7 the left multiplication $L(y) : ((y^a], \cdot) \to (yT(B), \cdot)$ is a homomorphism. It remains to show that it is one-to-one and onto.

First we show that this mapping is one-to-one. If the set $(y^a]$ has exactly one element, then it is obvious. So, we assume that $(y^a]$ has at least two elements. We consider the following two cases:

Case 1. Let $y^a \in JI(aT(B))$. Since $|(y^a]| \geq 2$, there exist a predecessor t of y^a in $(B, \leq)$. By definition of $y^a, yy^a \neq yt$. Now the proof follows immediately by Lemma 7.15.

Case 2. Let $y^a \notin JI(aT(B))$. Define $A := \{x \in JI(aT(B)) | x \leq y^a\}$. First we show that

(7.16.1) for every element x which is maximal in $(A, \leq)$ the mapping $L(y) : ((x], \cdot) \to (yT(B), \cdot)$ is one to one.

Let x be a maximal element in $(A, \leq)$. Since $y^a \notin JI(aT(B))$ there is an element t, such that t is a predecessor of y^a and $x \not\leq t$. By distributivity of the lattice $(aT(B), +, \cdot)$ it follows that xt is a predecessor of xy^a in $(B, \leq)$. Since $x \in A$, clearly, $xy^a = x$. To use Lemma 7.15 it suffices to show that $yx \neq yxt$. Suppose on the contrary that

$$yx = yxt. \tag{7.16.2}$$

By Lemma 7.10 we may assume that α is an element of $L(B)$ such that $\alpha x = a$ and $\alpha(xt) < b$. By Lemma 7.11 we have

$$\alpha(xt) = \Sigma(v \in aT(B) | vx = xt). \tag{7.16.3}$$

Since by Remark 1.2.5 $tx = xt$ and $t \in aT(B)$, from (7.16.3) it follows that $\alpha(xt) \geq t$ and consequently, by Remark 1.2.11, $y^a\alpha(xt) \geq y^a t$.

Because $t \leq y^a$, i.e. $y^a t = t$, we have

$$y^a\alpha(xt) \geq t. \tag{7.16.4}$$

By Remark 7.12, $\alpha(xt) \geq x$. Hence we conclude that

$$\alpha(xt) \not\geq y^a. \tag{7.16.5}$$

In consequence, since $y^a\alpha(xt) \leq y^a$, we obtain

$$y^a\alpha(xt) < y^a. \tag{7.16.6}$$

Indeed, if $y^a\alpha(xt) = y^a$, then by Remark 1.2.5, $\alpha(xt)y^a = y^a$, i.e. $y^a \leq \alpha(xt)$, contradicting (7.16.5). Since t is a predecessor of y^a, from (7.16.4) and (7.16.6) it follows that

$$y^a\alpha(xt) = t. \tag{7.16.7}$$

Then we have the following:

$y = yy^a$	by definition of y^a
$= yy^a a$	by Corollary 7.6
$= yy^a \alpha x$	since $\alpha x = a$
$= yy^a \alpha(yx)$	by Proposition 1.2.14
$= yy^a \alpha(yxt)$	by (7.16.2)
$= yy^a(xt)$	by Proposition 1.2.14
$= yt$	by (7.16.7).

Hence, by definition of $y^a, y^a \leq t$, contradicting the assumption $t < y^a$. This contradiction shows that $yx \neq yxt$. By Lemma 7.15 this completes the proof of (7.16.1).

Now let $u, v \in (y^a]$ and $u \neq v$. There exist $t \in JI(aT(B))$ such that $t \leq u$ and $t \not\leq v$, i.e. $tu = t$ and $zv < t$. Since $t \in JI(aT(B))$ and $t \leq y^a$, there exist an element z in A which is maximal in $(A, \leq)$ and such that $t \leq z$. Observe that $tu, tv \in (z]$ and $tu \neq tv$. In consequence, by (7.16.1) we obtain that $ytu \neq ytv$. This implies that $yu \neq yv$. Indeed, if $yu = yv$ then by Proposition 1.2.14, $ytu = ytyu = ytyv = ytv$, contradicting $ytu \neq ytv$. In this way we have proved that the left multiplication $L(y) : (y^a] \to yT(B)$ is one-to-one.

It remains to prove that $L(y)$ maps $(y^a]$ onto $yT(B)$.

Let $w \in yT(B)$. Obviously $y^a w \in (y^a]$. We show that $L(y)(y^a w) = w$. Note that

$L(y)(y^a w) = yy^a w = yy^a aw$	by Corollary 7.6
$L(y)(y^a w) = yy^a w = (yy^a)(yaw)$	by Proposition 1.2.7
$L(y)(y^a w) = yy^a w = y(yaw)$	by definition of y^a
$L(y)(y^a w) = yy^a w = yw$	by B2) and Corollary 7.6
$L(y)(y^a w) = yy^a w = w$	since $w \in y^T(B)$.

This implies that $L(y)$ maps $(y^a]$ onto $yT(B)$, what completes the proof. ∎

7.17. Lemma. *For every $y \in B$ and for every $w \in yT(B)$*

$$L(a)w = \Sigma(v \in (ay] | vy^a = u),$$

where u is the unique element of $(y^a]$ with the property $w = L(y)u$.

Proof. Let $y \in B$ and $w \in yT(B)$. By Lemma 7.16 there exists exactly one element, say u, in $(y^a]$ such that $w = yu$. Define $Z := \{v \in (ay] | vy^a = u\}$. It suffices to show that for every $v \in Z, v \leq L(a)w$ and $L(a)w \in Z$.

Let $v \in Z$. Then $vy^a = u$ and consequently, by Proposition 1.2.14, $vayy^a = vayvy^a = vayu = vaw = vL(a)w$. Since $yy^a = y$ and $v \in (ay]$, we have that $v = vay = vayy^a$. In consequence $v = vL(a)w$ and by Remark 1.2.5, $v = (L(a)w)v$, i.e. $v \leq L(a)w$.

Now we show that $L(a)w \in Z$. First note that $(ay)(L(a)w) = (ay)(aw) =$

$aw = L(a)w$. The second equality follows by Remarks 1.2.10, 1.2.11, and the assumption $w \in yT(B)$. So $L(a)w \in (ay]$. It remains to show that $(L(a)w)y^a = u$. Using Proposition 1.2.14, definition of y^a and Corollary 7.6 we obtain

$$y(L(a)w)y^a = a(aw)y^a = y(aw)yy^a = y(aw)y = yaw = yw = yyu = yu.$$

Hence, by Lemma 7.16, we have $(L(a)w)y^a = u$. Therefore $L(a)w \in Z$ and consequently, since $v \leq L(a)w$ for all $v \in Z$,

$$L(a)w = \Sigma(v \in (ay] | vy^a = u). \quad \blacksquare$$

7.18. LEMMA. *Let $x, y \in B$ be such that*

$$L(a)(yx') = L(a)(x') \text{ and } L(a)(xy') = L(a)(y')$$

for all $x' \leq x, y' \leq y$. Then $x = y$.

Proof. Suppose on the contrary that $x \neq y$. Then by Remark 1.2.10 $xT(B) \setminus yT(B) \neq \emptyset$ or $yT(B) \setminus xT(B) \neq \emptyset$. Without loss of generality we may assume that $xT(B) \setminus yT(B) \neq \emptyset$.

Let t be a minimal element in $(xT(B) \setminus yT(B), \leq)$. We will show that the elements t and yt satisfy the condition 5.1(ii). By assumption we have

$$(7.18.1) \qquad L(a)t = L(a)(yt)$$

and

$$(7.18.2) \qquad L(a)(ty) = L(a)(yty).$$

In view of Proposition 1.2.14, $yty = yt$. Therefore, from (7.18.1) and (7.18.2) it follows that $L(a)t = L(a)(ty)$. Hence, by Corollary 7.6 we obtain $t = ty$. But by Proposition 1.2.14, $tyt = ty$, so $tyt = t$. On the other hand, by Corollary 1.2.16, $(yt)t = yt$. Consequently

$$(7.18.3) \qquad (\{t, yt\}, \cdot) \text{ is a left zero semigroup.}$$

Now we will show that $(yt)T(B) \setminus tT(B) = \{yt\}$. First note that $yt \notin tT(B)$. Indeed, $yt \in tT(B)$ implies that $tyt = (yt)t$ and consequently, by (7.18.3), $t = yt$, contradicting the fact that $t \notin yT(B)$. Let $z \in (yt)T(B) \setminus tT(B)$ and $z \neq yt$. Then $z < yt$ and in consequence, by Remark 1.2.11,

$$(7.18.4) \qquad xz \leq xyt.$$

Moreover by Corollary 7.6, $z < yt$ implies that $az < ayt$. Note that since $z \leq y, axz = az$, and since $yt \leq y, axyt = ayt$. So we have $axz < axyt$ and consequently, by (7.18.4) we obtain

$$(7.18.5) \qquad xz < xyt.$$

Note that

$axyt = ayxyt$ since $xyt \leq x$

$axyt = ayxt$ by Proposition 1.2.14
$axyt = ayt$ since $t \leq x$
$axyt = at$ since $t \leq x$.
Hence, by Corollary 7.6, $xyt = t$ and in consequence, by (7.18.5),

$$xz < t. \tag{7.18.6}$$

Since t is minimal in $(xT(B) \setminus yT(B), \leq)$ (7.18.6) implies $xz \in yT(B)$. Recall that since $z \leq y, az = axz$. Consequently, by Corollary 7.6, $z = xz$. So, from (7.18.6) it follows that $z \in tT(B)$. This contradies the assumption that $z \in (yT)T(B) \setminus tT(B)$. We conclude that $(yt)T(B) \setminus tT(B) = \{yt\}$.

To prove that t and yt satisfy 5.1(ii) it remains to show that $tT(B) \setminus (yt)T(B) = \{t\}$. First we show that $t \in tT(B) \setminus (yt)T(B)$. Suppose on the contrary that $t \in (yt)T(B)$. Then by Remark 1.2.5, $tyt = (yt)t$ and consequently, by (7.18.3), $t = yt$ contradicting the fact that $t \notin yT(B)$. So $t \in tT(B) \setminus (yt)T(B)$. Now we show that t is the unique element of $tT(B) \setminus (yt)T(B)$. Suppose on the contrary that $u \in tT(B) \setminus (yt)T(B)$ and $u \neq t$. Then $u < t$ and consequently $tu = u$. Since t is minimal in $(xT(B) \setminus yT(B), \leq)$ we have that $u \in yT(B)$, whence $yu = u$. By (B5), $(yt)u = (yu)(tu)$. Therefore $(yt)u = uu = u$, i.e., $u \in (yt)T(b)$, contradicting the fact that $u \in tT(B) \setminus (yt)T(B)$. The last contradiction shows that $tT(B) \setminus (yt)T(B) = \{t\}$. This completes the proof that

(7.18.7) t and y satisfy the condition 5.1(ii).

By (7.18.7) and Lemma 7.7, it follows that $at \neq ayt$, a contradiction to (7.18.1). This contradiction shows that $x = y$. ∎

Let $\underline{L} := (aT(B), +, \cdot)$. An immediate consequence of Lemmas 7.3, 7.4 and 7.14 is the following

7.19. COROLLARY *The lattice $\underline{L}$ defined above is a finite distributive lattice with exactly one coatom.*

Let us define

$$R := \{(y^a, ay) | y \in B\}. \tag{7.20}$$

7.21. LEMMA. *The relation $R \subseteq B^2$ defined by (7.20) is contained in $\leq_L$ and satisfies the condition 7.0(i).*

Proof. First we show that $R \subseteq \leq_L$. Note that the lattice order $\leq_L$ is exactly the partial order $\leq$ of the bandoid $\underline{B}$, restricted to the set $aT(B)$. So we have to prove that $y^a \leq ay$. To do it, it suffices to show that $yay = y$. But this follows immediately from Corollary 7.6 and (B1). Therefore $R \subseteq \leq_L$.

In the next part of the proof we show that R satisfies the condition 7.0(i). Let $t \in JI(L) \setminus \{0\}$. If t is the unit of the lattice $\underline{L}$, i.e. $t = a$, then it suffices to put $n := 1, x_1 := a, z_1 := a$. Obviously $(a, a) \in R$, since $a = a^a = aa$.

Now we consider the case $t \neq a$. Let s be a predecessor of t in $(B, \leq)$. By Lemma 7.10 there exist $\alpha \in L(B)$ such that

$$\alpha t = a \text{ and } \alpha s < b. \tag{7.21.1}$$

Note that since $t \neq a$, $\alpha \neq \mathrm{id}_B$. So we may assume that for some $y_1, y_2, \dots, y_n \in B$

$$\alpha = L(y_n) \circ L(y_{n-1}) \circ \dots \circ L(y_1) \tag{7.21.2}$$

Let us define

$$x_i := (y_i y_{i-1} \dots y_1 t)^a \text{ and } z_i := a y_i y_{i-1} \dots y_1 t \tag{7.21.3}$$

for all $i \in \{1, 2, \dots, n\}$.

Obviously $(x_i, y_i) \in R$ for every $i = 1, 2, \dots, n$. By (7.21.1), (7.21.2) and (7.21.3) $z_n = a$. So, it remains to show that $t = x_1$ and for every $j = 1, 2, \dots, n-1$, $x_1 \leq x_{j+1} \leq z_j$.

First we show that $t = x_1$. Suppose on the contrary that $x_1 \neq t$, i.e.

$$(y_1 t)^a \neq t. \tag{7.21.4}$$

Note that $t \in aT(B)$ and by Corollary 1.2.16, $(y_1 t)t = y_1 t$. Hence $(y_1 t)^a \leq t$ and consequently, by (7.21.4), $(y_1 t)^a < t$. Since s is the only predecessor of t in $(B, \leq)$, the last inequality implies that $(y_1 t)^a \leq s$. So, by Remark 1.2.11, we have: $y_1 t = (y_1 t)(y_1 t)^a \leq (y_1 t)s \leq (y_1 t)t = y_1 t$. In consequence

$$(y_1 t)s = y_1 t. \tag{7.21.5}$$

On the other hand, by Propositions 1.2.12 and 1.2.7, we get

$$(y_1 t)s = (y_1 t)(y_1 s) = y_1(ts) = y_1 s. \tag{7.21.6}$$

From (7.21.5) and (7.21.6) it follows that $y_1 t = y_1 s$ and consequently, by (7.21.2) $\alpha t = \alpha s$, contradicting (7.21.1). Therefore

$$x_1 = t. \tag{7.21.7}$$

Now we prove that $x_1 \leq x_i$ for every $i = 1, 2, \dots, n$. Suppose on the contrary that $x_i \geq x_1$ for some $i \in \{1, 2, \dots, n\}$. Then, by (7.21.3) and (7.21.7), $(y_i y_{i-1} \dots y_1 t)^a \geq t$. Consequently

$$(y_i y_{i-1} \dots y_1 t)^a t \neq t. \tag{7.21.8}$$

By Proposition 1.2.14 and Remark 1.2.5 we have that

$$t(y_i y_{i-1} \dots y_1 t)^a t = t(y_i y_{i-1} \dots y_1 t)^a = (y_i y_{i-1} \dots y_1 t)^a t.$$

So $(y_i y_{i-1} \dots y_1 t)^a \leq t$ and in consequence, by (7.21.8), $(y_i y_{i-1} \dots y_1 t)^a < t$. Since s is the only predecessor of t we have $(y_i y_{i-1} \dots y_1 t)^a t \leq s$. Hence $(y_i y_{i-1} \dots y_1 t)^a t = s(y_i y_{i-1} \dots y_1 t)^a t = (y_i y_{i-1} \dots y_1 t)^a ts$ and consequently

$$y_i y_{i-1} \dots y_1 (y_i y_{i-1} \dots y_1 t)^a t = y_i y_{i-1} \dots y_1 (y_i y_{i-1} \dots y_1 t)^a ts$$

By Proposition 1.2.7, it follows that

$$(y_iy_{i-1}\dots y_1(y_iy_{i-1}\dots y_1t)^a)(y_iy_{i-1}\dots y_1t)$$
$$= (y_iy_{i-1}\dots y_1(y_iy_{i-1}\dots y_1t)^a)(y_iy_{i-1}\dots y_1t)(y_iy_{i-1}\dots y_1s).$$

Hence, by Remark 1.2.5, we obtain

$$(y_iy_{i-1}\dots y_1t)(y_iy_{i-1}\dots y_1(y_iy_{i-1}\dots y_1t)^a)$$
$$= ((y_iy_{i-1}\dots y_1t)(y_iy_{i-1}\dots y_1(y_iy_{i-1}\dots y_1t)^a)(y_iy_{i-1}\dots y_1s)$$

and in consequence, by Proposition 1.2.14

$$(y_iy_{i-1}\dots y_1t)(y_iy_{i-1}\dots y_1(y_iy_{i-1}\dots y_1t)(y_iy_{i-1}\dots y_1t)^a)$$
$$= ((y_iy_{i-1}\dots y_1t)y_iy_{i-1}\dots y_1(y_iy_{i-1}\dots y_1t)(y_iy_{i-1}\dots y_1t)^a)(y_iy_{i-1}\dots y_1s)$$

But $(y_iy_{i-1}\dots y_1t)(y_iy_{i-1}\dots y_1t)^a = y_iy_{i-1}\dots y_1t$, so

$$(y_iy_{i-1}\dots y_1t)(y_iy_{i-1}\dots y_1y_iy_{i-1}\dots y_1t)$$
$$= ((y_iy_{i-1}\dots y_1t)y_iy_{i-1}\dots y_1y_iy_{i-1}\dots y_1t)(y_iy_{i-1}\dots y_1s).$$

By Proposition 1.2.14

$$(y_iy_{i-1}\dots y_1)(y_iy_{i-1}\dots y_1t)$$
$$= (y_iy_{i-1}\dots y_1t)(y_iy_{i-1}\dots y_1t)(y_iy_{i-1}\dots y_1s)$$

whence, by Proposition 1.2.14 again

$$y_iy_{i-1}\dots y_1t = (y_iy_{i-1}\dots y_1)(y_iy_{i-1}\dots y_1s).$$

By Proposition 1.2.7 and since $s < t$ we have

$$(y_iy_{i-1}\dots y_1t)(y_iy_{i-1}\dots y_1s) = y_iy_{i-1}\dots y_1ts = y_iy_{i-1}\dots y_1s.$$

Therefore $y_iy_{i-1}\dots y_1t = y_iy_{i-1}\dots y_1s$ and consequently, by (7.21.2), $\alpha t = \alpha s$, a contradition to (7.21.1). This contradiction shows that $x_1 \le x_i$ for all $i = 1, 2, \dots, n$.

Now let $i \in \{1, 2, \dots, n-1\}$. To prove that $x_{i+1} \le z_i$, we note first the following:

$y_{i+1}y_i\dots y_1t = (y_{i+1}y_i\dots y_1t)(y_{i+1}y_i\dots y_1t)$ by (B1)
$y_{i+1}y_i\dots y_1t = (y_{i+1}y_i\dots y_1t)(y_{i+1}ay_i\dots y_1t)$ by Corollary 7.6
$y_{i+1}y_i\dots y_1t = (y_{i+1}y_i\dots y_1t)(y_{i+1}z_i)$ by (7.21.3)
$y_{i+1}y_i\dots y_1t = (y_{i+1}y_i\dots y_1t)z_i$ by Proposition 1.2.12.

Hence, by definition of y^a, we have $(y_{i+1}y_i\dots y_1t)^a \le z_i$, i.e. $x_{i+1} \le z_i$. This completes the proof of the fact that the relation R satisfy the condition 7.0(i) ■

For the lattice $\underline{L} = (aT(B), +, \cdot)$ and the relation R defined by (7.20) we consider the bandoid $\underline{B}(L, R)$ defined in Section 5. Let Ψ be the mapping defined as follows

$$(7.22) \qquad \Psi : B(L,R) \to B$$

and for every $x_{zt} \in B(L,R)$

$$\Psi x_{zt} = cx iff z = c^a \text{ and } t = ac.$$

7.23. LEMMA. *The mapping Ψ defined by (7.22) is a homomorphism from $(B(L,R),\cdot)$ to $(B,\cdot)$.*

Proof. Let $u_{xy}, w_{zt} \in B(L,R)$. By definition of R there exist elements, say c,d, in B such that $x = c^a$, $y = ac$, $z = d^a$, $t = ad$.

Then the following hold

$\Psi(u_{xy}w_{zt}) = \Psi(u_{xy}\phi_{ztxy}w_{zt})$ by Definition 1.2

$= \Psi(u_{xy}(x\Sigma(v \in (t]|vz = w))_{xy})$ by (2.3)

$= \Psi(u_{xy}(x\Sigma(v \in (ad]|vd^a = w))_{xy})$

$= \Psi(u_{xy}(xadw)_{xy})$ since by Lemma 7.17, $\Sigma(v \in (ad]|vd^a = w) = adw$

$= \Psi(uxadw)_{xy}$

$= cuxadw$ by (7.22)

$= c(ux)adw$ since $u, x, adw \in aT(B)$ and $(aT(B),\cdot)$ is a semilattice

$= cuadw$ since $u \leq x$

$= (cu)(cadw)$ by Proposition 1.2.7, since $u, adw \in aT(B)$

$= (cu)(cdw)$ by Corollary 7.6

$= (cu)(dw)$ by Proposition 1.2.12

$= \Psi u_{xy}\Psi w_{zt}$ by (7.22). ∎

7.24. LEMMA. *The mapping Ψ defined by (7.22) is onto.*

Proof. Let $x \in B$. Let $u = x^a, w = ax$. Obviously $(u,w) \in R$ and $u_{uw} \in B(L,R)$. Note that $\Psi u_{uw} = xu = xx^a = x$. So Ψ is onto. ∎

7.25. LEMMA. *The kernel $ker\Psi$ of the mapping Ψ is exactly the relation $\rho \subseteq B(L,R)^2$ defined by (4.1).*

Proof. First we show that $\rho \subseteq ker\Psi$. Let $u_{xy}, w_{zt} \in B(L,R)$ and $(u_{xy}, w_{zt}) \in \rho$. By definition of R, we may choose $c,d \in B$ such that $x = c^a, y = ac, z = d^a, t = ad$. By definition of ρ, since $(u_{xy}, w_{zt}) \in \rho we$ have

$$(7.25.1) \qquad au_{xy}w'_{xy} = au'_{xy} \text{ for every } u' \leq u.$$

We want to show that $\Psi u_{xy} = \Psi w_{zt}$, i.e. $cu = dw$. Let $v' \leq cu$. By Lemma 7.16 the left multiplication $L(c) : ((x],\cdot) \to (cT(B),\cdot)$ is an isomorphism. So

$v' = cu'$ for some $u' \leq u$. Note that

$$\begin{aligned}
a(dw)v' &= a(dw)(cu') \\
&= \Psi a \Psi w_{zt} \Psi u'_{xy} && \text{by (7.22)} \\
&= \Psi(a w_{zt} u'_{xy}) && \text{since } \Psi : B(L,R) \to (B,\cdot) \text{ is a homomorphism} \\
&= \Psi(a u'_{xy}) && \text{by definition (4.1) of } \rho, \text{ since } u_{xy}\rho w_{zt} \text{ and } u'_{xy} \leq u_{xy} \\
&= \Psi a \Psi u'_{xy} && \text{since } \Psi : B(L,R) \to (B,\cdot) \text{ is a homomorphism} \\
&= a(cu') && \text{by (7.22)} \\
&= av'.
\end{aligned}$$

So we have

(7.25.3) $\qquad a(dw)v' = av'$ for every $v' \leq cu$.

Analogously we show that

(7.25.4) $\qquad a(cu)v'' = av''$ for every $v'' \leq dw$.

By Lemma 7.18, (7.25.3) and (7.25.4) it follows that $cu = dw$. Therefore $\Psi u_{xy} = \Psi w_{zt}$ and consequently $(u_{xy}, w_{zt}) \in ker\Psi$. So $\rho \subseteq ker\Psi$.

Before showing the inverse inclusion, we prove that

(7.25.5) $\qquad$ for all $u_{xy}, w_{zt} \in B(L,R)$ if $\Psi u_{xy} = \Psi w_{zt}$, then $au_{xy} = aw_{zt}$.

Let $u_{xy}, w_{zt} \in B(L,R), \Psi u_{xy} = \Psi w_{zt}$ and c, d be such that $x = c^a, y = ac, z = d^a, t = ad$. Then $\Psi u_{xy} = cu$ and $\Psi w_{zt} = dw$. Since $\Psi u_{xy} = \Psi w_{zt}$, we have that $cu = dw$ and consequently, $acu = adw$. Note that

$$\begin{aligned}
acu &= \Sigma(v \in (y] | vx = u) && \text{by Lemma 7.17} \\
&= \phi_{xyaa} u_{xy} && \text{by (2.3), since a plays the role of unit in } \underline{L} \\
&= au_{xy} && \text{by Definition 1.2 and since } a \text{ is the unit in } \underline{L}.
\end{aligned}$$

Analogously we show that $adw = aw_{zt}$. Therefore $au_{xy} = aw_{zt}$, what completes the proof of (7.25.5).

Now we show that $ker\Psi \subseteq \rho$. Let $u_{xy}, w_{zt} \in B(L,R)$ and $\Psi u_{xy} = \Psi w_{zt}$. We want to show that $u_{xy}\rho w_{zt}$. Let $u' \leq u$. Then

$$\begin{aligned}
\Psi(w_{zt}, u_{xy}) &= \Psi w_{zt} \Psi u'_{xy} && \text{since } \Psi : \underline{B}(L,R) \to \underline{B} \text{ is a homomorphism} \\
&= \Psi u_{xy} \Psi u_{xy} && \text{since } \Psi u_{xy} = \Psi w_{zt} \\
&= \Psi(u_{xy} u'_{xy}) && \text{since } \Psi : \underline{B}(L,R) \to \underline{B} \text{ is a homomorphism} \\
&= \Psi u'_{xy} && \text{since } u'_{xy} \leq u_{xy}.
\end{aligned}$$

Hence, by (7.25.5), we have

(7.25.6) $\qquad aw_{zt}u'_{xy} = au'_{xy}$ for every $u' \leq u$.

Analogously we show that

(7.25.7) $$au_{xy}w'_{zt} = aw'_{zt} \text{ for every } w' \leq w.$$

By definition of ρ, from (7.25.6) and (7.25.7) it follows that $(u_{xy}, w_{zt}) \in \rho$. Therefore $\ker\Psi = \rho$. ∎

Proof of Theorem 7.0: Corollary 7.19, and Lemmas 7.21 through 7.25.

The following corollary will be useful in the next paper.

7.26. COROLLARY *Let $\underline{B}$ be a finite subdirectly irreducible left normal bandoid with a monolith $\Theta(a,b)$, where $b < a$. Let $\underline{L} = (aT(B), +, \cdot)$ and $R := \{(x^a, ax) | x \in B\}$. Then the mapping $\Phi : \underline{B} \to \underline{B}(L, R)$ such that for every $x \in B, \Phi x = x^{\rho}_{uw}$ iff $u = x^a, w = ax$, is an isomorphism.*

Proof. Let $x \in B$, $u = x^a$ and $w = ax$. By (7.22), $\Psi x_{uw} = xx = x$. Hence, using Lemmas 7.23 through 7.25 we conclude that Φ is an isomorphism from $\underline{B}$ to $\underline{B}(L, R)$. ∎

References

[1] E. Zajac: *Constructions of left normal bandoids*, Demonstratio Math. 24 (1991), 191–206.

[2] E. Zając: *Subdirectly irreducible left normal bandoids, I*, Demonstratio Math. 25 (1992), 927–946.

INSTITUTE OF MATHEMATICS,
PEDAGOGICAL UNIVERSITY OF KIELCE
ul. M. Konopnickiej 21,
25-406 KIELCE, POLAND

Received June 20, 1994.