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ON THE FORM OF FIXED POINTS FOR INVOLUTIONS IN A FREE ABELIAN GROUP

Dedicated to Professor Tadeusz Traczyk

If α is an automorphism of order two in an abelian group then each element of the form uu^α is a fixed point for α. The question is for which α each fixed point is of the form uu^α. There are examples of relatively free groups of rank two where the automorphism σ permuting generators has this property. We describe the automorphisms of order two with all fixed points of the form uu^α in a free abelian group of rank two.

Let F be a relatively free group of rank two generated by x, y, and σ be the automorphism of F, permuting the generators. If F is abelian, $w = x^s y^t$ then $w = w^\sigma$ implies $s = t$ and $w = (x^s)(x^s)^\sigma$, so each fixed point of σ has a form uu^σ, and hence the group of fixed points is cyclic, generated by xx^σ. It can be deduced from [5] that in a free two-nilpotent group of rank two the group of fixed points for σ is also cyclic, generated by $(xy)(xy)^\sigma$. In a free metabelian group of rank two the group of fixed points for σ also consists of elements uu^σ for all u in a commutator subgroup [3], and is infinitely generated, which also shows that the Gersten's Theorem, which says that in a free group of a finite rank each automorphism has a finitely generated group of fixed points [1], can not be extended to a two-generated free metabelian group. We describe here automorphisms α of order two in a free abelian group A generated by x and y, having all fixed points of the form uu^α. This property of fixed points is (or is not) satisfied for all conjugate automorphisms simultaneously. We show that there are three conjugacy classes for automorphisms of order two in A only one of which

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has all fixed points of the form uu^α (though in the free group of rank two there are four conjugacy classes of automorphisms of order two [2], none of which has all fixed points of the form uu^α).

Each automorphism α in A is given by $x^\alpha = x^k y^l$, $y^\alpha = x^m y^n$, and with respect to x, y, α is defined by the matrix $M = \begin{bmatrix} k & l \\ m & n \end{bmatrix}$. We assume that $\alpha^2 = id \neq \alpha$ then $M^2 = I$, and hence $k^2 = n^2 = 1 - ml$, $(k+n)l = (k+n)m = 0$. If $m = l = 0$ we get two not conjugate automorphisms of order two, $-id$ and δ defined by the matrix $D = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. Obviously, $-id$ has no fixed points ($\neq 1$) and δ has a cyclic groups of fixed points generated by x where only x^{2k} are of the form uu^α.

LEMMA 1. *In a free abelian group generated by x, y, an automorphism $\alpha\,(\alpha^2 = id \neq \alpha)$ is conjugate to either $-id$, or δ, or σ, respectively defined by matrices* $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, *and* $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

Proof. We note first that the matrix of σ is conjugate to that of δ over $\mathbf{Z}[1/2]$ by $C = \begin{bmatrix} 1 & 1 \\ -1/2 & 1/2 \end{bmatrix}$, but not over $\mathbf{Z}$. Let now $\alpha \neq -id, \delta$, then m, l are not both zeros, $n = -k$ and α is defined by $M = \begin{bmatrix} k & l \\ m & -k \end{bmatrix}$, $k^2 = 1 - ml$. Hence $\det(M) = -1$. Since $M^2 = I$, the Jordan form of M is diagonal and $det\, M = -1$ implies that M has eigenvalues 1 and -1. So α has a fixed point a, say. Since A is torsion free, it follows from the Theorem on Subgroups in Abelian Groups ([4], 3.5.2) that there exists b such that a, b form a base where α has the matrix $M_1 = \begin{bmatrix} 1 & 0 \\ t & -1 \end{bmatrix}$. If t is even, $t = 2n$ say, then for $C = \begin{bmatrix} 1 & 0 \\ n & -1 \end{bmatrix}$, $CM_1C^{-1} = D$ and hence α is conjugate to δ. If $t = 2n+1$, then $CM_1C^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ for $C = \begin{bmatrix} -n & 1 \\ 1+n & -1 \end{bmatrix}$ and α is conjugate to σ which finishes the proof. ■

LEMMA 2. *Each fixed point for α is of the form uu^α if and only if α is conjugate to σ.*

Proof. We assume that $\alpha \neq -id$, then as in Lemma 1, α is conjugate to an automorphism β with the matrix $M_1 = \begin{bmatrix} 1 & 0 \\ t & -1 \end{bmatrix}$, that is $x^\beta = x, y^\beta = x^t y^{-1}$. Let $w = x^r y^s$ be a fixed point for β, then $w = w^\beta = x^r x^{st} y^{-s}$, implies $s = 0$ and each fixed point is $x^r, r \in \mathbf{Z}$. So β would have all fixed points of the form uu^β if and only if there exists u such that $x = uu^\beta$. If

$u = x^\mu y^\nu$ then $x = uu^\beta = (x^\mu y^\nu)(x^\mu y^\nu)^\beta = x^\mu y^\nu x^\mu (x^t y^{-1})^\nu$ and hence $\{\mu, \nu\}$ should be an integer solution of the equation $1 = 2\mu + t\nu$ which exists if and only if t is coprime to 2. It follows from Lemma 1 that β, and hence α, is conjugate to σ as required. ■

The situation with fixed points for $-id$ and δ is clear, so we consider α defined by $M = \begin{bmatrix} k & l \\ m & -k \end{bmatrix}$, $k^2 = 1 - ml$, where m, l are not both zeros.We can assume that $m \neq 0$, and if m, l are not both even we assume that m is odd, because othervise we consider the automorphism $\sigma\alpha\sigma$ (obtained by interchanging x and y), with the matrix $\begin{bmatrix} k' & l' \\ m' & -k' \end{bmatrix} = \begin{bmatrix} -k & m \\ l & k \end{bmatrix}$, where $m' = l$, and use the fact that w is a fixed point for α if and only if w^σ is a fixed point for $\sigma\alpha\sigma$. In the following theorem we denote $d = \gcd(m, 1-k)$, $e = \gcd(m, l)$.

THEOREM. *In a free abelian group of rank two an automorphism α, given by $x^\alpha = x^k y^l$, $y^\alpha = x^m y^{-k}$, $k^2 + ml = 1$, $m \neq 0$, has the cyclic group of fixed points generated by $a = x^{m/d} y^{(1-k)/d}$, which is of the form $a = uu^\alpha$ if and only if e is odd. Then $u = x^\mu y^\nu$ for μ, ν satisfying $\mu(k+1) + \nu m = m/d$.*

Proof. The fixed points for α in A correspond to eigenvectors, with integer coordinates, for $\lambda = 1$. In our notation $x^\alpha = x^k y^l$ corresponds to $[0,1]\begin{bmatrix} k & l \\ m & -k \end{bmatrix} = (k, l)$. So, to find an eigenvector for $\lambda = 1$ we solve a system $\begin{cases} x(k-1) + ym = 0 \\ xl - y(k+1) = 0 \end{cases}$ which has an integer solution $[m, 1-k]$. To find an eigenvector for $\lambda = 1$ we solve a system with the determinant $\begin{vmatrix} k-1 & l \\ m & -k-1 \end{vmatrix} = 0$, which has an integer solution $\{m, 1-k\}$. So the group of fixed points is cyclic, generated by $a = x^{m/d} y^{(1-k)/d}$. If take $b = x^r y^s$, where s, r satisfies $sm - r(1-k) = d$, then since for $p = m/d$, $q = (1-k)/d$, we get $\begin{vmatrix} p & q \\ r & s \end{vmatrix} = ps - qr = 1$, the elements a, b form a new base, where α has the matrix $M_1 = CMC^{-1} = \begin{bmatrix} 1 & 0 \\ t & -1 \end{bmatrix}$ for $C = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$. By calculation we get $t = 2krs + s^2 m - r^2 l$. By Lemma 2, α has all fixed points of the form uu^α, if and only if t is odd, which is not so if m and l are both even. Let now m be odd, then d is odd. We denote $t_1 = s^2 m - r^2 l$ and consider $t_1 m = (sm)^2 - r^2(ml)$. As we know, $sm = r(1-k) + d$, and $ml = 1 - k^2$. This implies $t_1 m \equiv d^2 \bmod 2$ and hence t_1 (and t) is odd. So by Lemma 1, α is conjugate to σ, and hence, by Lemma 2, it has all fixed points of the form uu^α. We conclude that α has all fixed points of the form uu^α, if and only if m, l are not both even, that is e is odd. If now $a = uu^\alpha$, $u = x^\mu y^\nu$

then $a = uu^\alpha = (x^\mu y^\nu)(x^k y^l)^\mu (x^m y^n)^\nu$ and $\{\mu, \nu\}$ is an integer solution of a system which is equivalent to one equation $\mu(k+1) + \nu m = m/d$. ■

COROLLARY. *In a free abelian group of rank two an automorphism* α ($\neq id, -id$), *given by* $x^\alpha = x^k y^l$, $y^\alpha = x^m y^{-k}$, $k^2 + ml = 1$, *is conjugate to* δ *if* m, l *are both even, otherwise it is conjugate to* σ. ■

To give examples we denote $\alpha' = \sigma\alpha\sigma$. An element w is a fixed point for α if and only if w^σ is a fixed point for α' and $w = uu^\alpha$ if and only if $w^\sigma = (u^\sigma)(u^\sigma)^{\alpha'}$.

EXAMPLE 1. Let $x^\alpha = x^3 y^{-2}, y^\alpha = x^4 y^{-3}$. Here $m = 4, k = 3, d = 2$, so the subgroup of fixed points is generated by $a = x^{m/d} y^{(1-k)/d} = x^2 y^{-1}$ which is not of the form uu^α because m, l are both even. ■

EXAMPLE 2. Let $x^\alpha = x^6 y^{-7}$, $y^\alpha = x^5 y^{-6}$. Here $m = k - 1 = d = 5$, so the subgroup of fixed points is generated by xy^{-1}. Since m is odd $xy^{-1} = uu^\alpha$. To find u we solve $7\mu + 5\nu = 1$, so $u = x^3 y^{-4}$. ■

EXAMPLE 3. Let $x^\alpha = xy^3$, $y^\alpha = y^{-1}$. Since $m = 0$, we consider the automorphism $\alpha' = \sigma\alpha\sigma$ which maps $x \to x^{-1}, y \to x^3 y$. Here $m' = 3, k' = -1$, $d' = \gcd(3, -2) = 1$. The subgroup of fixed points for α' is generated by $a' = x^3 y^2$, which is of the form $uu^{\alpha'}$ where $u = y$. So for the initial automorphism α the subgroup of fixed points is generated by $a = x^2 y^3 = xx^\alpha$. ■

EXAMPLE 4. Let $x^\alpha = xy^2$, $y^\alpha = y^{-1}$. Since $m = 0$, we consider the automorphism $\alpha' = \sigma\alpha\sigma$ which maps $x \to x^{-1}, y \to x^2 y$. Here $m' = 2, k' = -1, d' = 2$, so the subgroup of fixed points for α' is generated by xy (the same for α), which is not of the form uu^a because m, l are both even. ■

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