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# ON VARIETIES OF GROUPOID MODES

*Dedicated to Professor Tadeusz Traczyk*

## 1. Introduction

In this paper we study varieties of groupoid modes, that is idempotent entropic algebras that can be defined by a single binary fundamental operation. For theory of modes and their applications in algebra and geometry see [33,34] and the references given therein.

From the equational point of view, groupoid modes are groupoids satisfying laws of idempotency and mediality:

(I) $$x^2 = x,$$

(M) $$(xy)(uv) = (xu)(yv).$$

and hence, they form a variety (equational class) of algebras. Ježek and Kepka [18] have described all varieties of commutative groupoid modes (They called them CIA–groupoids). The noncommutative case turn out to be much more complicated, and therefore other authors have focused attention on subvarieties defined by various simple additional identities like $(xy)y = x$, $(xy)y = y$, $(xy)x = y, \ldots$, etc. (see e.g., [5, 9, 14, 22, 25, 38, 40]).

In this paper we present more systematic approach to the subject based on the following classification of polynomials (terms) due to E. Marczewski (see [21]).

For binary polynomial symbol $\{\cdot\}$ we define $P_1 = \{x, y\}$ and $P_{k+1} = P_k \cup \{fg : f, g \in P_k\}$. Then $\bigcup_{k=1}^{\infty} P_k$ is the set of all binary polynomial symbols over $\{\cdot\}$. We say that an identity $f = g$ is of rank $k$, if $f, g \in P_k$ and at least one of them is not in $P_{k-1}$.

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Now, considering subvarieties defined by additional identities it is natural to start from identities of low rank. In this paper we consider nonregular identities, that is, having a variable, which occurs only on one side. For these we have:

THEOREM. *If $K$ is a nontrivial subvariety of the variety of groupoid modes defined by a single nonregular identity of rank 3, then $K$ is one of the following varieties (up to duality):*

$$\begin{aligned} \mathbf{R}: &\quad xy^2 = x; \\ \mathbf{RR}: &\quad (xy)z = yz \text{ and } x(yz) = y(xz); \\ \Delta: &\quad (xy)z = x(yz) = xz; \\ \mathbf{Q}: &\quad (xy)x = y; \\ \nabla: &\quad (xy)(yx) = x; \\ \mathbf{E}: &\quad (xy)(yx) = y. \end{aligned}$$

The proof of this theorem is given in Section 3. Then, in subsequent sections we deal with each variety in more detail. Earlier, for the reader convenience, we quote the results applied in this paper.

## 2. Terminology and applied results

In the earlier papers [11] and [12] we propose to classify modes by means of the number $p_2(G,\cdot)$ of essentially binary polynomials in $(G,\cdot)$. As a first step in this direction we have the following

THEOREM 2.1. *Let $(G,\cdot)$ be a proper medial idempotent groupoid, i.e., $xy$ is essentially binary. Then we have*

(i) $p_2(G,\cdot) = 1$ *if and only if $(G,\cdot)$ is either a semilattice or an affine space over $GF(3)$.*

(ii) $p_2(G,\cdot) = 2$ *if and only if $(G,\cdot)$ is either a diagonal semigroup or an $n$–polynomial groupoid or an affine space over $GF(4)$.*

(iii) $p_2(G,\cdot) = 3$ *if and only if $(G,\cdot)$ is either an affine space over $GF(5)$ or a nontrivial Płonka sum of affine spaces over $GF(3)$ which are not all singetons.*

(iv) $p_2(G,\cdot) = 5$ *if and only if $(G,\cdot)$ is either an affine space over $GF(7)$ or a nontrivial Płonka sum of affine spaces over $GF(5)$ which are not all one–element.*

Note that all the groupoids appearing in this theorem are well–known and the definitions and the basic characterizations of them are recalled in [11]. For the definition of the Płonka sum and its properties we refer the reader to [28]. Let us add that so far there is no such a description for medial idempotent groupoids $(G,\cdot)$ with $p_2(G,\cdot) = 4$ (comp. Theorem 2.1). In [12] and also the present paper we give some useful information on such groupoids. Here, as earlier, polynomially equivalent algebras (i.e., algebras having the same sets of polynomials) are treated as identical. Our terminology is standard (cf. [15]). Throughout the paper $xy^n$ denotes the polynomial $(\ldots(xy)\ldots)y$, where $y$ appears $n$ times and dually ${}^n yx$ stands for $y(y(\ldots(yx)\ldots))$ ($n$ times $y$). By $Q_{m,n}$ we denote the variety of all groupoids $(G,\cdot)$ satisfying $xy^m = x$ and ${}^n yx = x$ for fixed nonnegative integers $m$ and $n$ (see [8]).

THEOREM 2.2. *Let $(G,\cdot)$ be a commutative idempotent groupoid. Then we have*

(i) *If* $\operatorname{card} G > 1$, *then the polynomials $xy^k$ and $y$ are distinct for all $k$ (see Theorem* 1 *of* [7]).

(ii) *If $(G,\cdot)$ is medial, then we have*

(a) *$(G,\cdot)$ satisfies $xy^n = x$ for some $n \geq 1$ if and only if $(G,\cdot)$ is an affine module over $\mathbf{Z}_d = (\{0,\ldots,d-1\},\overset{+}{_d},\overset{\cdot}{_d})$, where $d$ is a divisor of the number* $2^n - 1$ (Theorem 1 of [8], see also [6]).

(b) *$(G,\cdot)$ satisfies $xy^{n+1} = xy$ for some $n \geq 1$ if and only if $(G,\cdot)$ is a Plonka sum of affine modules $\mathbf{Z}_d$ with $d$ dividing* $2^n - 1$ (for details see Theorems 2, 3 and Corollary from [8]).

(c) *$(G,\cdot)$ is an affine space over $GF(3)$ if and only if $(G,\cdot)$ satisfies* $xy^2x = x$ (see (i) of Lemma 3.4 in [11]).

(d) *$(G,\cdot)$ is an affine space over $GF(5)$ if and only if $(G,\cdot)$ satisfies* $xy^2x = y$ (see (ii) of Lemma 3.4 in [11]).

(iii) *If $(G,\cdot)$ satisfies $xy^n = x$ for some $n \geq 1$, then the clone of $(G,\cdot)$ is minimal if and only if $(G,\cdot)$ is an affine space over $GF(p)$, where $p$ divides the number* $2^n - 1$ (see the Proposition in [10]).

THEOREM 2.3. *Let $(G,\cdot)$ be an idempotent groupoid. Then we have*

(i) *If $(G,\cdot)$ is distributive (or medial), then $(G,\cdot)$ is a diagonal semigroup if and only if $(G,\cdot)$ satisfies $(xy)x = x$ (or dually $x(yx) = x$)* (see Lemma 4.1 of [11]).

(ii) *If $(G,\cdot)$ is medial with $p_2(G,\cdot) = 2$, then $(G,\cdot)$ is a nontrivial affine space over $GF(4)$ if and only if $(G,\cdot)$ satisfies $(xy)x = y$* (see §4.3 of [11] and (ii) of Theorem 2.1).

(iii) *The following are equivalent:*

(a) *$(G,\cdot)$ satisfies $(xy)z = yz$ and $x(yz) = y(xz)$;*

(b) *$(G,\cdot)$ is a medial groupoid satisfying $xy^2 = y$;*

(c) *$(G,\cdot)$ is a distributive groupoid satisfying $xy^2 = y$*

(The dual version is also true, for details see Lemma 3.1 of [12] or Proposition 1.1 in [35])

(iv) *If* $(G,\cdot) \in Q_{m,n}$, *then* $(G,\cdot)$ *is a quasigroup* (see e.g. in [8]).

(v) *If* $(G,\cdot) \in Q_{2,n}$ *for some* $n \geq 2$, *then the polynomial* $x + y = {}^{n-1}yx$ *is commutative*, $x + (n-1)y = yx$, $x + ny = x$ *and consequently the groupoids* $(G,\cdot)$ *and* $(G,+)$ *are polynomially equivalent* (see the proof of Theorem 9 of [8]).

## 3. A characterization

In this section we prove the result quoted in Introduction. We start form

LEMMA 3.1. *For a binary (idempotent) polynomial symbol* $\{\cdot\}$ *we have, for a nonregular identity of rank* 3 *the following posibilities*:

(1) $(xy)y = x$, (2) $(xy)y = y$, (3)$(xy)x = x$, (4) $(xy)x = y$,
(5) $x(xy) = x$, (6) $x(xy) = y$, (7) $x(yx) = x$, (8) $x(yx) = y$,
(9) $(xy)(yx) = x$ and (10)$(xy)(yx) = y$.

Proof. Immediately.

We say that a groupoid $(G,\circ)$ is dual to the groupoid $(G,\cdot)$ if $x \circ y = yx$ holds for all $x, y \in G$ and in this context e.g., the identities (1) and (6) in the above lemma are dual. So we have

LEMMA 3.2. *The following identities of the preceding lemma are dual*: (1) *with* (6), (2) *with* (5), (3) *with* (7), (4) *with* (8). *We also have that in any groupoid the identities* $(xy)x = y$, $x(yx) = y$ *are equivalent and such a groupoid* $(G,\cdot)$ *(satisfying* $(xy)x = y$*) is a quasigroup.*

Note that as show the examples of [3] the identities $(xy)x = x$ and $x(yx) = x$ are not equivalent.

According to Lemma 3.2 we shall further consider the following single identities of rank 3, namely: $xy^2 = x$, $xy^2 = y$, $(xy)x = x$, $(xy)x = y$, $(xy)(yx) = x$ and $(xy)(yx) = y$. It is now easy to see that the proof of Theorem from Introduction follows from (i), (iii) of Theorem 2.3 and Lemma 3.2.

The variety **R** was considered by B. Roszkowska in her Ph.D thesis (see [38]). She described the lattice of all subvarieties of the variety **R**. Some of subvarieties of **RR** and the variety itself were considered by many authors e.g., [27, 32, 35, 36, 37]. In [35] and [36] a characterization of groupoids from the variety **RR** is given. The most well-known variety is the variety $\Delta$ of all diagonal semigroups. It appears in many papers and books e.g., in [14, 20, 26, 29, 41, 42].

The identity $(xy)x = y$ can be found e.g., in [5, 22, 25 and 40].

In [14] T. Evans investigated the variety of all groupoids $(G,\cdot)$ defined by a single identity $(xy)(yz) = y$. Obviously any such a groupoid satisfies $(xy)(yx) = y$ but Evans groupoids are nonmedial and also nonidempotent. A little is known about the varieties $\mathbf{E}$ and $\nabla$ (see Section 5).

## 4. Medial idempotent groupoids with $xy^2 = x$

In this section we deal with the variety $\mathbf{R}$. We start from

THEOREM 4.1. *Let $(G,\cdot)$ be a medial idempotent groupoid satisfying $xy^2 = x$. Then we have*

(i) $p_2(G,\cdot) = 1$ *iff $(G,\cdot)$ is a nontrivial affine space over $GF(3)$.*
(ii) $p_2(G,\cdot) = 2$ *iff $(G,\cdot)$ is a proper groupoid satisfying $x(yz) = xy$ and $(xy)z = (xz)y$.*
(iii) $p_2(G,\cdot) = 3$ *iff $(G,\cdot)$ is a nontrivial affine space over $GF(5)$.*
(iv) *There exist groupoids $(G,\cdot)$ with $p_2(G,\cdot) = 4$.*
(v) $p_2(G,\cdot) = 5$ *iff $(G,\cdot)$ is a nontrivial affine space over $GF(7)$.*

Proof. The condition (i) follows from (i) of Theorem 2.1.

(ii) If $(G,\cdot)$ is a proper groupoid satisfying $x(yz) = xy$ and $(xy)z = (xz)y$, then one can check that $p_2(G,\cdot) = 2$. To prove the converse we consider the polynomial $x(xy)$. Using $xy^2 = x$ we see that $(G,\cdot)$ is right cancellative and hence $x(xy) \neq xy$. If $x(xy) = yx$, then $(yx)y = (x(xy))y = (xy)((xy)y) = (xy)x$ which proves that $(xy)x$ is commutative and hence $p_2(G,\cdot) \geq 3$, a contradiction. If $x(xy) = y$, then $y(xy) = x(xy)^2 = x$ and hence $yx = (x(xy))x = xy$ which proves that $(G,\cdot)$ is a Steiner quasigroup which contradicts $p_2(G,\cdot) = 2$. If $x(xy) = x$, then we apply the dual version of (iii) of Theorem 2.3 to get our requirement.

(iii) First observe that an affine space $(G,\cdot)$ over $GF(5)$ satisfies $xy^2 = x$. Indeed, we have $(G,\cdot) = (G, 3x+3y) = (G, 4x+2y)$, where $(G,+)$ is an abelian group of exponent 5. We see that the groupoid $(G, 4x+2y)$ is a medial idempotent groupoid satisfying $xy^2 = x$. Further the proof follows from (iii) of Theorem 2.1 and the fact that $xy^2 = x$ is a nonregular identity but Płonka's sums preserve only regular identities (see [28]).

(iv) Let $G_1$ be a nontrivial affine space over $GF(3)$ and $G_2$ a non–one–element semigroup with $xy = x$. Take $G = G_1 \times G_2$. Then the groupoid $(G,\cdot)$ is medial, idempotent, satisfying $xy^2 = x$, $x(yz) = x(zy)$, $(xy)x = x(xy)$, $x(x(xy)) = xy$ and consequently $p_2(G,\cdot) = 4$.

(v) Observe that if $(G,\cdot)$ is an affine space over $GF(7)$, then $(G,\cdot) = (G, 4x+4y) = (G, 5x+3y) = (G, 6x+2y)$, where $(G,+)$ is an abelian group of exponent 7. The last groupoid, i.e., $(G, 6x+2y)$ satisfies $xy^2 = x$ and obviously $p_2(G,\cdot) = 5$. Further, the proof follows from (iv) of Theorem 2.1

and we use the same arguments as in the proof of (iii) which completes the proof of the theorem.

THEOREM 4.2. *Let $(G,\cdot)$ be a groupoid. Then*

(i) *If $(G,\cdot) \in Q_{m,n}$, then $(G,\cdot)$ is a quasigroup.*

(ii) *If $(G,\cdot) \in Q_{2,n}$ with $2 \le n$, then the polynomial $x+y = {}^{n-1}yx$ is commutative, $x+(n-1)y = yx$, $x+ny = x$ and consequently $(G,+)$ and $(G,\cdot)$ are polynomially equivalent. If additionally $(G,\cdot)$ is medial, then also the groupoid $(G,+)$ is medial.*

(iii) *Let $(G,\cdot)$ be a medial idempotent groupoid. Then $(G,\cdot) \in Q_{2,n}$ for some $n \ge 2$ iff $(G,\cdot)$ is an affine module over $\mathbf{Z}_d$ with $d \,|\, 2^n - 1$.*

(iv) *If $(G,\cdot)$ is an idempotent groupoid satisfying $xy^2 = x$ which is left-sided cancellative, then either $(G,\cdot) \in Q_{2,n}$ for some $n \ge 2$ or $(G,\cdot)$ is polynomially infinite, i.e., $p_m(G,\cdot)$ is infinite for all $m \ge 2$.*

Proof. (i) and the first part of (ii) follow from Theorem 2.3. To complete the proof of (ii) we assume that $(G,\cdot)$ satisfies $y(yx) = x$ and $xy^n = x$ for some $n \ge 2$. (We use the dual version). Suppose that $(G,\cdot)$ is medial. Then we prove that $x+y = xy^{n-1}$ is also medial. Using the simple induction we show that $(xy^k)(uv^k) = (xu)(yv)^k$ holds in any medial groupoid $(G,\cdot)$ for all $k$. Using this identity we have to compute out the expression $(x+y)+(u+v) = (xy^{n-1})(uv^{n-1})^{n-1} = (((xy^{n-1})(uv^{n-1}))(uv^{n-1})^{n-2} = (((xu)(yv^{n-1}))(uv^{n-1}))(uv^{n-1})^{n-3} = ((xu^2)(yv^2)^{n-1})(uv^{n-1})^{n-3} = \ldots = ((xu^k)(yv^k)^{n-1})(uv^{n-1})^{n-1-k} = \ldots = (((xu^{n-2})(yv^{n-2})^{n-1})(uv^{n-1}) = (xu^{n-1})(yv^{n-1})^{n-1} = (x+u)+(y+v)$, as required.

(iii) Let now $(G,\cdot)$ be a medial idempotent groupoid such that $(G,\cdot) \in Q_{2,n}$, where $n \ge 2$. Using (i) the groupoid $(G,\cdot)$ is polynomially equivalent with a medial commutative idempotent groupoid $(G,+)$ satisfying $x+ny = x$. Now the proof follows from Theorem 2.2 (see (a) of (i)). Suppose that $(G,\cdot)$ is an affine module over $\mathbf{Z}_d$, where $d$ is a divisor of $2^n-1$. Then we have $(G,\cdot) = (G, \frac{d+1}{2}(x+y))$, where $(G,+)$ is an abelian group of exponent $d$. (for details see [30]). Take $(G,\circ)$, where $x \circ y = (d-1)x + 2y$. We get $(x \circ y) \circ y = (d-1)((d-1)x+2y)+2y = x$. Obviously $(G,\circ)$ is medial and idempotent. Further we have $y \circ (y \circ x) = 3(d-1)y + 2^2x$, ${}^3y \circ x = (2^3-1)(d-1)y + 2^3x, \ldots, {}^ky \circ x = (2^k-1)(d-1)y + 2^kx$, ${}^{n-1}y \circ x = (2^{n-1}-1)(d-1)y + 2^{n-1}x = 2^{n-1}x + (1-2^{n-1})y$ and ${}^ny \circ x = (2^n-1)(d-1)y + 2^nx$. Since $2^n \equiv 1 \pmod{d}$ we get ${}^ny \circ x = x$. By the same reason we get ${}^{n-1}y \circ x = \frac{d+1}{2}(x+y)$. Thus the groupoid $(G,\cdot) = (G,\circ)$ is in $Q_{2,n}$ which completes the proof of condition (iii).

(iv) Take the polynomial ${}^kyx$ and consider the mapping $k \to {}^kyx$. If this mapping is one–to–one, then the polynomials ${}^kyx$ are essentially binary for all $k$ and by the main result of [19] we infer that $(G,\cdot)$ is polynomially infinite which contradicts the assumption. If ${}^a yx = {}^b yx$ for some $a, b$ such that $a > b$, then using the left-hand-side cancellation law we deduce that there exists an $n$ such that ${}^n yx = x$ which completes the proof of the theorem.

THEOREM 4.3. *Let $(G,\cdot) \in Q_{2,n}$ for some $n \geq 2$. Then the clone of $(G,\cdot)$ is minimal if and only if $(G,\cdot)$ is a nontrivial affine space over $GF(p)$, where $p$ is a prime divisor of $2^n - 1$.*

Proof. The fact that the clone of a nontrivial affine space over $GF(p)$ is minimal one can find e.g., in [31]. Let $(G,\cdot) \in Q_{2,n}$. Applying (ii) of the preceding theorem we see that $(G,\cdot) = (G,+)$, where $(G,+)$ is a commutative (and clearly idempotent by the minimality of the clone) groupoid satisfying $x + ny = x$. Applying (iii) of Theorem 2.2 we get our requirement, completing the proof.

For more information about minimal clones see in [2, 10, 13, 23, 24 and 39].

## 5. On the identity $(xy)x = y$

In this section we deal with groupoids $(G,\cdot)$ satisfying $(xy)x = y$. We start with easy to prove

LEMMA 5.1. *If $(G,\cdot)$ is a distributive cancellative groupoid, then $(G,\cdot)$ is idempotent.*

LEMMA 5.2. *Let $(G,\cdot)$ be a groupoid satisfying $(xy)x = y$. Then we have*

(i) *$(G,\cdot)$ satisfies also $x(yx) = y$ and the identities $(xy)x = y$, $x(yx) = y$ are equivalent.*
(ii) *$(G,\cdot)$ is a quasigroup and therefore $(G,\cdot)$ is cancellative.*
(iii) *For each $n \geq 1$ we have $p_n(G,\cdot) \leq p_{n+1}((G,\cdot)$.*
(iv) *If $(G,\cdot)$ is distributive, then $(G,\cdot)$ is in the variety $Q_{6,6}$.*

Proof. We prove only (iv). We have $(xy)(yx) = ((xy)y)((xy)x) = xy^3$ and hence $xy^4 = ((xy)(yx))y = ((xy)y)x = y(yx)$, $xy^5 = yx$ and $xy^6 = x$. Analogously we show that ${}^6yx = x$ and therefore $(G,\cdot) \in Q_{6,6}$, completing the proof.

LEMMA 5.3. *Let $(G,\cdot)$ be a medial idempotent groupoid satisfying $(xy)x = y$. Then $(G,\cdot)$ is an affine space over $GF(7)$ iff $(G,\cdot)$ satisfies $xy^2 = yx^2$ (or dually ${}^2yx = {}^2xy$).*

Proof. If $(G,\cdot)$ is an affine space over $GF(7)$, then $(G,\cdot) = (G, 4x + 4y) = (G, 5x + 2y) = (G, 6x + 2y)$, where $(G,+)$ is an abelian group of exponent 7. We see that the polynomial $xy = 5x + 3y$ satisfies $xy^2 = yx^2$, $(xy)x = y$ and obviously $(G, xy)$ is medial and idempotent. Let now $xy^2 = yx^2$ and put $x + y = xy^2$. Using (iv) of Lemma 5.2 we see that $x + 3y = x$ and also $(x+2y)+x=(xy^4)x^2=(y(yx))x^2=((yx)(yx^2))x=((yx)(xy^2))x=((y(xy))(xy))x=(x(xy))x=xy$ and hence $(x+2y)+x=xy$ which proves that $(G,\cdot)$ and $(G,+)$ are polynomially equivalent. It is easy to check that $(G,+)$ is medial. Applying (a) of (ii) of Theorem 2.2 we deduce that $(G,\cdot)$ is an affine space over $GF(7)$, completing the proof of the lemma.

THEOREM 5.4. *Let $(G,\cdot)$ be a medial idempotent groupoid satisfying $(xy)x=y$. Then we have*

(i) $\operatorname{card} G=1$ *iff one of the following conditions holds*: $p_2(G,\cdot)=0$ *or $xy$ is a projection or also $xy^2\in\{y,xy\}$.*
(ii) *The following are equivalent*:
(a) *$(G,\cdot)$ a nontrivial affine space over $GF(3)$*;
(b) $p_2(G,\cdot)=1$;
(c) *$xy^2=x$ and card$G>1$.*
(iii) *The conditions are equivalent*;
(a) *$(G,\cdot)$ is a nontrivial affine space over $GF(4)$*;
(b) $p_2(G,\cdot)=2$.
(c) *$(G,\cdot)$ satisfies $xy^2=yx$ and card$G>1$.*
(iv) *There are no groupoids $(G,\cdot)$ with $p_2(G,\cdot)\in\{3,4\}$.*
(v) *The following conditions are equivalent*:
(a) *$(G,\cdot)$ is a nontrivial affine space over $GF(7)$*;
(b) $p_2(G,\cdot)=5$;
(c) *$(G,\cdot)$ satisfies $xy^2=yx^2$ and card$G>1$.*

Proof. (i) If $\operatorname{card} G=1$, then obviously each of the conditions is fulfiled. If $p_2(G,\cdot)=0$, then $xy$ is a projection and hence $xy^2=xy$. If again $(G,\cdot)$ satisfies $xy^2=xy$ or $xy^2=y$, then by (ii) of Lemma 5.2 we get $x=y$.

(ii) (a)⇒(b) is obvious. If $p_2(G,\cdot)=1$, then $(G,\cdot)$ is commutative and hence $y=(xy)x=yx^2$. Thus we get (b)⇒(c). We prove (c)⇒(a). If $xy^2=x$, then using $y=(xy)x$ we get $yx=(xy)x^2=xy$ and therefore $(G,\cdot)$ is a medial commutative idempotent groupoid satisfying $xy^2=x$. Using (a) of (ii) of Theorem 2.2 we infer that $(G,\cdot)$ is an affine space over $GF(3)$.

(iii) First we prove (a)⇒(b). If $(G,\cdot)$ is an affine space over $GF(4)$, then $(G,\cdot)=(G,ax+by)$ where $G$ is a vector space over a four-element field $K=\{0,1,a,b\}$. If $\operatorname{card} G>1$, then it is easy to check that $(G,\cdot)=2$.

(b)⇒(c). If $(G,\cdot)$ is a medial idempotent groupoid satisfying $(xy)x=y$, then using (ii) of Lemma 5.2 we see that $xy^2\notin\{y,yx\}$. By (c) of (ii) (see above) we get $xy^2\neq x$. Since $p_2(G,\cdot)=2$ we infer that $xy^2=yx$, as required.

(c)⇒(a). If (c) holds, then one shows that $xy, yx$ are the only essentially binary polynomials over $(G,\cdot)$. Applying (ii) of Theorem 2.3 we get (a).

(iv) Let $p_2(G,\cdot) = 3$. Using (iii) of Theorem 2.1 we infer that $(G,\cdot)$ is either a nontrivial Płonka sum of affine spaces over $GF(3)$ being not all one–element or a nontrivial affine space over $GF(5)$. Since $(G,\cdot)$ satisfies $(xy)x = y$ we infer that the first case cannot happen (Płonka's sums preserve only regular identities, see [28]). If $(G,\cdot)$ is an affine space over $GF(5)$, then $(G,\cdot) = (G, 3x + 3y) = (G, 4x + 2y)$, where $(G,+)$ is an abelian group of exponent 5. We see that none of the polynomials $3x + 3y$, $4x + 2y$ satisfies the identity $(xy)x = y$.

Now we prove that there is no medial idempotent groupoid $(G,\cdot)$ satisfying $(xy)x = y$ and $p_2(G,\cdot) = 4$. Assume that such a groupoid $(G,\cdot)$ there exists. Consider $xy^2$. By (i), (c) of (ii) and (c) of (iii) of this theorem we infer that $xy^2 \notin \{x, y, xy, yx\}$. If $xy^2 = yx^2$, then applying Lemma 5.3 we see that $(G,\cdot)$ is an affine space over $GF(7)$. It is clear that in this case $p_2(G,\cdot) \neq 4$. Since in a nontrivial affine space over $GF(7)$ the polynomials $2x + 6y$, $2y + 6x$, $3x + 5y$, $3y + 5x$ and $4x + 4y$ are the only essentially binary polynomials over $(G,\cdot) = (G, 4x + 4y)$ (see e.g., in [1]). Thus further we may assume that the polynomials $xy$, $yx$, $xy^2$, $yx^2$ are the only essentially binary polynomials in $(G,\cdot)$. By the same arguments as above (using the dual versions of the results) we infer that $xy$, $yx$, ${}^2xy$ and ${}^2yx$ are essentially binary and pairwise distinct. Assume that $(G,\cdot)$ satisfies $(xy)y = y(yx)$. Then we obtain $(xy)z = (xz)(yz) = ((xz)y)((xz)z) = ((xz)y)(z(zx)) = ((xz)z)(y(zx)) = (z(zx))(y(zx)) = (zy)(zx) = z(yx)$. Thus we get $(xy)z = z(yx)$ which proves that $(G,\cdot)$ is commutative, a contradiction. Suppose that $(xy)y = x(xy)$ holds. Then $xy = (x(xy))x = ((xy)y)x = ((xy)x)(yx) = y(yx)$, a contradiction (in this case, applying (c) of (iii) (the dual version) we see that $(G,\cdot)$ is an affine space over $GF(4)$ which contradicts the assumption $p_2(G,\cdot) = 4$), completing the proof of (iv).

(v) The implication (a)$\Rightarrow$(b) is obvious.

(b)$\Rightarrow$(c). Since $p_2(G,\cdot) = 5$ and $(G,\cdot)$ satisfies a nonregular identity $(xy)x = y$ we infer, applying (iv) of Theorem 2.1 that $(G,\cdot)$ is a nontrivial affine space over $GF(7)$ and therefore $(G,\cdot)$ satisfies the condition (c) (see the begining of the proof of Lemma 5.3).

(c)$\Rightarrow$(a) follows from Lemma 5.3. This completes the proof of the theorem.

We also have

THEOREM 5.5. *Affine spaces over $GF(q)$ with $q \in \{3, 4, 7\}$ are medial idempotent groupoids $(G,\cdot)$ satisfying the identity $(xy)x = y$. Moreover there exist medial idempotent groupoids $(G,\cdot)$ satisfying $(xy)x = y$ which are not such affine spaces.*

Proof. The first part easily follws from the preceding theorem. To prove the last statement we take two groupoids $G_1 = (\{0,1,2\}, 2x \overset{+}{_3} 2y)$, $G_2 = (\{0,1,a,b\},\cdot)$, where $xy = ax + by$ and $K = \{0,1,a,b\}$ is a four–element field. Then we check that the Cartesian product $G = G_1 \times G_2$ is a proper medial noncommutative idempotent groupoid satisfying $(xy)x = y$, $xy^k$ are essentially binary for $k = 1, \dots, 5$, $xy^2$ is noncommutative and different from $xy$, $yx$ (see the preceding theorem), completing the proof.

Further let $(G,\cdot)$ be a medial idempotent groupoid. We define a binary relations $q$ in $(G,\cdot)$ as follows: if $a, b \in G$, then $aqb \Leftrightarrow (ab)a = b \not\subset (ba)b = a$. Then we have

THEOREM 5.6. *If $(G,\cdot)$ is a medial idempotent groupoid, then the relation $q$ in $(G,\cdot)$ is a congruence relation and consequently $G = \bigcup_{t\in T} G_t$, where $G_t$ is a subgroupoid of $G$ which $t \in T$ is a quasigroup and $G_t \cap G_{t'} = \emptyset$ for $t \neq t'$ $(t, t' \in T)$.*

Proof. If $a, b, c \in G$ and $aqb$, $bqc$, then we have $(ac)a = (ac)((ba)b) = (a(ba))(cb) = b(cb) = c$. Analogously we get $(ca)c = a$. If $a_1qb_1$ and $a_2qb_2$, then $((a_1a_2)(b_1b_2))(a_1a_2) = ((a_1b_1)(a_2b_2))(a_1a_2) = ((a_1b_1)a_1)((a_2b_2)a_2) = b_1b_2$. Similarly we prove $((b_1b_2)(a_1a_2))(b_1b_2) = a_1a_2$. Further the proof follows from Lemma 5.2.

Note that we have also a similar result for medial commutative idempotent groupoids, namely we have.

THEOREM 5.7. *If $(G,\cdot)$ is a medial commutative idempotent groupoid, then for any positive integer the relation $q_n$ defined as follows: if $a, b \in G$, then $aq_nb \Longleftrightarrow ab^n = a \not\subset ba^n = b$ is a congruance relation in $(G,\cdot)$ and then $G = \bigcup_{t\in T} G_t$, where $G_t \cap G_{t'} = \emptyset$ for $t \neq t'$ and each subgroupoid $G_t$ $(t \in T)$ is an affine module over $\mathbf{Z}_d$ with $d$ dividing the number $2^n - 1$.*

Proof. The second fact follows from (a) of (ii) of Theorem 2.2. To get the first statement we use repeatedly the medial and the distributive laws and we omit the proof.

## 6. On the varieties RR, $\Delta$, $\nabla$ and E

In this section we deal with the remaining varieties appearing in Theorem of Section 1. First we prove

THEOREM 6.1. *Let $(G,\cdot)$ be an idempotent groupoid. Then we have*

(a) *The following are equivalent:*
(i) *$(G,\cdot)$ satisfies $(xy)z = yz$ and $x(yz) = y(xz)$ i.e., $(G,\cdot) \in$ **RR**;*
(ii) *$(G,\cdot)$ is medial and $xy^2 = y$;*
(iii) *$(G,\cdot)$ is distributive and $xy^2 = y$*

(b) *If* $(G,\cdot) \in \mathbf{RR}$, *then* $p_2(G,\cdot)$ *is even or infinite.*

(c) *Let* $(G,\cdot)$ *be in* $\mathbf{RR}$. *Then* $p_2(G,\cdot) = 4$ *iff* $(G,\cdot)$ *satisfies either* $x(x(xy)) = y$ *or* $x(x(xy)) = xy$ *or* $x(x(xy)) = x(xy)$ *(and in each case the polynomial* $x(xy)$ *is essentially binary and different from* $xy$*).*

Proof. (a) follows from (iii) of Theorem 2.3.

(b) If $(G,\cdot) \in \mathbf{RR}$, then using the identities of the variety $\mathbf{RR}$ we see that

$$\{x, y, xy, yx, x(xy), y(yx), \ldots, {}^k xy, {}^k yx, \ldots\}$$

is the set $\mathbf{A}^{(2)}(G,\cdot)$ of all binary polynomials over $(G,\cdot)$. So every binary polynomial into two variables $x, y$ is of the form ${}^k xy$ and it depends on $y$ (if $\operatorname{card} G > 1$). Using the identity $(xy)z = yz$ we infer that none of the polynomials ${}^k xy$ is commutative and therefore if $p_2(G,\cdot)$ is finite, then $p_2(G,\cdot)$ is even.

(c) Let $(G,\cdot) \in \mathbf{RR}$ and $p_2(G,\cdot) = 4$. Then $p_2(G,\cdot) = 4$ implies that $x(xy)$ is essentially binary, noncommutative and $x(xy)$ is different from $xy, yx$. Thus $xy, yx, x(xy)$ and $y(yx)$ are the only essentially binary polynomials over $(G,\cdot)$. Consider the polynomial $x(x(xy))$. Using $(xy)z = yz$ we infer that ${}^3 xy$ is different from $yx, y(yx)$ and depends on $y$ and therefore ${}^3 xy \in \{y, xy, x(xy)\}$. To prove the converse we use the formula of a description of the set $\mathbf{A}^{(2)}(G,\cdot)$ of all binary polynomials over $(G,\cdot)$, the fact that $x(xy)$ is essentially binary and different form $xy$, completing the proof.

Now we present some well-known facts on the variety $\Delta$. We have

THEOREM 6.2. *Let* $(G,\cdot)$ *be a proper idempotent groupoid. Then the following conditions are equivalent:*

(i) $(G,\cdot)$ *is a diagonal semigroup, i.e.,* $(G,\cdot)$ *is an idempotent semigroup satisfying* $xyz = xz$;

(ii) $G = A \times B$ *for some non-one-element sets* $A, B$, *where the fundamental operation on* $G$ *is defined as follows: if* $(a,b), (c,d) \in G$, *then* $(a,b)(c,d) = (a,d)$;

(iii) $(G,\cdot)$ *is a semigroup with* $xyx = x$;

(iv) $(G,\cdot)$ *is medial and* $(G,\cdot)$ *satisfies* $(xy)x = x$;

(v) $(G,\cdot)$ *is distributive and* $(G,\cdot)$ *satisfies* $(xy)x = x$;

(vi) *For some* $n \geq 3$, *the following polynomials* $(\ldots(x_1x_2)\ldots x_{n-1})x_n$ *and* $x_1(x_2(\ldots(x_{n-1}x_n)\ldots))$ *are not essentially* $n$*-ary*;

(vii) $p_m(G,\cdot) = 0$ *for some* $m \geq 3$;

(viii) $(G,\cdot)$ *is a noncommutative groupoid with* $p_k(G,\cdot) < k$ *for some* $k \geq 3$.

The proof of this theorem can be deduced from the results of [4,9,20] and Theorem 2.3 (i). See also [17].

Now we deal with medial idempotent groupoids $(G,\cdot)$ satisfying $(xy)(yx) = x$ i.e., we consider the variety $\nabla$. It is clear that any diagonal semigroup $(G,\cdot)$ is in $\nabla$ and therefore $\Delta \subset \nabla$. Further observe that any affine space over $GF(4)$ is also in $\nabla$. Indeed, if $(G,\cdot)$ is such an affine space, then $(G,\cdot)$ satisfies $(xy)x = y$ and $(xy)z = (zy)x$. Using these identities we get the medial law and $(xy)(yx) = ((yx)y)x = x$.

Now we give some examples from $\nabla$.

EXAMPLE 6.3. Let $(G,+)$ be an abelian group of exponent 4. Putting $xy = 2x + 3y$. Then we check that $(G,\cdot)$ is a medial idempotent groupoid satisfying $(xy)(yx) = x$.

EXAMPLE 6.4. Consider the variety $\Delta$ and the variety **Aff**(4) of all affine spaces over $GF(4)$ and take the direct product of these varieties, i.e., $\Delta \times$ **Aff** (4). According to the above remarks we see that $\Delta \times$ **Aff** (4) is a subclass of $\nabla$. Using Theorem 1 of [16] we infer that this class is a subvariety of $\nabla$. The required binary polynomial needed in [16] is simply $(xy)x$. In the first variety $\Delta$ we have $(xy)x = x$ and in **Aff** (4) we have $(xy)x = y$.

We have the following easy to prove

THEOREM 6.5. *Let $(G,\cdot) \in \nabla$. Then we have*

(i) *$(G,\cdot)$ is a diagonal semigroup iff $(G,\cdot)$ satisfies $(xy)x = x$.*
(ii) *$(G,\cdot)$ is affine space over $GF(4)$ iff $(xy)x = y$, i.e., $(G,\cdot) \in Q$.*

EXAMPLE 6.6. Let **C** be the complex field. Take $(\mathbf{C},\cdot)$, where $xy = \frac{x+y}{2} + \frac{x-y}{2}\sqrt{3}i$. Then the groupoid $(\mathbf{C},\cdot)$ is a medial idempotent groupoid satisfying $(xy)x = y$ i.e., $(G,\cdot) \in Q$, the polynomial $(xy)(yx)$ is essentially binary, noncommutative and different from $xy, yx$.

Now we give two examples of medial idempotent groupoids $(G,\cdot)$ satisfying the identity $(xy)(yx) = y$, i.e., $(G,\cdot) \in \mathbf{E}$.

EXAMPLE 6.7. Let $(G,+)$ be an abelian group of exponent 5. Take $(G,\cdot)$, where $xy = 4x + 2y$ i.e., $(G,\cdot)$ is an affine space over $GF(5)$. Then the groupoid $(G,\cdot)$ is medial, idempotent, $(G,\cdot)$ satisfies $(xy)(yx) = y$ and the polynomial $(xy)x$ is commutative.

This example shows that the variety **Aff** (5) of all affine spaces over $GF(5)$ is a subvariety of the variety **E**. The next example shows that **Aff** (5) is properly contained in **E**.

EXAMPLE 6.8. Let **C** be the complex field. Then the groupoid $(\mathbf{C},\cdot)$, where $xy = x + (x - y)i$ is a proper noncommutative medial idempotent groupoid satisfying $(xy)(yx) = y$ in which $(xy)x$ is essentially binary, noncommutative and different form $xy$ and $yx$.

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