DEMONSTRATIO MATHEMATICA
Vol. XXVII No 3–4 1994

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THE GROUP OPERATIONS ON A GROUP WITH A FREE SUBGROUP

Dedicated to Professor Tadeusz Traczyk

A binary operation of the form $a \circ b = a^{\alpha_1} b^{\beta_1} a^{\alpha_2} b^{\beta_2} \cdots a^{\alpha_n} b^{\beta_n}$ on a group G is called a group operation iff a pair $(G, \circ)$ is a group and $a \cdot b = a_\circ^{\gamma_1} \circ b_\circ^{\delta_1} \circ a_\circ^{\gamma_2} \circ b_\circ^{\delta_2} \circ \cdots \circ a_\circ^{\gamma_n} \circ b_\circ^{\delta_n}$ for some integers $\gamma_1, \gamma_2, \ldots, \gamma_n, \delta_1, \delta_2, \ldots, \delta_n$. If there exists a free not cyclic subgroup of the group G then every group operation $a \circ b$ has one of the forms: $a \circ b = a \cdot b$ or $a \circ b = b \cdot a$.

Let G be a group. Let us consider binary operations $\circ : G \times G \longrightarrow G$ of the form

$$a \circ b = a^{\alpha_1} b^{\beta_1} a^{\alpha_2} b^{\beta_2} \cdots a^{\alpha_n} b^{\beta_n},$$

where $\alpha_1, \alpha_2, \ldots, \alpha_n, \beta_1, \beta_2, \ldots, \beta_n$ are any integers. We call it a group operation iff the following two conditions are satisfied:

1. The pair $G_\circ = (G, \circ)$ is a group.
2. The equation:

$$a \cdot b = a_\circ^{\gamma_1} \circ b_\circ^{\delta_1} \circ a_\circ^{\gamma_2} \circ b_\circ^{\delta_2} \circ \cdots \circ a_\circ^{\gamma_n} \circ b_\circ^{\delta_n},$$

   holds for some integers $\gamma_1, \gamma_2, \ldots, \gamma_n, \delta_1, \delta_2, \ldots, \delta_n$, where $a_\circ^{\gamma}$ means the power of a in sense of the group $G_\circ$.

(see [1], [5], [2]).

On each group there are at least two group operations: $a \circ b = a \cdot b$ and $a \circ b = b \cdot a$. These group operations are called trivial (see [2]).

THEOREM 1. *Let G be a group. If there is a free subgroup $F < G$ which is not cyclic, then all group operations on G are trivial.*

This paper has been presented at the Conference on Universal Algebra and its Applications, organized by the Institute of Mathematics of Warsaw University of Technology held at Jachranka, Poland, 8–13 June 1993.

Proof. Let $a \circ b = a^{\alpha_1} b^{\beta_1} a^{\alpha_2} b^{\beta_2} \cdots a^{\alpha_n} b^{\beta_n}$, be a group operation on a group G. It is clear that the operation $\circ$, restricted to the subgroup F, is a group operation on the group F.

It was proved by Hanna Neumann [4] that there are only trivial group operations on a free group. So, for $a, b \in F$ we have $a \circ b = a \cdot b$ or $a \circ b = b \cdot a$. Because the group F is not cyclic, we can put as a and b the different free generators of the group F and hence the equality is the equality of words. But this means that such equality is true on each group, especialy on the group G and so the theorem follows. ∎

The group $G = Sl(n, Z)$ of matrices with determinant equal to 1 gives an example of such a group containig a free noncyclic subgroup (see [3]).

References

[1] A. Goetz, *On weak isomorphisms and weak homomorphisms of abstract algebras*, Colloq. Math. 14 (1966), 163–167.

[2] A. Hulanicki, S. Świerczkowski, *On group operarions other then xy or yx*, Publ. Math. Debrec. 9 (1962), 142–148.

[3] M. Kargapaow, Ju. Merzliakow, *Osnowy tieori grupp*,m Moskwa 1982.

[4] H. Neumann, *On a question of Kertész*, Publ. Math. Debrecen 8 (1961), 75–78.

[5] A. P. Street, *Subgroup — determining functions on groups*, Illinois J. Math. 12 (1968), 99–120.

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Received November 12, 1993.