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Lucyna Żurawska

THE INJECTIVE DIAGONALIZABLE ALGEBRAS

Dedicated to Professor Tadeusz Traczyk

The notion of a diagonalizable algebra was introduced by Magari ([5]). In that paper the author described some algebraic properties of these algebras.

The aim of this paper is to characterize all injective diagonalizable algebras.

An algebra $\mathbf{A} = \langle A, \vee, \wedge, \neg, \tau, 0, 1\rangle$ of type $\langle 2,2,1,1,0,0\rangle$ is called a *diagonalizable algebra*, if it satisfies the following axioms:

A1. $\langle A, \vee, \wedge, \neg, \tau, 0, 1\rangle$ is a Boolean algebra,

A2. $\tau(x \wedge y) = \tau(x) \wedge \tau(y)$,

A3. $\tau(\tau(x) \to x) = \tau(x)$, where the symbol $x \to y$ denotes $\neg x \vee y$,

A4. $\tau(1) = 1$.

If $\mathbf{A}$ is a diagonalizable algebra, we write $\mathbf{A} = \langle \underline{A}, \tau\rangle$, where $\underline{A}$ is a Boolean algebra.

Let $\underline{A}$ be a Boolean algebra. We can define τ on A by $\tau(x) = 1$ for each $x \in A$. Then $\langle A, \tau\rangle$ is a diagonalizable algebra. This algebras are call "trivial".

If $\mathbf{A} = \langle \underline{A}, \tau\rangle$ is a diagonalizable algebra such that $\underline{A}$ is a complete Boolean algebra, we say that $\mathbf{A}$ is "complete".

It is known (see [5]) that the following properties hold in every diagonalizable algebra:

W1. $\tau(x) \leq \tau(\tau(x))$,

W2. if $x \leq y$ then $\tau(x) \leq \tau(y)$,

W3. if $\tau(x)$ then $x = 1$,

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W4. $\tau(\neg\tau(0)) = \tau(0)$.

Let $\mathbb{K}$ be a class of algebras of the same type. An algebra $\mathbb{C}$ is *injective* in $\mathbb{K}$ if and only if, for each $\mathbb{A}, \mathbb{B} \in \mathbb{K}$ a monomorphism i from $\mathbb{A}$ into $\mathbb{B}$ and a homomorphism h from $\mathbb{A}$ into $\mathbb{C}$, there exists a homomorphism φ from $\mathbb{B}$ into $\mathbb{C}$ satisfying $\varphi \circ i = h$.

It is known ([4]) that a Boolean algebra $\underline{B}$ is injective in the class of Boolean algebra if and only if $\underline{B}$ is complete Boolean algebra (Sikorski's theorem).

Let the symbol $\mathbb{D}$ denotes the class of all diagonalizable algebras.

THEOREM 1. *Each diagonalizable algebra which is complete and trivial is an injective diagonalizable algebra in the class* $\mathbb{D}$.

Proof. Let $\mathbb{C} = \langle \underline{C}, \tau \rangle$, $\mathbb{C} \in \mathbb{D}$ be trivial and complete, $\mathbb{A} = \langle \underline{A}, \tau \rangle$, $\mathbb{B} = \langle \underline{B}, \tau \rangle$, $\mathbb{A}, \mathbb{B} \in \mathbb{D}$, i a monomorphism from $\mathbb{A}$ into $\mathbb{B}$, h a homomorphism from $\mathbb{A}$ into $\mathbb{C}$. Then $h(\tau(x)) = \tau(h(x)) = 1$ for each $x \in A$. An algebra $\mathbb{C}$ is complete, so by Sikorski's theorem $\underline{C}$ is injective in the class of Boolean algebras. Hence there exists a Boolean homomorphism φ from $\underline{B}$ into $\underline{C}$ such that $\varphi(i(x)) = h(x)$ for $x \in A$. We will show, that $\varphi(\tau(x)) = \tau(\varphi(x))$ for $x \in B$. An algebra $\mathbb{C}$ is trivial, so $\tau(\varphi(x)) = 1$ for $x \in B$ and $\varphi(\tau(x)) \geq \varphi(\tau(0)) = \varphi(1(\tau(0)) = h(\tau(0)) = \varphi(h(0)) = 1$. So $\varphi(\tau(x)) = 1$.

An algebra $\mathbb{C}$ is called a *retract* of an algebra $\mathbb{B}$ if and only if there exist a monomorphism f from $\mathbb{C}$ into $\mathbb{B}$ and a homomorphism g from $\mathbb{B}$ into $\mathbb{C}$ such that $g \circ f = \mathrm{id}|_C$.

We say that an algebra $\mathbb{C}$ is an *absolute subretract*, if $\mathbb{C}$ is a retract of every algebra including $\mathbb{C}$.

It is known:

P1. (see [4]) Each algebra, which is a retract of a complete Boolean algebra, is a complete algebra.

P2. (see [2]) For each diagonalizable algebra $\mathbb{C}$, there exists an embedding from $\mathbb{C}$ into a complete, atomic diagonalizable algebra.

It is easy to see, that:

P3. Each injective algebra in a class of algebras $\mathbb{K}$ is an absolute subretract.

P4. A retract of a complete diagonalizable algebra is a complete diagonalizable algebra.

The following theorem is a consequence of P1–P4.

THEOREM 2. *Every injective diagonalizable algebra is a complete diagonalizable algebra.*

THEOREM 3. *Every injective diagonalizable algebra is trivial.*

Proof. Let $\mathbb{C} = \langle \underline{C}, \tau \rangle$ be an injective diagonalizable algebra. Let us suppose, that $\mathbb{C}$ is not trivial, that means $\tau(0) \neq 1$.

We consider two cases:

I. $\tau^{n+1}(0) = 1$ for some $n \in N$, $n \geq 1$.

Let $\tau(0) = a_1, \tau^2(0) = a_2, \ldots, \tau^n(0) = a_n, \tau^{n+1}(0) = 1$.

Now, let $\underline{A}$ be a Boolean subalgebra od $\underline{C}$ generated by $\{a_1, \ldots, a_n\}$. It is obvious that $\mathrm{card}(A) = 2^{n+1}$. Let $a_n, b_1, \ldots, b_n$ are co-atoms of $\underline{A}$ such that $a_n \wedge b_1 = a_{n-1}, a_n \wedge b_1 \wedge b_2 = a_{n-2}, \ldots, a_n \wedge b_2 \wedge \ldots \wedge b_{n-1} = a_1, b_n = a_1'$.

We will show, that $\tau(x) \in A$ for $x \in A$. We have $\tau(a_n) = 1$, $\tau(a_{n-1}) = a_n$ and $a_n \wedge b_i = a_{n-i}$, so $\tau(b_1) = a_n$. Because $a_{n-2} = a_n \wedge b_1 \wedge b_2$ so $\tau(a_{n-2}) = \tau(a_n) \wedge \tau(b_1) \wedge \tau(b_2)$ and $a_{n-1} = a_n \wedge \tau(b_2)$. Therefore $\tau(b_2) \geq a_{n-1}$ and it is not true that $\tau(b_2) \geq a_n$. So $\tau(b_2) = a_{n-1}$ or $\tau(b_2) = b_1$. Let $\tau(b_2) = b_1$. Then $\tau(b_2) \leq \tau(\tau(b_2)) = \tau(b_1) = a_n$ it is impossible. Hence $\tau(b_2) = a_{n-1}$ and if $a_{n-2} \leq x \leq b_2$ then $\tau(x) = a_{n-1}$ for $x \in A$. If we suppose for $k \leq n - 2\tau$ $(b_k) = \tau(a_{n-k}) = a_{n-k+1}$, we obtain $a_{n-k} = \tau(a_{n-k-1}) = \tau(b_{k+1}) \wedge a_{n-k+1}$ so $\tau(b_{k+1}) \geq a_{n-k}$ and it is not true, that $\tau(b_{k+1}) \geq a_{n-k+1}$ and $\tau(b_{k+1}) \leq \tau(\tau(b_{k+1})) = a_{n-k+1}, \tau(b_{k+1}) = a_{n-k}$.

Hence for each $1 \leq k \leq n, \tau(b_k) = a_{n-k+1}$.

We proved, that:

1°. $\langle \underline{A}, \tau \rangle$ is a subalgebra of $\mathbb{C}$.

2°. τ on A satisfies the following condition:

$$\tau(x) = \begin{cases} a_i & \text{if } x \geq a_{i-1} \text{ and } x \not\geq a_i \text{ for } i = 1, \ldots, n+1, \\ 1 & \text{if } x = 1. \end{cases}$$

Now, we construct Boolean algebra $\underline{B} = 2^{\underline{C}}$. In $\underline{B}$ we take the chain $\L = \{a_0 = 0, a_1, \ldots, a_n, a_{n+1} = 1\}$, where a_n is a co-atom in $\underline{B}$. On B we describe operation τ by the following condition:

$$\tau(x) = \begin{cases} a_i & \text{if } x \geq a_{i-1} \text{ and } x \not\geq a_i \text{ for } i = 1, \ldots, n+1 \\ 1 & \text{if } x = 1. \end{cases}$$

It is easy to prove, that $\mathbb{B} = \langle \underline{B}, \tau \rangle$ is a diagonalizable algebra. Let $[\L]$ denotes Boolean subalgebra of $\underline{B}$ generated by Ł. It is obvious that $\langle [\L], \tau \rangle$ is a subalgebra of $\langle \underline{B}, \tau \rangle$ and there is an isomorphism f from $\langle \underline{A}, \tau \rangle$ into $\langle [\L], \tau \rangle$ such that $f(a_i) = a_i$ for $i = 1, \ldots, n$. An algebra $\mathbb{C}$ is injective, so there exists a homomorphism φ from $\mathbb{B}$ into $\mathbb{C}$ such that $(*) \varphi \circ f = \mathrm{id}|_A$ and $(**)$ $\varphi(\tau(x)) = \tau(\varphi(x))$ for $x \in B$. We will prove, that $\ker \varphi = \{0\}$.

Suppose, that $x \in B - \{1\}$ and $\varphi(x) = 1$. Then $\tau(\varphi(x)) = 1$ and by $(**)$ $\tau(\varphi(x)) = \varphi(\tau(x)) = \varphi(f(\tau(x')))$ where $\tau(x) = f(\tau(x'))$ for some $x' \in A$. By $(*) \varphi(f(\tau(x')) = \tau(x')$, so $\tau(x') = 1$ and $\tau(x) = f(1) = 1$. Therefore $x = a_n$. But $a_n = f(a_n)$ and $1 = \varphi(x) = \varphi(a_n) = \varphi(f(a_n)) = a_n$. It is a contradiction. So $\{x \in B : \varphi(x) = 1\} = \{1\}$ and $\ker \varphi = \{0\}$. We obtained that φ is a monomorphism from $\mathbb{B}$ into $\mathbb{C}$ and $\mathrm{card}(B) > \mathrm{card}(C)$. What is impossible.

II. Let $\tau^n(0) \neq 1$ for each $n \in N$.

Now, let $\mathbb{A} = \langle \underline{A}, \tau \rangle$ be the subalgebra of $\underline{C}$ generated by $\{\tau(0)\}$. It is know (see [1], [5]) that each element $x \in A$ can be written by the following form:

$$x = (a_{i1} \wedge \neg a_{ji}) \vee \ldots \vee (a_{ik} \wedge \neg a_{jk}) \text{ or } x = (a_{si} \vee \neg a_{ki}) \wedge \ldots \wedge (a_{sl} \vee \neg a_{kl})$$
$$\text{and } \tau((a_{i1} \wedge \neg a_{j1}) \vee \ldots \vee (a_{ik} \wedge \neg a_{jk})) = \tau(0)$$
$$\tau((a_{si} \vee \neg a_{ki}) \wedge \ldots \wedge (a_{sl} \vee \neg a_{kl})) = a_{n+1},$$

where $a_n = \min\{a_{si}, \ldots, a_{sl}\}$.

It is easy to see, that τ satisfies condition:

$$\tau(x) = \begin{cases} a_i & \text{if } x \geq a_{i-1} \text{ and } x \ngeq a_i \text{ for } i \in N, \\ i & \text{if } x = 1. \end{cases}$$

We construct a Boolean algebra $\underline{B}$ such that $card(B) > \text{card}(C)$ and $\underline{A}$ is a subalgebra of $\underline{B}$. Such an algebra exists. It suffices to take a Boolean product of algebras $\underline{A}$ and $2^{\underline{C}}$. The operation τ define as follows:

$$\tau(x) = \begin{cases} a_i & \text{if } x \geq a_{i-1} \text{ and } x \ngeq a_i \text{ for } i \in N, \\ 1 & \text{if } x = 1. \end{cases}$$

Of course $\mathbb{B} = \langle \underline{B}, \tau \rangle$ is a diagonalizable algebra and analogously to I. we obtain, that there is not a homomorphism φ from $\mathbb{B}$ into $\mathbb{C}$ such that $\varphi|_A = \text{id}$.

COROLLARY 1. *An algebra $\mathbb{C}$ is injective in the class of all diagonalizable algebras if and only if $\mathbb{C}$ is a complete and trivial.*

We say, that the class $\mathbb{K}$ of algebras of the same type is *enough injective*, if every algebra of $\mathbb{K}$ can be embedded into an injective algebra in $\mathbb{K}$.

COROLLARY 2. *The class of diagonalizable algebras is not enough injective.*

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INSTITUTE OF MATHEMATICS, PEDAGOGICAL UNIVERSITY
Konopnickiej 21,
25-406 KIELCE, POLAND

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