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QUASI-TOPOLOGICAL STONE SPACES

Dedicated to Professor Tadeusz Traczyk

Introduction

The class of quasi-topological Boolean algebras (QTBA's) and some of its subclasses were investigated in [5], [6]. These algebras appear in the natural way in the algebraic semantics of the quasi-topological strengthening of the sentential calculus with identity (SCI, cf. [4]).

Relational and quasi-topological representations of total complete atomic QTBA's (TCA-QTBA's) were considered in [6] and [7]. The present paper deals with dual spaces of QTBA's which contain both topological Stone spaces as well as total quasi-topological ones. Spaces of this sort will be called here quasi-topological Stone spaces (QTSS's). The idea of a construction of such spaces comes from Suszko and Quackenbush who applied similar dual spaces (consisting of two topologies) for topological Boolean algebras (TBA's, cf. [2]). The class of QTSS's is a generalization of the Suszko–Quackenbush spaces (SQS's) in the sense that an additional topological space is replaced by a total quasi-topological space. Obviously every SQS is a QTSS but not conversely.

The paper consists of three sections. The first one concerns with basic properties pertaining to QTSS's and presents some relationships between QTBA's and QTSS's. The second section investigates relations which hold between homomorphisms of QTBA's and two-continuous mappings of QTSS's. The third section in turn considers categories of QTBA's, QTSS's and some functors between them. It is shown that categories of QTBA's and QTSS's are dually equivalent.

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The paper uses notation and terminology of [6] and [7].

1. Basic properties

In this section we investigate fundamental properties concerning quasi-topological Stone's spaces (QTSS's), topological spaces formed on Q-closed subbsets and quasi-topological Stone spaces constructed on topologically closed subsets of selfconjugate QTSS's.

An abstract algebra $\mathcal{A} = \langle A, -, \cup, \cap, C\rangle$ is said to be a quasi-topological Boolean algebra (QTBA) if $\langle A, -, \cup, \cap\rangle$ is a Boolean algebra and $a \mapsto C(a)$ is an additional quasi-closure (Q-closure) operation in $\mathcal{A}$. A QTBA $\mathcal{A}$ is called complete and atomic if its Boolean reduct is complete and atomic (cf. [3]). The class of QTBA's contains topological Boolean algebras (TBA's), monadic algebras, Henle's algebras and obviously Boolean algebras. A pair $\mathcal{X} = \langle X, C\rangle$, where X is a non-empty set and C is a Q-closure operation on it is called a quasi-topological space (QTS). If $F(X)$ is a Boolean field of subsets of X with respect to the set-theoretical operations $-, \cup, \cap$ and C is a Q-closure operation on $F(X)$, then the algebra $F(X) = \langle F(X), -, \cup, \cap, C\rangle$ is said to be a quasi-topological field (QTF). QTF's in QTBA's play similar role as the usual field of sets play in the theory of Boolean algebras. As it is known ([6]), every QTBA is isomorphic to some QTF. A complete and atomic QTBA $\mathcal{A} = \langle A, -, \cup, \cap, C\rangle$ is called total (TCA-QTBA) if $C(a) = \bigcup^{\mathcal{A}}\{C(x) : x \leq a, x \in At(\mathcal{A})\}$ for every $a \in A$, where $At(\mathcal{A})$ is the set of atoms in $\mathcal{A}$. Analogously, a QTS $\mathcal{X} = \langle X, C\rangle$ is total (TQTS) whenever $C(Y) = \bigcup_{y\in Y} C(\{y\})$ for every subset $Y \subseteq X$. A structure $\mathcal{X} = \langle X, B(X), C\rangle$ will be called a *quasi-topological Stone space* (QTSS) if X is the Stone topological space and its basis $B(X)$ with respect to the set-theoretical operations $-, \cup, \cap$ and a Q-closure C is a QTF. Note that $\mathcal{P}(X) = \langle P(X), -, \cup, \cap, C_x\rangle$ is a QTF in which $\langle P(X), -, \cup, \cap\rangle$ is the Boolean field of all subsets of the set X and C_x is a Q-closure operation ([6]) defined by the formula:

$$C_x(Y) = \bigcap\{C(Z) : Y \subseteq Z, Z \in B(X)\} \tag{1.1}$$

for every $Y \subseteq X$. Since $C_x(Y) = C(Y)$ for every $Y \in B(X)$, it follows that $\langle B(X), -, \cup, \cap, C\rangle$ is a quasi-topological subfield of $\mathcal{P}(X)$. The QTF $\mathcal{P}(X)$ is *total* because by (1.1), $C_x(Y) = \bigcup_{y\in Y} C_x(\{y\})$ for every $Y \subseteq X$. It is easy to see that if Y is a clopen subset in X, then $C_x(Y)$ is also clopen. For every element $y \in X$ subsets of the form $C_x(\{y\})$ are always closed since they are intersections of clopen subsets belonging to $B(X)$. A straightforward calculation shows that if Y is an open subset in X, then $C_x(Y)$ is also open. Similarly, if Y is a closed subset in X, the $I_x(Y)$ is closed, where I_x is a Q-interior operation dual to C_x.

Let us denote by $D(X)$ the class of all closed subsets in X and by $O(X)$ the class of all open subsets in X. By virtue of the above remarks it is clear that $\mathcal{D}(X) = \langle D(X), -, \cup, \cap, I_x\rangle$ and $\mathcal{O}(X) = \langle O(X), -, \cup, \cap, C_x\rangle$ are distributive quasi-topological lattices with the smallest and greatest elements. The function $f : O(X) \to D(X)$ such that $f(Y) = -Y$ for every $Y \subseteq X$ is a dual isomorphism from $\mathcal{O}(X)$ onto $\mathcal{D}(X)$.

If $\mathcal{X}$ is a given QTSS, then the class $\mathcal{C}(\mathcal{X}) = \{Y \subseteq X : C(Y) = Y\}$ forms a total topological space on X. This follows from the fact that $\emptyset, X \in \mathcal{C}(X)$ and for any indexed family $(Y_i)_{i\in I}$ of elements of $\mathcal{C}(\mathcal{X})$, $\bigcup_{i\in I} Y_i \in \mathcal{C}(\mathcal{X})$ and $\bigcap_{i\in I} Y_i \in \mathcal{C}(\mathcal{X})$. The subsets of $\mathcal{X}$ of the form $C^\infty(\{x\}) = \bigcup_{n=1}^{\infty} C^{(n)}(\{x\})$, where $C^{(n)}(\{x\}) = C(C^{(n-1)}\{x\})$ are open subsets in that topology for every $x \in X$. Moreover, if $G(x)$ is a minimal open subset in $\mathcal{C}(\mathcal{X})$ containing $x \in X$, then the following formula holds:

$$G(x) = C^\infty(\{x\}) \text{ for every } x \in X. \tag{1.2}$$

To show this, let us assume that $y \in C^\infty(\{x\})$. Then $y \in C^{(n)}(x)$ for some natural number n. From this, there exist $z_1 \in C^{(n-1)}(\{x\})$, $z_2 \in C^{(n-2)}(\{x\}), \ldots, z_{n-1} \in C^{(1)}(\{x\})$ such that $y \in C(\{z_1\})$, $z_1 \in C(\{z_2\}), \ldots$ $\ldots, z_{n-1} \in C(\{x\})$. Since $G(x)$ is open, $y \in G(x)$. Hence, $C^\infty(\{x\}) \subseteq G(x)$. But $G(x)$ is a minimal open set containing x. Therefore, $G(x) \subseteq C^\infty(\{x\})$. Putting these facts together, $G(x) = C^\infty(\{x\})$ for every $x \in X$. Thus we see that the topology $\mathcal{C}(\mathcal{X})$ is determined by a Q-closure operator C and minimal open subsets of elements $x \in X$. Furthermore it is easy to show that $Y = \bigcup_{x\in Y} G(x)$ for every $Y \in \mathcal{C}(\mathcal{X})$. This means that the family $B(\mathcal{C}(\mathcal{X})) = \{G(x) : x \in X\}$ is a basis of that topology $\mathcal{C}(\mathcal{X})$.

Let us recall ([6]) that $\mathcal{X} = \langle X, B(X), C\rangle$ is a self-conjugate QTSS (SC-QTSS) whenever the following equivalence

$$C(Z) \subseteq -Y \text{ iff } C(Y) \subseteq -Z \tag{1.3}$$

holds for every $Y, Z \subseteq X$. Observe that with any SC-QTSS $\mathcal{X}$ is associated a topological space $\mathcal{C}(\mathcal{X})$, whose basis sets satisfy the following two conditions:

$$x \in G(y) \text{ implies } y \in G(x), \tag{1.4}$$

$$G(x) \cap G(y) \neq \emptyset \text{ implies } G(x) = G(y), \tag{1.5}$$

for every element $x, y \in X$. In fact, making use of (1.2), (1.3) one proves (1.4). The property (1.5) follows directly from (1.4). So it is clear that if X is a SC-QTSS, then $\mathcal{C}(\mathcal{X})$ is a partitional topology on X.

It turns out that a Q-closure operator C in a SC-QTSS is closely related to the Vietoris topology formed on the class of all non-empty topologically closed subsets $D(X)$ of X. Let us recall that the Vietoris topology formed on $D(X)$ of the Stone space X is the Stone space in which families $B(Y) =$

$\{D \in D(X) : D \subseteq Y\}$ for every $Y \in B(X)$ generate a basis consisting of all clopen subsets of $D(X)$. Since for every $y \in X$ the set $C(\{y\})$ is closed in X, one can define a function $d : X \to D(X)$ such that $d(y) = C(\{y\})$. A straightforward calculation shows that d is a continuous mapping from X into $D(X)$. Indeed, $d^{-1}(-B(-Y)) = \{y \in X : d(y) \notin B(-Y)\} = \{y \in X : C(y) \cap Y \neq \emptyset\} = \bigcup_{z \in Y} C(\{z\}) = C(Y)$ and similarly, $d^{-1}(B(Y)) = \{y \in X : d(y) \in B(Y)\} = I(Y)$. But $C(Y)$ as well as $I(Y)$ belong to $B(X)$ for every $Y \in B(X)$. Thus d is a topologically continuous mapping from X into $D(X)$. The basis $B_V(D(X))$ of the Vietoris space with respect to the set-theoretical operations $-, \cup, \cap$ and with a Q-closure operation C_V such that $C_V(B(Y)) = B(C(Y))$ is a QTF. It is easy to see that the function g: $B_V(D(X)) \to B(X)$ defined by $g(D) = d^{-1}(D)$ for every $D \in B_V(D(X))$ is a homomorphism from $B_V(D(X))$ into $B(X)$.

A triple $\mathcal{X} = \langle X, B(X), C\rangle$ will be called a Stone space with a total self-conjugate quasi-topology (SS-TSCQT) if X is the Stone topological space, $B(X)$ is its basis and C is a total self-conjugate Q-closure operator on X. Clearly $\mathcal{X}$ becomes a SC-QTSS if $\langle B(X), C\rangle$ is a QTF. With the help of standard arguments one can show that any SS-TSCQT $\mathcal{X} = \langle X, B(X), C\rangle$ is a SC-QTSS iff the sets $C(\{y\})$ are closed in X and d is a continuous mapping from X into $D(X)$. So SC-QTSS's can be characterized by means of a continuous topological mapping from the Stone space X into its Vietoris topology $D(X)$.

Now the concept of a 2-continuous mapping between QTSS's will be introduced. Let $\mathcal{X}_1 = \langle X_1, B(X_1), C_1\rangle$ and $\mathcal{X}_2 = \langle X_2, B(X_2), C_2\rangle$ be two QTSS's. Then any function $g : X_1 \to X_2$ is said to be a 2-continuous mapping from $\mathcal{X}_1$ into $\mathcal{X}_2$ if the following two conditions are satisfied:

$$f^{-1}(Y') \in B(X_1), \tag{1.6}$$

$$f^{-1}(C_2(Y')) = C_1(f^{-1}(Y')), \tag{1.7}$$

for every $Y' \in B(X_2)$. The first condition means that f is a topological continuous mapping from X_1 into X_2, whereas the second one means that f^{-1} is a homomorphism from a QTF $B(X_2)$ into a QTF $B(X_1)$. It is not hard to verify that the mapping d is an example of a 2-continuous mapping from $\mathcal{X} = \langle X, B(X), C\rangle$ into $\mathcal{X}_V = \langle D(X), B_V(D(X)), C_V\rangle$. Note that the notion of a 2-continuous mapping coincides with a 2-continuous mapping in the sense of Quackenbush–Suszko (cf. [2]) when the structure of a quasi-topological space is replaced by the McKinsey–Tarski topology. In particular if C_1 and C_2 in $\mathcal{X}_1$ and $\mathcal{X}_2$, respectively, are identity operators, then $\mathcal{X}_1$ and $\mathcal{X}_2$ become the usual Stone topological spaces and any 2-continuous mapping between them passes to the well-known topological condition of continuity.

2. Quasi-topological Stone spaces of QTBA's

This section presents constructions of QTSS's corresponding to QTBA's, examines relationships between QTBA's, normal QTBA's and their dual counterparts as well as describes connections between homomorphisms of QTBA's and 2-continuous mappings of QTSS's.

Let $\mathcal{A} = \langle A, -, \cup, \cap, C_A \rangle$ be a QTBA. Denote by $X(\mathcal{A})$ the set of all ultrafilters in $\mathcal{A}$ and by $h_A(\mathcal{A}) = \{h_A(a) : a \in A\}$, where $h_A(a) = \{\nabla \in X(\mathcal{A}) : a \in \nabla\}$. As it is known ([3], [6]), $h_A(\mathcal{A})$ with respect to the set-theoretical operations $-, \cup, \cap$ and with a Q-closure C_A^* defined by $C_A^*(h_A(a)) = h_A(C_A(a))$ for every $a \in A$ is a QTF isomorphic to $\mathcal{A}$. The family $h_A(\mathcal{A})$ is a basis of the Stone topology on $X(\mathcal{A})$ (cf. [3]). According to the extension theorem (cf. [6]), a Q-closure operation C_A^* on $h_A(\mathcal{A})$ one can extend to the Q-closure operation C on $X(\mathcal{A})$ such that

$$C(Y) = \bigcup_{\nabla' \in Y} \bigcap_{a \in \nabla'} h_A(C_A(a)) \tag{2.1}$$

for every subset Y of $X(\mathcal{A})$.

By virtue of a straightforward verification one shows that $C(h_A(a)) = C_A^*(h_A(a))$ for every $a \in A$, i.e. the operations C and C_A^* coincide on the quasi-topological subfield $h_A(\mathcal{A})$. Moreover, from (2.1) it follows that C is a total Q-closure. Thus with every QTBA $\mathcal{A}$ is associated a QTSS of the form $\mathcal{X}_A = \langle X(\mathcal{A}), h_A(\mathcal{A}), C \rangle$, where C is defined by (2.1). This space will be called a QTSS of $\mathcal{A}$. Also to $\mathcal{X}_A$ is assigned the QTBA $\mathcal{A}_{\mathcal{X}_A} = h_A(\mathcal{A})$ which will be called the dual QTBA of $\mathcal{X}_A$. Obviously $\mathcal{A}$ is isomorphic to $\mathcal{A}_{\mathcal{X}_A}$.

Now let $\mathcal{X} = \langle X, B(X), C \rangle$ be any QTSS. Then $\mathcal{A}_{\mathcal{X}} = B(X)$ is its dual QTBA. This algebra determines in turn the QTSS $\mathcal{X}_{\mathcal{A}_{\mathcal{X}}} = \langle X(B(X)), h(B(X)), C \rangle$, where $X(B(X))$ is the set of all ultrafilters in the QTF $B(X)$, $h(B(X))$ consists of elements of the form $h(Y)$ for $Y \in B(X)$ and C^* is a Q-closure on $X(B(X))$ which extends of C_B^* defined by $C_B^*(h(Y)) = h(C(Y))$ for every $Y \in B(X)$. Observe that a mapping $g : X \to X(B(X))$ such that $g(x) = \nabla_x$ for every $x \in X$ (where ∇_x is an ultrafilter in $B(X)$ determined by a point $x \in X$) is a bijective 2-continuous mapping (2-homeomorphism) from $\mathcal{X}$ onto $\mathcal{X}_{\mathcal{A}_{\mathcal{X}}}$. In fact, if $x \neq y$, then there exists a clopen subset $Y \in B(X)$ such that $y \in Y$ and $x \notin Y$. Hence, $Y \in \nabla_y$ and $Y \notin \nabla_x$. This implies that $\nabla_x \neq \nabla_y$ which means that $g(x) \neq g(y)$. So g is $1-1$. To prove the property "onto", let ∇ be any ultrafilter in $X(B(X))$. Since $B(X)$ is a perfect QTF, it follows that there exists $x \in X$ such that $\nabla = \nabla_x$. Hence, there exists $x \in X$ for which $g(x) = \nabla_x = \nabla$. Thus g is an epifunction. Moreover, for every $x \in X$ and every $Y \in h(B(X))$ the following equivalences hold: $x \in g^{-1}(h(Y))$ iff $g(x) = \nabla_x \in h(Y)$ iff $Y \in \nabla_x$ iff $x \in Y$ and $x \in g^{-1}(C^*(h(Y))$

iff $\nabla_x \in h(C(Y))$ iff $C(Y) \in \nabla_x$ iff $x \in C(Y) = C(g^{-1}(h(Y)))$. Thus we have shown that g is a bijective 2-continuous mapping (2-homeomorphism) from $\mathcal{X}$ onto $\mathcal{X}_{\mathcal{A}_\mathcal{X}}$. Putting these facts together we receive the following theorem.

THEOREM 2.1. *There exists a one-to-one correspondence between the classes of QTBA's and QTSS's.*

Let us confine ourselves to the subclass of normal QTBA's. Recall that a QTBA $\mathcal{A} = \langle A, -, \cup, \cap, C_A\rangle$ is normal if there exists a normal ultrafilter in $\mathcal{A}$. As it is known (cf. [5]), any Boolean ultrafilter ∇ is normal if $C_A(a) \in \nabla$ iff $a \neq 0$ for every $a \in A$. It will be shown that dual spaces corresponding to normal QTBA's are strongly compact QTSS's. A QTSS $\mathcal{X} = \langle X, B(X), C\rangle$ is said to be *strongly compact* if $\bigcap_{i\in I} C(Y_i) \neq \emptyset$ for every indexed family $(Y_i)_{i\in I}$ of non-empty subsets of $\mathcal{X}$ (cf. [6]).

LEMMA 2.1. *Let $\mathcal{A} = \langle A, -, \cup, \cap, C_A\rangle$ be a normal QTBA. Then $\mathcal{X}_A = \langle X(A), h(A), C\rangle$ is a strongly compact QTSS such that its dual QTBA $\mathcal{A}_{\mathcal{X}_A}$ is isomorphic to $\mathcal{A}$.*

Proof. If $\mathcal{A} = \langle A, -, \cup, \cap, C_A\rangle$ is normal, then there exists a normal ultrafilter ∇ in $\mathcal{A}$. The corresponding to $\mathcal{A}$ a QTSS is of the form $\mathcal{X}_A = \langle X(\mathcal{A}), h(\mathcal{A}), C\rangle$. Let $\bigcap_{i\in I} h_A(C_A(a_i) = \emptyset$. Then $\nabla \notin \bigcap_{i\in I} h_A(C_A(a_i))$. This implies that there exists $i \in I$ such that $h_A(C_A(a_i)) \notin \nabla$. By normality of ∇, $h_A(a_i) = \emptyset$ for some $i \in I$. Thus the set $D(h(\mathcal{A})) = \{h_A(C_A(a)) : h_A(a) \neq \emptyset\}$ has the intersection property (intersections are non-empty). Hence, for every element $a \in \nabla \in X(\mathcal{A})$ we have $\bigcap_{a\in\nabla} h_A(C_A(a)) = C(\nabla) \neq \emptyset$. Note that $\bigcap_{i\in I} C(\nabla_i) = \bigcap_{i\in I} \bigcup_{a\in\nabla_i} h_A(C_A(a))$ is not empty, because $a \neq 0$. Finally a simple calculation shows that $\bigcap_{i\in I} C(Y_i) = \bigcap_{i\in I} \bigcup_{\nabla\in Y_i} C(\nabla) = \bigcup_{\alpha\in\prod_{i\in I} Y_i} \bigcap_{i\in I} C(\alpha(i)) \neq \emptyset$ for every indexed family $(Y_i)_{i\in I}$ of non-empty subsets in $X(\mathcal{A})$. So $\mathcal{X}_A$ is a strongly compact QTSS. This space determines the dual QTBA $\mathcal{A}_{\mathcal{X}_A} = h(\mathcal{A})$ that is isomorphic to $\mathcal{A}$.

Now if we start with any strongly compact QTSS, then we get the following lemma.

LEMMA 2.2. *Let $\mathcal{X} = \langle X, B(X), C\rangle$ be a strongly compact QTSS. Then it determines a normal QTBA $\mathcal{A}_\mathcal{X} = B(X)$ which determines in turn a strongly compact QTSS $\mathcal{X}_{\mathcal{A}_\mathcal{X}}$ that is 2-homeomorphic to $\mathcal{X}$.*

Proof. If $\mathcal{X}$ is a strongly compact QTSS, then in $B(X)$ the set $D(B(X)) = \{C(Y) : Y \neq \emptyset\}$ has FIP (finite intersection property). Hence, the set $D(B(X))$ generates a filter which can be extended to an ultrafilter $\nabla(D)$. Observe that $\nabla(D)$ is a normal ultrafilter in $B(X)$. Thus $\mathcal{A}_\mathcal{X} = B(X)$ is a normal QTBA. By Theorem 2.1, $\mathcal{X}_{\mathcal{A}_\mathcal{X}} = \langle X(B(X)), h(B(X)), C\rangle$ is a QTSS of $\mathcal{A}_{\mathcal{X}}$. According to Lemma 2.1, $\mathcal{X}_{\mathcal{A}_{\mathcal{X}}}$ is a strongly compact QTSS of $\mathcal{A}_{\mathcal{X}}$. As it was shown above the function $g : X \to X(B(X))$ defined by $g(x) = \nabla_x$ for every $x \in X$ is a bijective 2-continuous mapping (2-homeomorphism) from $\mathcal{X}$ onto $\mathcal{X}_{\mathcal{A}_{\mathcal{X}}}$.

The next theorem presents main relationships between the subclass of normal QTBA's and the subclass of strongly compact QTSS's.

THEOREM 2.2. *The classes of normal QTBA's and strongly compact QTSS's are in a one-to-one correspondence.*

Proof. By Lemmas 2.1 and 2.2.

For every QTBA $\mathcal{A} = \langle A, -, \cup, \cap, C_A \rangle$ one can construct a normal QTBA $\mathcal{A}^* = \mathcal{A} \otimes 2 = \langle A \times \{0,1\}, -, \cup, \cap, C^* \rangle$, in which the Boolean operations are defined componentwise and a Q-closure C^* is defined by the formula $C^*(a,x) = (C_A(a), 1)$ whenever $(a,x) \neq (0,0)$ and $C^*(a,x) = (0,0)$ whenever $(a,x) = (0,0)$ for every $(a,x) \in A \times \{0,1\}$ (cf. [5]). By virtue of Theorem 2.2, corresponding to $\mathcal{A}^*$ the QTSS $\mathcal{X}_{\mathcal{A}^*} = \langle X(\mathcal{A}^*), h(\mathcal{A}^*), C^* \rangle$ is strongly compact. The dual counterpart of the algebraic construction of normal QTBA's is a quasi-topological construction of strongly compact QTSS's. Namely, if $\mathcal{X} = \langle X, B(X), C \rangle$ is a QTSS, then $\mathcal{X}^* = \langle X^*, B(X^*), C^* \rangle$, where $X^* = X \cup \{x^*\}$, $x^* \notin X$, $B(X^*) = B(X) \cup P(\{x^*\})$ and

$$C^*(Y) = \begin{cases} C(Y) \cup \{x^*\}, & \emptyset \neq Y \subseteq X^* \\ \emptyset, & Y = \emptyset \end{cases}$$

for every $Y \subseteq X^*$, is a strongly compact QTSS obtained from $\mathcal{X}$. A QTSS $\mathcal{X}^*$ constructed from $\mathcal{X}$ will be called a 1-point strong compactification (1-PSC) of $\mathcal{X}$.

Let us apply now the 1-PSC to $\mathcal{X}_{\mathcal{A}}$ of $\mathcal{A}$. Then we get a strongly compact QTSS of the form $\mathcal{X}^*_{\mathcal{A}} = \langle X^*(\mathcal{A}), h^*(\mathcal{A}), C^* \rangle$, where $X^*(\mathcal{A}) = X(\mathcal{A}) \cup \{1\}$, $h^*(\mathcal{A}) = h_A(\mathcal{A}) \cup P(\{1\}) = \{h_A(a) \cup h_2(x) : a \in A, x \in \{0,1\}\}$ and h_A, h_2 are the Stone isomorphisms of $\mathcal{A}$ and 2, respectively. With the help of standard calculations one shows that the function $f : X(\mathcal{A}^*) \to X^*(\mathcal{A})$ such that $f(\nabla, \{1\}) = \{\nabla\} \cup \{1\}$ for every $\nabla \in \mathcal{X}(\mathcal{A})$ is a 2-homeomorphism from $\mathcal{X}_{\mathcal{A}^*}$ onto $\mathcal{X}^*_{\mathcal{A}}$. Thus for every QTBA $\mathcal{A}$ one can construct a strongly compact QTSS $\mathcal{X}^*_{\mathcal{A}}$ using 1-PSC as well as a strongly compact QTSS $\mathcal{X}_{\mathcal{A}}$ using the algebraic normal construction. These two spaces up to 2-homeomorphism can be treated as identical.

Connections between homomorphisms of QTBA's and 2-continuous mappings of QTSS's are described int the following two theorems.

THEOREM 2.3. *Let $h : A_1 \to A_2$ be a homomorphism from a QTBA $\mathcal{A}_1 = \langle A_1, -, \cup, \cap, C_1 \rangle$ into a QTBA $\mathcal{A}_2 = \langle A_2, -, \cup, \cap, C_2 \rangle$. Then the mapping*

$g : X(\mathcal{A}_2) \to X(\mathcal{A}_1)$ defined by the formula:

$$g(\nabla') = h^{-1}(\nabla') \text{ for every } \nabla' \in X(\mathcal{A}_2) \tag{2.2}$$

is a 2-continuous mapping from a QTSS $\mathcal{X}_{\mathcal{A}_2} = \langle X(\mathcal{A}_2), h_2(\mathcal{A}_2), C_2\rangle$ into a QTSS $\mathcal{X}_{\mathcal{A}_1} = \langle X(\mathcal{A}_1), h_1(\mathcal{A}_1), C_1\rangle$.

Proof. We have to show two conditions: $g^{-1}(Y) \in h_2(\mathcal{A}_2)$ and $g^{-1}(C_1(Y)) = C_2(g^{-1}(Y))$ for every $Y \in h_1(\mathcal{A}_1)$. Since $Y \in h_1(\mathcal{A}_1)$, there exists $a \in A_1$ such that $Y = h_1(a)$. Hence, $g^{-1}(h_1(a)) = \{\nabla' \in X(\mathcal{A}_2) : g(\nabla') \in h_1(a)\} = \{\nabla' \in X(\mathcal{A}_2) : h(a) \in \nabla'\} = h_2(h(a)) \in h_2(\mathcal{A}_2)$. This means that $g^{-1}(Y) \in h_2(\mathcal{A}_2)$ for every $Y \in h_1(\mathcal{A}_1)$. Note that for every $\nabla' \in X(\mathcal{A}_2)$ and every $a \in A_1$ the following equivalences hold: $\nabla' \in g^{-1}(C_1(h_1(a)))$ iff $g(\nabla') \in C_1(h_1(a))$ iff $C_1(a) \in g(\nabla') = h^{-1}(\nabla')$ iff $h(C_1(a)) = C_2(h(a)) \in \nabla'$ iff $\nabla' \in h_2(C_2(h(a))) = C_2(h_2(h(a))) = C_2(g^{-1}(h_1(a)))$. So the second condition is satisfied.

THEOREM 2.4. *If $g : X_1 \to X_2$ is a 2-continuous mapping from a QTSS $\mathcal{X}_1 = \langle X_1, B(X_1), C_1\rangle$ into a QTSS $\mathcal{X}_2 = \langle X_2, B(X_2), C_2\rangle$, then the mapping $h : B(X_2) \to B(X_1)$ such that*

$$h(Y') = g^{-1}(Y') \text{ for every } Y' \in B(X_2) \tag{2.3}$$

is a homomorphism from a QTF $B(X_2)$ into a QTF $B(X_1)$.

Proof. Standard calculations show that h preserves Boolean operations in $B(X_2)$ and $B(X_1)$. Moreover, for every $Y' \in B(X_2), h(C_2(Y')) = g^{-1}(C_2(Y')) = C_1(g^{-1}(Y')) = C_1(h(Y'))$.

Thus we see that dual counterparts of homomorphisms in QTBA's are 2-continuous mappings of QTSS's. By virtue of Theorems 2.1, 2.2 and simple calculations one can show that dual counterparts of isomorphisms between QTBA's are 2-homeomorphisms between QTSS's.

If h is a monomorphism from $\mathcal{A}_1 = \langle A_1, -, \cup, \cap, C_1\rangle$ into $\mathcal{A}_2 = \langle A_2, -, \cup, \cap, C_2\rangle$, then the corresponding 2-continuous mapping $g : X(\mathcal{A}_2) \to X(\mathcal{A}_1)$ defined by (2.2) is a surjection. Indeed, if h is a monomorphism, then $h(\nabla) \in X(\mathcal{A}_2)$ for every $\nabla \in X(\mathcal{A}_1)$. From this, for every $\nabla \in X(\mathcal{A}_1)$ there exists $h(\nabla) \in X(\mathcal{A}_2)$ such that $g(h(\nabla)) = h^{-1}(h(\nabla)) = \nabla$.

If h is an epimorphism from $\mathcal{A}_1$ onto $\mathcal{A}_2$, then the function g is an injective 2-continuous mapping from $\mathcal{X}_{\mathcal{A}_2}$ into $\mathcal{X}_{\mathcal{A}_1}$. In fact, let us assume that $g(\nabla') = g(\nabla'')$ for any $\nabla', \nabla'' \in X(\mathcal{A}_2)$. Hence, by (2.2), $h^{-1}(\nabla') = h^{-1}(\nabla'')$ which implies that $h(h^{-2}(\nabla') = h(h^{-1}(\nabla''))$. By hypothesis that h is an epimorphism we get $\nabla' = \nabla''$. From this and from Theorem 2.3 it follows that g is an injective 2-continuous mapping.

By the above remarks it is seen that to monomorphisms (epimorphisms) of QTBA's correspond surjective (injective) 2-continuous mappings of

QTSS's. Similarly it may be shown that if g is a surjective (injective) 2-continuous mapping from a QTSS $\mathcal{X}_1 = \langle X_1, B(X_1), C_1\rangle$ onto (into) a QTSS $\mathcal{X}_2 = \langle X_2, B(X_2), C_2\rangle$, then the corresponding dual mapping $h : B(X_2) \to B(X_1)$ defined by (2.3) is a monomorphism (epimorphism) from $B(X_2)$ into (onto) $B(X_1)$.

As it is known for every QTBA $\mathcal{A} = \langle A, -, \cup, \cap, C_A\rangle$ one can construct a normal QTBA $\mathcal{A}^* = \langle A^*, -, \cup, \cap, C^*\rangle$. Since the mapping $\Pi : A^* \to A$ such that $\Pi(a, x) = a$ for every $(a, x) \in A^*$ is an epimorphism from $\mathcal{A}^*$ onto $\mathcal{A}$, it follows that the mapping $g : X(\mathcal{A}) \to X(\mathcal{A}^*)$ defined by $g(\nabla) = \Pi^{-1}(\nabla)$ for every $\nabla \in X(\mathcal{A})$ is the corresponding injective 2-continuous mapping from $\mathcal{X}_A = \langle X(A), h(A), C\rangle$ into $\mathcal{X}_A^* = \langle X^*(\mathcal{A}), h^*(\mathcal{A}), C^*\rangle$. In fact, the mapping $h : A^* \to h^*(\mathcal{A})$ such that $h(a, x) = h_A(a) \cup h_2(x)$ for every $(a, x) \in A^*$ is an isomorphism from $\mathcal{A}^*$ onto $h^*(\mathcal{A})$. Furthermore, $\nabla \in g^{-1}(h_A(a) \cup h_2(x))$ iff $g(\nabla) \in h(a, x)$ iff $(a, x) \in \Pi^{-1}(\nabla)$ iff $a \in \nabla$ iff $\nabla \in h_A(a)$ as well as $\nabla \in g^{-1}(C^*(h_A(a) \cup h_2(x))$ iff $g(\nabla) \in (h_A(C_A(a), \{1\}) = h(C_A(a), 1)$ iff $(C_A(a), 1) \in \Pi^{-1}(\nabla)$ iff $C_A(a) \in \nabla$ iff $\nabla \in h_A(C_A(a)) = C(h_A(a)) = C(g^{-1}(h_A(a) \cup h_2(x))$ for every $\nabla \in X(A)$ and every $a \in A, x \in \{0, 1\}$. But g is an injection. Therefore g is an injective 2-continuous mapping from $\mathcal{X}_A$ into $\mathcal{X}_A^*$.

Let us take now any QTSS $\mathcal{X} = \langle X, B(X), C\rangle$. Then $\mathcal{X}^* = \langle X^*, B(X^*), C^*\rangle$ is a strongly compact QTSS such that the identity function $g(x) = x$ for every $x \in X$ is a bijective 2-continuous mapping (2-embedding) from $\mathcal{X}$ into $\mathcal{X}^*$. This mapping determines in turn an epimorphism $\Pi : B(X^*) \to B(X)$ such that $\Pi(Z) = g^{-1}(Z)$ for every $Z \in B(X^*)$.

It is worth to note that not every subclass of QTSS's is closed under the operation of taking the 1-PSC. For instance if in $\mathcal{X} = \langle X, B(X), C\rangle$ a Q-closure operator satisfies the following condition:

$$C(Y) \cap C(-C(Y)) = \emptyset \text{ for every } Y \subseteq X, \tag{2.4}$$

then the strongly compact QTSS $\mathcal{X}^* = \langle X^*, B(X^*), C^*\rangle$ does not satisfy (2.4). Indeed, $C^*(Y \cup \{x^*\}) \cap C^*(-C^*(Y \cup \{x^*\})) = (C(Y) \cup \{x^*\}) \cap C^*(-(C(Y) \cap \{x^*\}) = (C(Y) \cap C(-C(Y)) \cup \{x^*\} = \{x^*\}$. Hence $\mathcal{X}^*$ does not belong to the subclass of QTSS's satisfying (2.4). Moreover, this result implies that the corresponding subclass of QTBA's is not closed under the construction of normal QTBA's. Let us add that if a subclass of QTBA's is closed under the operation of taking algebras of the form $\mathcal{A}^*$, then free algebras are normal. This fact is important in the Boolean strangthening of the sentential calculus with identity since normal algebras are just models of that logic ([4], [5]).

3. Connections between categories of QTBA's and QTSS's

The goal of this section is to describe some relationships between QTBA's and their QTSS's in terms of categories and functors.

Let $K(QTBA)$ be a category of QTBA's. Objects are QTBA's and morphisms are homomorphisms between QTBA's. The dual counterpart of this category is a category $K(QTSS)$ of QTSS's the objects of which are QTSS's and morphisms are 2-continuous mappings between QTSS's. Making use of Theorem 2.3 and applying standard calculations one shows that $F : K(QTBA) \to K(QTSS)$ such that $F(\mathcal{A}) = \mathcal{X}_{\mathcal{A}}$ and $F(g)(\nabla') = g^{-1}(\nabla')$ for any QTBA $\mathcal{A}$ and every $\nabla' \in X(\mathcal{A}_2)$ is a contravariant functor from $K(QTBA)$ into $K(QTSS)$, where g is a homomorphism from a QTBA $\mathcal{A}_1$ into a QTBA $\mathcal{A}_2(g \in \mathrm{Hom}(\mathcal{A}_1, \mathcal{A}_2)$ and $F(g)$ is a 2-continuous mapping from $\mathcal{X}_{\mathcal{A}_2}$ into $\mathcal{X}_{\mathcal{A}_1}(F(g) \in \mathrm{Cont}_2(\mathcal{X}_{\mathcal{A}_2}, \mathcal{X}_{\mathcal{A}_1})$.

Since the restriction of F to $\mathrm{Hom}(\mathcal{A}_1, \mathcal{A}_2)$ is an injective mapping for every QTBA $\mathcal{A}_1, \mathcal{A}_2$, it follows that F is a faithful functor. Furthermore, F transforms injective (surjective) homomorphisms of QTBA's into surjective (injective) 2-continuous mappings of QTSS's. In particular, F transforms isomorphisms between QTBA's into 2-homeomorphisms between QTSS's.

Let $\mathcal{A}^*$ be the normal algebra constructed from a QTBA $\mathcal{A}$. Then the function $\Pi : A^* \to A$ such that $\Pi(a, x) = a$ for every $(a, x) \in A^*$ is an epimorphism from $\mathcal{A}^*$ onto $\mathcal{A}$. Hence $F(\Pi)$ is an injective 2-continuous mapping from $\mathcal{X}_{\mathcal{A}}$ into $\mathcal{X}_{\mathcal{A}^*}$. But up to isomorphism $\mathcal{X}_{\mathcal{A}^*} = \mathcal{X}_{\mathcal{A}}^*$. Therefore, $F(\mathcal{A}^*) = F(\mathcal{A})^*$ for every QTBA $\mathcal{A}$. Note that if $h \in \mathrm{Hom}(\mathcal{A}_1, \mathcal{A}_2)$, where $\mathcal{A}_1, \mathcal{A}_2$ are normal QTBA's and ∇' is a normal ultrafilter in $\mathcal{A}_2$, then $F(h)(\nabla')$ does not have to be a normal ultrafilter in $\mathcal{A}_1$. This means that $F(h)$ does not preserve normality of ultrafilters. It can be shown that $F(h)$ preserves the normal property of ultrafilters iff h is a monomorphism.

Let us take now under consideration a function $G : K(QTSS) \to K(QTBA)$ such that $G(\mathcal{X}) = B(X)$ for every QTSS $\mathcal{X} = \langle X, B(X), C \rangle$. Moreover, if f is any 2-continuous mapping from a QTSS $\mathcal{X}_1 = \langle X_1, B(X_1), C_1 \rangle$ into a QTSS $\mathcal{X}_2 = \langle X_2, B(X_2), C_2 \rangle$, then $G(f) \in \mathrm{Hom}(B(X_2), B(X_1))$ is defined by the formula $G(f)(Y') = f^{-1}(Y')$ for every $Y' \in B(X_2)$. Observe that G is a contravariant functor from $K(QTSS)$ into $K(QTBA)$. This functor transforms injective (surjective) 2-continuous mappings into surjective (injective) homomorphisms. Since every TQTSS $\mathcal{X}$ is 2-embedded in the strongly compact QTSS $\mathcal{X}^*$, it follows that G is commutative with the operation "*", i.e. $G(\mathcal{X}^*) = G(X)^*$ (up to isomorphism). Furthermore, if g is a 2-embedding from $\mathcal{X}$ into $\mathcal{X}^*$, then $G(g)$ is the corresponding epimorphism from $B(X^*)$ onto $B(X)$ such that $G(g)(Y) = g^{-1}(Y)$ for every $Y \in B(X^*)$. Also it is easy to see that if f is a 2-homeomorphism from $\mathcal{X}_1 = \langle X_1, B(X_1), C_1 \rangle$ onto $\mathcal{X}_2 = \langle X_2, B(X_2), C_2 \rangle$, then $G(f)$ is the corresponding isomorphism from $B(X_2)$ onto $B(X_1)$.

Denote by FG a covariant functor which is the composition of F and G. We will show that FG is naturally equivalent to the identity functor on the category of QTSS's.

LEMMA 3.1. *Let f be a 2-continuous mapping from a QTSS $\mathcal{X}_1 = \langle X_1, B(X_1), C_1\rangle$ into a QTSS $\mathcal{X}_2 = \langle X_2, B(X_2), C_2\rangle$. Then there are 2-homeomorphisms $g_1 : \mathcal{X}_1 \to FG(\mathcal{X}_1)$ and $g_2 : \mathcal{X}_2 \to FG(\mathcal{X}_2)$ such that $g_2 \circ f = FG(f) \circ g_1$.*

Proof. Let us consider the function $g_1 : \mathcal{X}_1 \to FG(\mathcal{X}_1) = \langle X(B(X_1)), B_{FG}(X_1), C_{FG}\rangle$ defined by the formula $g_1(x_1) = \nabla_{x_1}$ for every $x_1 \in X_1$, where ∇_{x_1} is a principal ultrafilter generated by $x_1 \in X_1$. The function g_1 is an injection since the field $B_{FG}(X_1)$ is irreducible. But $B_{FG}(X_1)$ is also perfect. Therefore g_1 is an epimorphism. It remains to be shown yet that g_1 is a 2-continuous mapping. Elements of $B_{FG}(X_1)$ will be denoted by $h(Y) = \{\nabla \in X(B(X_1)) : Y \in \nabla\}$ for every $Y \in B(X_1)$. Then, $g_1^{-1}(h(Y)) = \{x_1 \in X_1 : g_1(x_1) = \nabla_{x_1} \in h(Y)\} = \{x_1 \in X_1 : Y \in \nabla_{x_1}\} = \{x_1 \in X_1 : x_1 \in Y\} = Y \in B(X_1)$ and $g_1^{-1}(C_{FG}(h(Y))) = g_1^{-1}(h(C_1(Y))) = C_1(Y) \in B(X_1)$. Thus g_1 is a 2-homeomorphism. Making use similar reasonings one shows that the function $g_2 : \mathcal{X}_2 = \langle X_2, B(X_2), C_2\rangle \to FG(\mathcal{X}_2) = \langle X(B(X_2)), B_{FG}(X_2), C_{FG}\rangle$ such that $g_2(x_2) = \nabla_{x_2}$ for every $x_2 \in X_2$ is a 2-homeomorphism. Moreover, $(g_2 \circ f)(x_1) = g_2(f(x_1)) = \nabla_{f(x_1)} = G(f)^{-1}(\nabla_{x_1}) = FG(f)(g_1(x_1))$ for every $x_1 \in X_1$. Thus, $g_2 \circ f = FG(f) \circ g_1$.

It turns out that the covariant functor GF is also naturally equivalent to the identity functor on the category K(QTBA). This is shown in the following lemma.

LEMMA 3.2. *Let h be a homomorphism from a QTBA $\mathcal{A}_1 = \langle A_1, -, \cup, \cap, C_1\rangle$ into a QTBA $\mathcal{A}_2 = \langle A_2, -, \cup, \cap, C_2\rangle$. Then there exist isomorphisms $f_1 : \mathcal{A}_1 \to GF(\mathcal{A}_1)$ and $f_2 : \mathcal{A}_2 \to GF(\mathcal{A}_2)$ such that $f_2 \circ h = GF(h) \circ f_1$.*

Proof. If $h \in \mathrm{Hom}(\mathcal{A}_1, \mathcal{A}_2)$, then by Theorem 2.3 the function $F(h) : \mathcal{X}_{\mathcal{A}_2} = \langle X(\mathcal{A}_2), h_2(\mathcal{A}_2), C_{\mathcal{A}_2}\rangle \to \mathcal{X}_{\mathcal{A}_1} = \langle X(\mathcal{A}_1), h_1(\mathcal{A}_1), C_{\mathcal{A}_1}\rangle$ such that $F(h)(\nabla') = h^{-1}(\nabla')$ for every $\nabla' \in X(\mathcal{A}_2)$ is a 2-continuous mapping. Let us define $f_1 : \mathcal{A}_1 \to GF(\mathcal{A}_1) = h_1(\mathcal{A}_1)$ by $f_1(a) = h_1(a)$ for every $a \in A_1$ and define $f_2 : \mathcal{A}_2 \to GF(\mathcal{A}_2) = h_2(\mathcal{A}_2)$ by $f_2(a') = h_2(a')$ for every $a' \in A_2$. Then f_1 as well as f_2 are isomorphisms. Furthermore, $(f_2 \circ h)(a) = f_2(h(a)) = h_2(h(a)) = \{\nabla' \in X(\mathcal{A}_2) : h(a) \in \nabla'\} = \{\nabla' \in X(\mathcal{A}_2) : F(h)(\nabla') \in h_1(a)\} = F(h)^{-1}(h_1(a)) = GF(h)(h_1(a)) = GF(h)(f_1(a))$ for every $a \in A_1$. Hence, $f_2 \circ h = GF(h) \circ f_1$.

From above two lemmas we get the following theorem.

THEOREM 3.1. *Categories K(QTBA) and K(QTSS) are dually equivalent.*

Proof. By Lemma 3.1, FG is naturally equivalent to the identity functor $I_{K(QTSS)}$. By Lemma 3.2, GF is naturally equivalent to the identity functor $I_{K(QTBA)}$. From these facts it follows that $K(QTBA)$ is dually equivalent to K(QTSS).

Let us note finally that if $\mathcal{A} = \langle A, -, \cup, \cap, C_A \rangle$ is a total complete and atomic QTBA (TCA-QTBA), then the corresponding QTSS $\mathcal{X}_{\mathcal{A}}$ becomes a total quasi-topological space (TQTS). As it is known (cf. [7]), TCA-QTBA's form a category K(TCA-QTBA) which is a full subcategory of K(TQTBA). Also TQTS's constitute a full subcategory of K(QTSS). Thus confining ourselves to the subclasses of TCA-QTBA's and TQTS's Theorem 3.1 passes to the dual equivalence between subcategories K(TCA-QTBA) and K(TQTS).

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