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DISSEMILATTICES WITH MULTIPLICATIVE REDUCTS CHAINS

Dedicated to Professor Tadeusz Traczyk

In the years 1980–1990 a number of papers appeared investigating the structure, properties and meaning of meet-distributive bisemilattices, in which the multiplication distributes over the addition. (See the references at the end of the paper.) The algebras were subsequently [RS3] referred to as dissemilattices. Among them, those having the most accessible structure are the distributive dissemilattices (also called distributive quasilattices), in which the addition also distributes over the multiplication. They are all Płonka sums of distributive lattices [Pl]. This class is now known very well. As years of investigation have shown, the structure of dissemilattices is much more complicated, and there is no uniform structure theorem for them. However, we have a quite elegant structural description of free dissemilattices over a semilattice, and in particular over a set [R3, RS1, RS3, RS4, RS5]. This description is based on some versions of a construction introduced in [RS3] under the name of "Lallement sum". The construction is also used to describe some other classes of bisemilattices in [R5], [R6] and [R7], and involves some intriguing combinatorics. However, there are dissemilattices that cannot be described in a simple way as Lallement sums of simpler but well-known dissemilattices. Examples are furnished by some dissemilattices having at least one semilattice reduct a chain, and by known subdirectly irreducible dissemilattices [R2, R4]. In some cases it is however possible to describe such algebra in a simple pictorial manner, introducing certain spe-

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cial transformations of the graph of the multiplicative reduct to obtain the graph of the additive reduct [R2, R4, R5, RS3, RS5]. In particular this concerns dissemilattices with additive reduct a chain, with both reducts chains, and with multiplicative reduct a Boolean lattice [R2]. In this paper we continue this approach to studying dissemilattices. We recall some descriptions of bisemilattices based on graph manipulations, and observe some new properties for this class. This is done in Section 2, following Section 1, where we briefly recall the necessary definitions and notation. In Section 3, we describe the structure of dissemilattices with multiplicative reduct a chain. The intriguing aspect of the main result is the correspondence between the meet-distributive identity and geometrical correspondences between the two graphs of semilattices.

Before we present the result, let us mention that during the last ten years, dissemilattices have shown to be a very usefull tool in the investigation of the structure theory of modals [RS3, RS4, RS5], and have recently attracted the serious attention of computer scientists [L], [Pu], [RT].

1. Preliminaries

A *bisemilattice* is a set B with two semilattice operations, $\cdot$ of meet and $+$ of join. Each of these operations yields a partial order on B by setting

$$x \leq_{\cdot} y \quad \text{iff} \quad xy = x\,,$$
$$x \leq_{+} y \quad \text{iff} \quad x + y = y\,.$$

Examples are furnished by lattices $(L, \vee, \wedge)$ with the usual meet and join operations (for which the two partial orders $\leq_{\wedge}$ and $\leq_{\vee}$ coincide with the usual order relation) and "stammered" semilattices $(S, \cdot, \cdot)$ obtained from a semilattice $(S, \cdot)$ by taking the same underlying set S with the semilattice operation considered twice, once as a meet and once as a join.

Among many classes of bisemilattices investigated in recent years, the class of *meet-distributive* bisemilattices, in which the meet operation $\cdot$ distributes over the join operation $+$:

$$(\cdot\, D) \qquad x(y+z) = xy + xz\,,$$

plays a quite important rôle. As examples one has distributive lattices, stammered semilattices and *distributive* bisemilattices, in which also the join operation $+$ distributes over the meet operation $\cdot$:

$$(+D) \qquad x + yz = (x+y)(x+z)\,.$$

Distributive bisemilattices are also known under the name of "distributive quasilattices" [B], [N], [Pł].

It is well-known that the distributive bisemilattices are Płonka sums of distributive lattices [Pł], [RS3], and that in bisemilattices the distributive law ($\cdot\, D$) does not imply ($+D$) [R2]. As in [RS3], meet-distributive bisemilattices are called *dissemilattices* in this paper. If both semilattice reducts of a dissemilattices $\underline{B} = (B, +, \cdot)$ are chains, $\underline{B}$ is called a *bichain*. If neither of the distributive laws ($\cdot\, D$) and $+D$ is satisfied in a bisemilattice, it is called *nondistributive*.

We use notation as in [R4], to which this paper may be considered as a sequel. Let $\circ$ denote $\cdot$ or $+$. We write $x \prec_\circ y$ if y covers x in the reduct $(B, \circ)$ of $\underline{B}$. The symbol $x \leftrightarrow_\circ y$ means that x and y are comparable in $(B, \circ)$, and $x \underset{\circ}{\|} y$ that they are not. In the pictures, the left hand diagram always represents the order $\leq_\cdot$, and the right hand the order $\leq_+$ of $\underline{B}$.

Let B_1 and B_2 be subsets of B. Let a be in B_1 and b in B_2. If a $<_\circ x$ for all x in B_2, then we write $a <_\circ B_2$. If $y <_\circ b$ for all y in B_1, we write $B_1 <_\circ b$. If $x \leq_\circ y$ for all x in B_1 and all y in B_2, we write $B_1 \leq_\circ B_2$.

A subsemilattice $\underline{A}$ of a semilattice $\underline{S}$ is called *Boolean* if it is reduct of a Boolean lattice. Stammered semilattices are called briefly semilattices.

A subset C of a bisemilattice $\underline{B}$ is called a *convex* subalgebra, if C is a convex subsemilattice of both semilattice reducts of $\underline{B}$.

The symbols $\underline{2}$ and $\underset{\sim}{2}$ denote the two element lattice and two element (stammered) semilattice, respectively.

We refer the reader to the list of references at the end of the paper for further information concerning dissemilattices, and other concepts and results not recalled here.

2. Semilattice reducts of some dissemilattices

We start with some known properties of dissemilattices.

LEMMA 2.1 [R2, R4]. *Let $\underline{B} = (B, +, \cdot)$ be a bisemilattice.*

(i) *Let $B = \{a, b, c\}$ and $a <_\cdot b <_\cdot c$. If $(B, +)$ is a chain, then $(B, \leq_+)$ has one of the following forms:*

a) *$a <_+ b <_+ c$, whence $\underline{B}$ is a lattice;*
b) *$c <_+ b <_+ a$, whence $\underline{B}$ is a semilattice;*
c) *$b <_+ c <_+ a$, in which case $\underline{B}$ is distributive;*
d) *$a <_+ c <_+ b$, in which case $\underline{B}$ is meet-distributive;*
e) *$b <_+ a <_+ c$, in which case $\underline{B}$ is join-distributive;*
f) *$c <_+ a <_+ b$, in which case $\underline{B}$ is nondistributive.*

(ii) *Let $B = \{a, b, c\}$ and let $\underline{B}$ be a dissemilattice, but not a bichain. Then either $\underline{B}$ is a semilattice or $a <_\cdot b <_\cdot c$ and $a + c = b$, or $a <_+ b <_+ c$ and $ac = b$. In all these cases $\underline{B}$ is distributive.*

(iii) *Let $B = \{a, b, c, d\}$ and let $\underline{B}$ be a dissemilattice. If $a <_\cdot b <_\cdot c <_\cdot d$ and $(B, +)$ is Boolean, then $d + b = c$ and $a <_+ d, b$.*

(iv) *Let $B = \{a, b, c, d\}$. If $(B, \cdot)$ is Boolean and $(B, +)$ is a chain, then $\underline{B}$ is nondistributive or join-distributive.*

(v) *If $\underline{B}$ is a dissemilattice, then for a, b in B with $a \multimap_{\circ} b$, one has $a \leftrightarrow_{\circ} b$.*

(vi) *If $\underline{B}$ is a dissemilattice, $(B, \cdot)$ consists of elements 0, 1, x_i for $i = 1, \ldots, n$ with $0 \prec\!\!\cdot\, x_i \prec\!\!\cdot\, 1$ and $x_i \,\|\, x_j$ for $i \neq j$, then $\underline{B}$ is a semilattice.* ∎

THEOREM 2.2 [R4]. *Let $\underline{B} = (B, +, \cdot)$ be a bisemilattice with both semilattice reducts chains. Then $\underline{B}$ is a dissemilattice if and only if one of the following holds:*

(i) *$\underline{B}$ is a lattice;*
(ii) *$\underline{B}$ is a semilattice;*
(iii) *$(B, +)$ can be divided into two convex intervals B_1 and B_2 such that $B_1 \leq_+ B_2$. Moreover, $(B_1, +, \cdot)$ forms a lattice and $(B_2, +, \cdot)$ a semilattice.* ∎

THEOREM 2.3 [R4]. *Let $\underline{B} = (B, +, \cdot)$ be a bisemilattice with the reduct $(B, +)$ a chain. Then $\underline{B}$ is a dissemilattice if and only if either:*

(i) *$\underline{B}$ is a bichain; or*
(ii) *the reduct $(B, \cdot)$ is a tree, in which each chain forms a bichain in $\underline{B}$, moreover:*
   a) *if $(a_i)_{i \in I}$ is a family of elements of B pairwise non-comparable in $(B, \cdot)$ such that $a_i a_j = b$, then I has at most two elements;*
   b) *if a in B is meet reducible, then the set $A := \{x \in B | a \leq_{\cdot} x\}$ is a convex subalgebra of $\underline{B}$, and $B - A <_+ A$ or $A <_+ B - A$.* ∎

THEOREM 2.4 [R4]. *Let $\underline{B} = (B, +, \cdot)$ be a dissemilattice. If the reduct $(B, \cdot)$ is Boolean, and isomorphic to $\underset{\sim}{2}^n$, then the reduct $(B, +)$ is also Boolean. Moreover, $\underline{B}$ is isomorphic to one of the following distributive dissemilattices: $\underline{2}^n$, $\underline{2}^{n-1} \times \underset{\sim}{2}$, $\underline{2}^{n-2} \times \underset{\sim}{2}^2, \ldots, \underline{2} \times \underset{\sim}{2}^{n-1}$, $\underset{\sim}{2}^n$.* ∎

LEMMA 2.5 [R1]. *The free dissemilattice on two generators x and y has five elements in the form presented in Fig. 1.* ∎

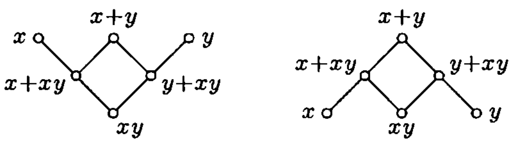


Fig. 1

LEMMA 2.6. *The free dissemilattice with unity* 1 *over the meet semilattice generated by two elements x and y has* 13 *elements in the form presented in Fig* 2.

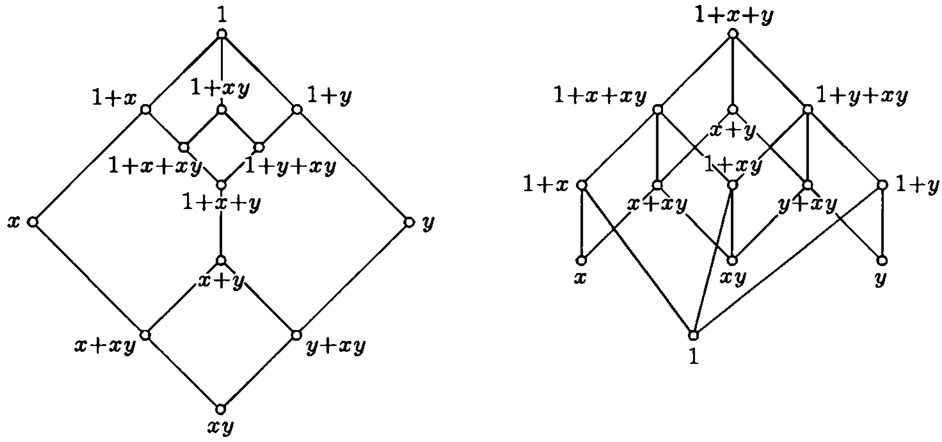


Fig. 2

P r o o f. This follows by the characterization of free dissemilattices over semilattices in [RS1] and [RS3]. ∎

COROLLARY 2.7. *The free dissemilattice with unity* 1 *over the meet semilattice generated by two elements x and y with x, $y <_+ 1$ is a five element lattice presented in Fig* 3. ∎

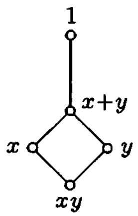


Fig. 3

COROLLARY 2.8. *The free dissemilattice with unity* 1 *over the meet semilattice generated by two elements x and y with $1 <_+ x$, y is a semilattice presented in Fig* 4. ∎

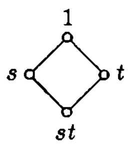


Fig. 4

LEMMA 2.9. *For elements x, y, a, b of a dissemilattice $\underline{B} = (B, +, \cdot)$, the following hold:*

(i) *if $x, y \leq_{\cdot} a$, then $x + y \leq_{\cdot} a$;*
(ii) *if $b \leq_{+} x, y$, then $b \leq_{+} xy$.*

Proof. For (i) see [R4]. To prove (ii), note that by Lemma 2.1(ii), if $b \leq_{+} x$, then $b \leq_{+} bx \leq_{+} x$, and if $b \leq_{+} y$, then $b \leq_{+} by \leq_{+} y$. Hence $by + xy = (b + x)y = xy$ and $bx + yx = (b + y)x = xy$. Consequently, $b \leq_{+} bx, by \leq_{+} xy$. ∎

COROLLARY 2.10. *In a dissemilattice $\underline{B} = (B, +, \cdot)$, all intervals of $(B, \cdot)$ and all intervals of $(B, +)$ are subalgebras of $\underline{B}$.* ∎

3. Dissemilattices with multiplicative reducts chains

We now describe a certain family of (join) semilattices that will play a special rôle in the main theorem of this section. Each such semilattice $(S, +)$ is a disjoint union of a chain $(M, +)$ called a *mast* and family of

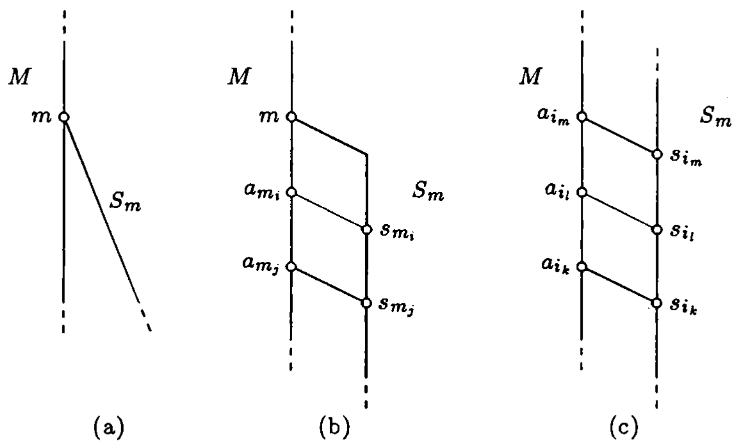
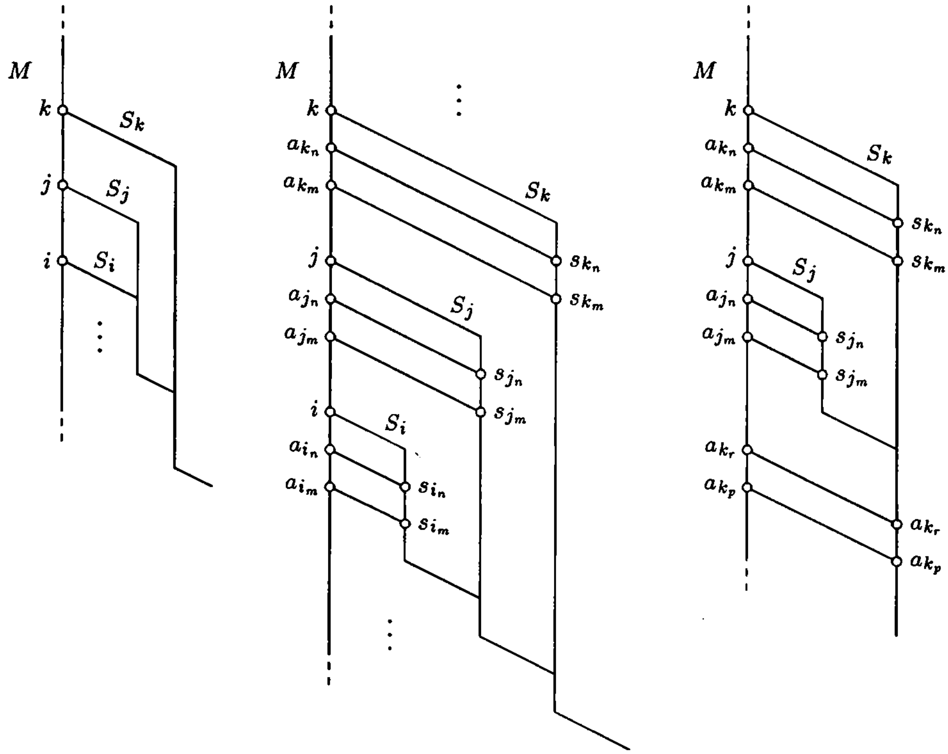


Fig. 5

chains $(S_i, +)$ for i in some set I, called *stripes*. Some elements s_{ij} of a stripe S_i may be covered by elements a_{ij} of the mast. We say that the elements a_{ij} are the *points of attachment* of S_i, and the elements s_{ij} *are attached* to the mast. If there is a least element m in M with $S_i <_{+} m$, we usully denote the stripe S_i by S_m and call m the *main* point of attachment of S_m.

A single stripe with one or more elements attached to the mast is called a *simple flag*. Examples of simple flags are shown in Fig. 5. Note that a simple flag $(F, +)$ may not be bounded from above. In this case, it has infinitely many attachment points, as in Fig. 5(c).

A *composed flag* consists of more than one simple stripe. Moreover, each two simple stripes, say S_j and S_k, in a composed flag are *related*, meaning that one of them, say S_k, contains an element that is less than all elements of S_j, and all points of attachment of S_k are above and sometimes also below all points of attachment of S_j. Examples of composed flags are shown in Fig. 6



(a) (b) (c)

Fig. 6

A semilattice $(S,+)$ constructed from a mast $(M,+)$ and (simple and composed) flags $(F_j,+)$ for j in some set J, is called a *flagstaff*, if it satisfies the following two condition:

(FS1) If an element m of the mast $(M,+)$ is join-reducible, and there is a family $\{a_k\}_{k\in K}$ of pairwise non-comparable elements of S with $a_k + a_l = m$ for $k \neq l$, then K has exactly two elements.

(FS2) If $(F_1,+)$ and $(F_2,+)$ are two flags of $(S,+)$, and A_1 and A_2 are the sets of attachment points of F_1 and F_2, respectively, then either $A_1 <_+ A_2$, or $A_2 <_+ A_1$.

If $A_1 <_+ A_2$, then we say that the *flag* F_1 *is below the flag* F_2. In this case we say, that the flags F_1 and F_2 are attached to the mast in a *disjoint way*. Similarly, if A_j and A_k are the sets of attachment points of stripes S_j and S_k, and $A_j <_+ A_k$, then we say that S_j is *below* S_k. On the other hand, if S_j and S_k are situated as in Fig. 6(c), then we say that S_j is *inside* S_k.

PROPOSITION 3.1. *Let* $\underline{B} = (B, +, \cdot)$ *be a bisemilattice. Let* $(B, \cdot)$ *be a chain, and* $(B, +)$ *be a flagstaff consisting of a mast* $(M, +)$ *and one simple or composed flag* $(F, +)$. *Assume that* $(B, +)$ *satisfies the following conditions.*

(i) $(M, +, \cdot)$ *is a semilattice.*

(ii) *If* $(F, +)$ *consists of stripes* $(S_i, +)$ *for* i *in* I, *then each* $(S_i, +, \cdot)$ *is a lattice.*

(iii) *If for* i, j *in* I, *the stripe* $(S_i, +)$ *is below the stripe* $(S_j, +)$, *then* $S_j <_\cdot S_i$.

(iv) *If for* i, j *in* I, *the stripe* $(S_i, +)$ *is inside the stripe* $(S_j, \cdot)$, *then all attachment points of* S_j *that are above* S_i *in* $(B, +)$, *are below* S_i *in* $(B, \cdot)$ *(Cp. Fig. 7).*

Then $\underline{B}$ *is a dissemilattice.*

Proof. If $(F, +)$ consists of one stripe $(S_i, +)$ attached to the mast at the point i, then 3.1 follows by Lemma 2.1 and Theorem 2.2. Assume now that $(S_i, +)$ is not necessarily bounded from above, and has at least two attachment points s_i' and s_i''. Then each triple of elements of B belongs either to a subbisemilattice of the type already considered or generates a four element subbisemilattice with Boolean additive reduct. The last one is meet-distributive by Lemma 2.1(iii).

Now let $(F, +)$ be a composed flag. Let a, b, c be elements of B. The cases, when a, b, c are all in $M \cup S_i$ for some i in I, were considered before. Let S_i, S_j, S_k be three different stripes of $(B, +)$. Without loss of generality we can assume that $(S_i, +)$ is below $(S_j, +)$, that $(S_j, +)$ is below $(S_k, +)$, and that the remaining cases are the following depicted in Fig. 8

Using Lemma 2.1 again, one can easily check that the elements a, b, c, j, k or a, b, c, i, j, k form a meet-distributive subbisemilattice in each of these cases. ∎

COROLLARY 3.2. *Let* $\underline{B} = (B, +, \cdot)$ *be a bisemilattice. Let* $(B, \cdot)$ *be a chain and* $(B, +)$ *a flagstaff with all flags satisfying conditions* (i)–(iv) *of* 3.1. *If any two flags of* $(B, +)$ *are attached to the mast in disjoint way, and for a flag* $(F_i, +)$ *below a flag* $(F_2, +)$, $F_2 <_\cdot F_1$, *then* $\underline{B}$ *is a dissemilattice.* ∎

THEOREM 3.3. *Let $\underline{B} = (B, +, \cdot)$ be a bisemilattice with the reduct $(B, \cdot)$ being a chain. Then $\underline{B}$ is a dissemilattice if and only if the reduct $(B, +)$ is a flagstaff satisfying all conditions of Corollary* 3.2.

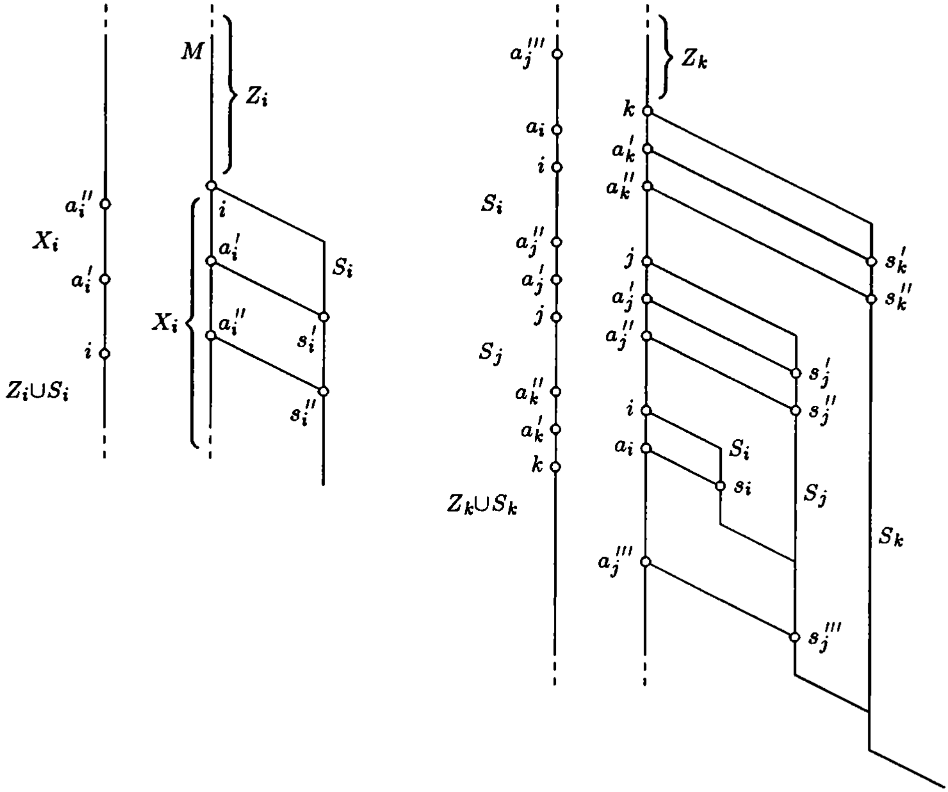


Fig. 7

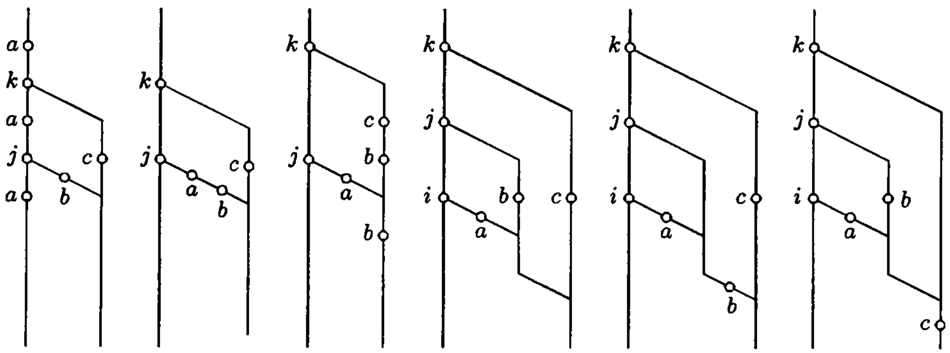


Fig. 8

P r o o f. Corollary 3.2 provides the proof of the sufficiency. If remains to prove the necessity. So let us assume that $\underline{B}$ is a dissemilattice and $(B, \cdot)$ is a chain.

A. Let a, b, c, d be in B with $a+b = a+d = b+d = c$. Then Lemma 2.1 implies that $d,\ b <\cdot\ c <\cdot\ a$, whence $b <\cdot\ c <\cdot\ d$ and $d <\cdot\ c <\cdot\ b$, or $a <\cdot\ c <\cdot\ b,\ d$, whence again $d <\cdot\ c <\cdot\ b$ and $b <\cdot\ c <\cdot\ d$. This gives a contradiction in each case and proves condition (FS1).

B. Let n be a join-reducible element of B. Suppose X, Y, Z are subsets of B with $X \underset{+}{\|} Y$, moreover $X,\ Y <_{+} n <_{+} Z$, and $x + y = n$ for each x in X and y in Y. Then Lemma 2.1 implies that either $X <\cdot\ n <\cdot\ Y$ or $Y <\cdot\ n <\cdot\ X$. Without loss of generality assume $Y <\cdot\ n <\cdot\ X$. By Lemma 2.1 again, one has $Y \cup Z <\cdot\ n <\cdot\ X$. By Corollary 2.7, $(Y \cup \{n\}, +, \cdot)$ is a lattice, and hence a chain. By Corollary 2.8, $(Z \cup \{n\}, +, \cdot)$ is a semilattice and hence a chain. See Fig. 9

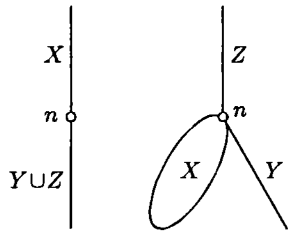


Fig. 9

C. In the context of $\mathbb{B}$, let $X_n := \{x \in X \mid j <_{+} x$ for each join-reducible element j of $X\}$, $Y'_n := \{y \in Y \mid$ there is no $x \in X$ with $y <_{+} x\}$ and $Z_n := Z$. It follows by B, that Z_n forms a semilattice, and Y'_n forms a lattice. Again by $\mathbb{B}$, it is clear that if $k <_{+} m <_{+} n$ are join reducible elements of $(B, +)$, then $Y'_n \cup Z_n <\cdot\ n <\cdot\ X_n \cup Y'_m <\cdot\ m <\cdot\ X_m \cup Y'_k <\cdot\ k <\cdot\ X_k$. See Fig. 10.

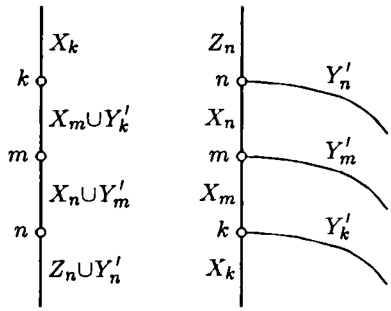


Fig. 10

D. Let us consider the configuration depictured in Fig. 10, where the set Y'_m may be emply. Suppose there is c in B with $c <_+ Y'_m \cup \{m\}$ and $c <_+ Y'_n$. Let x_n be in $X_n \cup \{m\}$ and y_n be in Y'_n. Then $x_n + y_n = n$, and x_n, n, y_n, c form a Boolean semilattice. By Lemma 2.1(iii) and $\mathbb{C}$, $c <_\cdot y_n <_\cdot n <_\cdot x_n$. Let $Y''_n := \{c \in B | c <_+ Y'_n$ and $c <_+ Y'_m \cup \{m\}$ for $m <_+ n\}$. Let $Y_n := Y'_n \cup Y''_n$. Then it is easy to see that $Y_n <_\cdot n$, and by Corollary 2.7, $(Y_n, +, \cdot)$ is a lattice. It follows, that the chains Y_n and Y_m may be placed with respect to each other in three possible ways depicted in Fig. 11, Fig. 12 and Fig. 13.

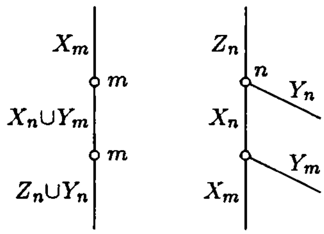


Fig. 11

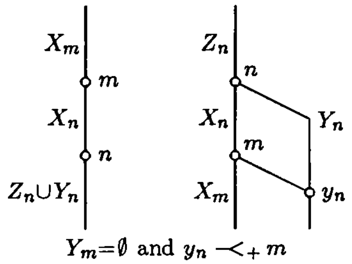


Fig. 12

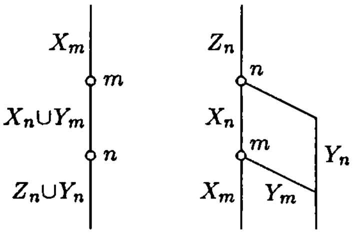


Fig. 13

E. Let J be the set of all join-reducible elements of $(B, +)$. Let $M' := \cup(Z_j \mid j \in J)$. By $\mathbb{B}$ and $\mathbb{C}$, the set M' forms a semilattice in $(B, +, \cdot)$. If there is a subset $X_0 \subseteq B$ with $X_0 <_+ J$, then X_0 must form a bichain, and by Theorem 2.2, $(X_0, +)$ decomposes into two convex intervals X'_0 and X^2_0 such that X'_0 forms a lattice, X^2_0 forms a semilattice and $X'_0 <_+ X^2_0$. We define the mast M of $(B, +)$ to be $M := X^2_0 \cup M'$.

F. For j in J, the reducts $(Y_j, +)$ of lattices $(Y_j, +, \cdot)$ are stripes attached to the mast $(M, +)$ at least at points j of M. If there is no element y_j of Y_j with $y_j <_+ y_k$ for some y_k in Y_k, where $k \in J$, then $(Y_j, +)$ forms a simple flag, not related to others. If a simple flag $(F, +)$ is not bounded from above, then it has infinitely many attachment points and diagrams as in Fig. 14. Moreover $F = X \cup F$, $(X, +, \cdot)$ is a semilattice and $(Y, +, \cdot)$ a lattice.

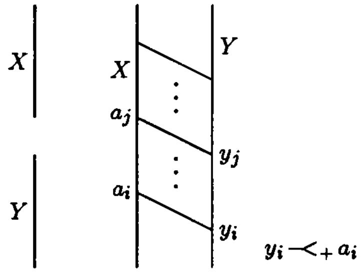


Fig. 14

G. Now suppose two stripes $(Y_m, +)$ and $(Y_n, +)$, where $m <_+ n$ in $(M, +)$, form simple flags. Suppose further that a_m in Y_m and b_n in Y_n are attached to M at a and b, respectively and $a <_+ b <_+ m$. Then obviously, $a_m + b_n = b$ and by $\mathbb{D}$, $b_n <\cdot\, a_m <\cdot\, b$. By Lemma 2.1(ii), the elements a_m, b_n and b form a non-distributive triple, contradicting the distributivity of $(B, +, \cdot)$. The case $b <_+ a$ gives a similar contradiction. If follows that in $(B, +)$, all points of attachment of Y_n are above all points of attachment of Y_m. Similar result is obtained if at least one of $(Y_m, +)$ and $(Y_n, +)$ is unbounded from above. See Fig. 15.

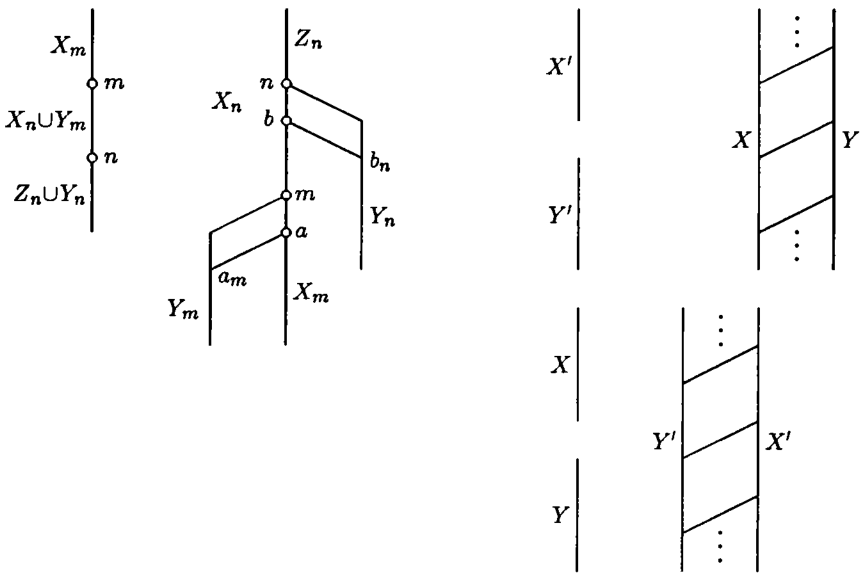


Fig. 15

H. Now suppose $m <_+ n$ in $(M, +)$ and the stripes $(Y_m, +)$ and $(Y_n, +)$ are related. An argument similar to that in $\mathbb{G}$ shows that in this case, all attachment points of Y_n are above or above and below all attachment points of Y_m. In the last case, the attached points of Y_n attached to M below all attachment points of Y_m, are all below the whole Y_m. Moreover, Lemma 2.1(ii) implies that in $(B, \cdot)$, all elements of Y_m are above all points of attachment of Y_n that are in X_n. See Fig. 16. Each two related flags belong to one composed flag.

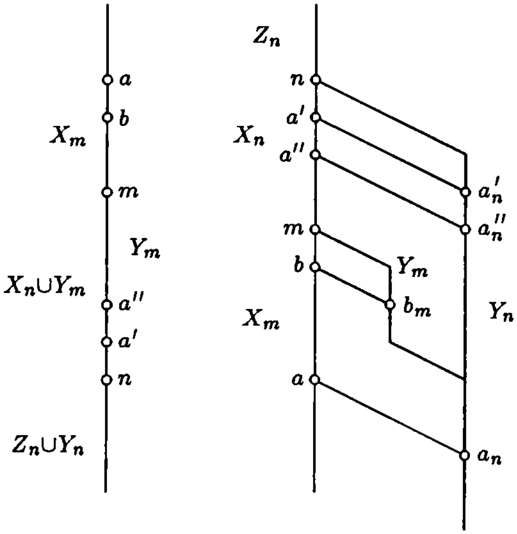


Fig. 16

I. It remains to show that any two flags $(F_1, +)$ and $(F_2, +)$ are attached to the mast in a disjoint way. This follows by $\mathbb{G}$ in the case both $(F_1, +)$ and $(F_2, +)$ are simple flags. So suppose $(F_2, +)$ is composed, and let $(Y_m, +)$ be a stripe in $(F_1, +)$, and $(Y_k, +)$ and $(Y_n, +)$ with $k <_+ n$ be two related

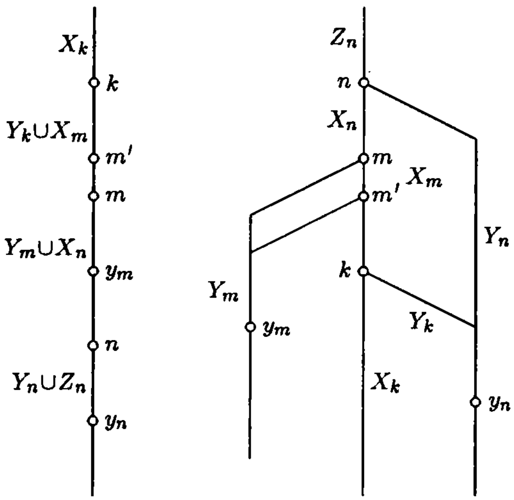


Fig. 17

stripes in $(F_2, +)$. Let $y_n \in Y_n$ and $y_m \in Y_m$ with $y_n <_+ Y_k$. See Fig. 17.

If $k <_+ m' \leq_+ m$, then $y_n + y_m = m'$ and $y_n <_\cdot y_m <_\cdot m'$, which contradicts Lemma 2.1(ii). The case $m' <_+ k$ gives a similar contradiction. It follows, that all attachment points of F_1 are above or below all attachment points of F_2. ∎

As an immediate corollary from Theorem 2.3 and 3.3, one obtains the following theorem proved directly in [R2].

THEOREM 3.4. *Let $(B, +, \cdot)$ be a subdirectly irreducible dissemilattice. Then the reduct $(B, +)$ is a chain if and only if the reduct $(B, \cdot)$ is a chain as well.* ∎

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