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AN AXIOMATIZATION FOR ABELIAN RELATIVE INVERSES

Dedicated to Professor Tadeusz Traczyk

Introduction

In this article we define RI-sets, which provide a common axiomatic base for commutativity in groups and lattices. Relative inverses are either $b - a$ or $b \wedge a'$. Abelian groups, D-posets, orthoalgebras, orthoposets, Boolean algebras and rings of sets are special examples of RI-sets. These structures can be obtained from partial order. It is mentioned that the notion of a homomorphism is for RI-sets closely related with the notion of an additive map.

1. RI-sets

DEFINITION 1.1. Let X be a set with a special element 0 and $\uplus$ be a partially defined binary operation on X. We call $(X; \uplus, 0)$ an RI-set if the binary operation $\uplus$ satisfies the following rules:

(i) $a \uplus 0$ is defined for all $a \in X$ and $a \uplus 0 = a$.
(ii) $a \uplus a = 0$ for all $a \in X$.
(iii) If $b \uplus a$ is defined then $b \uplus (b \uplus a)$ is defined.
(iv) If in the equation $(a \uplus b) \uplus c = (a \uplus c) \uplus b$ one side is defined then both sides are defined and the equation holds.
(v) If $b \uplus a, c \uplus b$ are defined then $c \uplus a$ is defined.

We call $b \uplus a$ a *relative inverse of a in b*. If $a = b$ then in the following equations $a \uplus c = b \uplus c$ and $c \uplus a = c \uplus b$, if one side is defined then both sides are defined and the equation holds.

If we have the additional property

(vi) $c \uplus a = d \uplus a$ implies $c = d$,

then we call $(X; \uplus, 0)$ an *abelian RI-set.*

Examples which fulfill these rules are:

a) Abelian groups $(G; +, -, 0)$ with $-$ replacing $\uplus$. They are abelian RI-sets. Also the additive structure of rings is included here. In particular the real numbers $\mathbb{R}$ and complex numbers $\mathbb{C}$ are abelian RI-sets.

b) The natural numbers $\mathbb{N}$ together with 0 and $-$ replacing $\uplus$ defined for $n, m \in \mathbb{N}$ with $n \geq m$ is an abelian RI-set.

c) Multiplicative abelian groups $(H; \circ, ()^{-1}, 1)$ with 1 replacing 0 and $\uplus$ the usual division, $b \uplus a = ba^{-1}$ are abelian RI-sets.

d) D-posets X which have an additional largest element $1 \in X$ and where $b \uplus a$ is defined iff $a \leq b$ and $\ominus$ replaces $\uplus$. They are also abelian RI-sets.

e) Ortholattices with $b \wedge a'$ replacing $b \uplus a$ are RI-sets. This example includes power sets and Boolean algebras.

Boolean algebras are also D-posets, but are separately listed, because for D-posets the existence of $b \uplus a$ means $a \leq b$, but $b \uplus a$ always exists in its above definition for Boolean algebras.

f) Suppose X is a ring of subsets of a given set E (i.e. $0 \in X$ and X is closed under the formation of set theoretic differences and finite unions).

(1) X with the set theoretic difference replacing $\uplus$ is an RI-set which evidently is not abelian.

(2) X with the set theoretic symmetric difference ($A \triangle B = (A \setminus B) \cup (B \setminus A)$) replacing $\uplus$ is an abelian RI-set.

These examples (in case $E \notin X$) cannot be included in d) or e).

g) Other finite examples of RI-set may be constructed by using tables for $\uplus$ or by writing a computer program which checks possible tables whether or not they fulfill the axioms (i)–(v).

h) In g) one may take for instance X as the set of 4 elements $0, a, b, c$ with $x \uplus x = 0$ and $x \uplus 0 = x$ for all $x \in X$ and define $b \uplus a = c$, $b \uplus c = a$, $c \uplus a = b$, $c \uplus b = a$. This example can be embedded into the abelian group $Z_2 \times Z_2$.

i) On the poset X containing as elements and as order: $0 < c, d$; $c, d < e < b, a$; $b, a < 1$ there does not exist an operation $\uplus$ which fulfills $b \uplus a$ iff $a \leq b$ and (i)–(vi), [6]. But there exists, for instance, the partially defined operation $x \uplus x = 0$, $x \uplus 0 = x$, $b \uplus e = b \uplus d = e = d \uplus b = d \uplus e$, $x \in X$, which shall satisfy the rules (i)–(v) of an RI-set. The axioms (iv), (v) do not produce new elements $x \uplus y$.

Notice that we have chosen such axioms of RI-sets which enable us to introduce notions of subalgebras, homomorphisms, additive maps, isomorphisms and direct products by a unique simple requirement to inherit the operation $\uplus$ (see part 4). Moreover, by this way defined subalgebras, isomorphic images and products of many examples of RI-sets (e.g. abelian RI-posets, abelian groups, D-posets, orthomodular posets, Boolean orthoposets and Boolean algebras) inherit their own algebraic structures (see part 4). To show that we are going to derive (in parts 2. and 3.) necessary and sufficient conditions for RI-sets under which they become algebraic structures listed above. Those conditions are inherited always when the operation $\uplus$ is inherited.

PROPOSITION 1.2. *Let X be an abelian RI-set. Then*

(i) *if $b \uplus a$ is defined then $b \uplus (b \uplus a) = a$,*
(ii) *$a \uplus b = a \uplus c$ implies $b = c$,*
(iii) *$b \uplus a = 0$ implies $a = b$.*

Proof. (i) By 1.1(ii), $(b \uplus a) \uplus (b \uplus a) = 0$, which by 1.1(iv) implies $(b \uplus (b \uplus a)) \uplus a = 0 = a \uplus a$. By 1.1(vi) it holds $b \uplus (b \uplus a) = a$.

(ii) We have $a \uplus b = a \uplus c$, which implies $a \uplus (a \uplus b) = a \uplus (a \uplus c)$ and $b = c$.

(iii) $b \uplus a = 0$ implies $a = b \uplus (b \uplus a) = b \uplus 0 = b$.

PROPOSITION 1.3. *Let X be an abelian RI-set and $a, b, c \in X$. Then:*

(i) *$b \uplus a$ is defined and $b \uplus a = c$ iff $b \uplus c$ is defined and $b \uplus c = a$.*
(ii) *If $c \uplus a, c \uplus b$ are defined and one side of the equality $(c \uplus a) \uplus (c \uplus b) = b \uplus a$ is defined, then both sides are defined and the equality holds.*
(iii) *If $b \uplus a, c \uplus b$ are defined then $(c \uplus a) \uplus (b \uplus a) = c \uplus b$.*
(iv) *If $b \uplus a$ and $0 \uplus (b \uplus a)$ are defined then $a \uplus b$ is defined and $0 \uplus (b \uplus a) = a \uplus b$.*

Proof. (i) It is sufficient to assume that $b \uplus a$ is defined and $b \uplus a = c$. We have by 1.1(ii) $(b \uplus a) \uplus c = 0$, by 1.1(iv) $(b \uplus c) \uplus a = 0$ and by 1.2(iii) $b \uplus c = a$.

(ii) and (iii). Assume first $b \uplus a, c \uplus b$ are defined. Then $b \uplus a = (c \uplus (c \uplus b)) \uplus a = (c \uplus a) \uplus (c \uplus b)$ by 1.2(i) and 1.1(iv). By (i) $(c \uplus a) \uplus (b \uplus a) = c \uplus b$. If $(c \uplus a) \uplus (c \uplus b)$ is defined then it equals $b \uplus a$ by 1.1(iv) and 1.2(i).

(iv) By 1.1(ii) and 1.1(iv) we have $0 \uplus (b \uplus a) = (b \uplus b) \uplus (b \uplus a) = (b \uplus (b \uplus a)) \uplus b$, which by 1.2(i) equals $a \uplus b$.

2. Abelian RI-posets

An *abelian partial semigroup* is a set X with a special element 0 and with a partially defined operation $\uplus$ which satisfies the commutative and associative laws (if one side of these equations is defined) and absorbs a zero element.

DEFINITION 2.1. For an abelian RI-set X we define the partial operation $\uplus$ by

$$a \uplus b = c \quad \text{iff} \quad c \boxminus a = b \text{ exists for } a, b, c \in X.$$

By 1.1(vi), $\uplus$ is well-defined.

PROPOSITION 2.2. *For an abelian RI-set* $(X; \boxminus, 0)$ *the structure* $(X; \uplus, 0)$, *defined in* 2.1, *is an abelian partial semigroup. If all operations* $b \boxminus a$ *are defined then* $(X; \uplus, 0)$ *is an abelian group and* $-$ *coincides with* $\boxminus$.

P r o o f. (i) The commutative law holds for $\uplus$, since $a \uplus b = c$ means that $c \boxminus a = b$, which by 1.3(i) is equivalent to $c \boxminus b = a$ and hence $b \uplus a = c$.

(ii). The associative law holds for $\uplus$, since for $u = (a \uplus b) \uplus c$ we have the following equivalences: $u = c \uplus (a \uplus b)$ by (i), $u \boxminus c = a \uplus b$ by the definition of $\uplus$, $(u \boxminus c) \boxminus a = b$, $(u \boxminus a) \boxminus c = b$ by 1.1(iv), $c \uplus b = u \boxminus a$, which is equivalent to $u = a \uplus (c \uplus b) = a \uplus (b \uplus c)$ by (i).

(iii) Assume all operations $b \boxminus a$ are defined. Then $0 \uplus a = a \uplus 0 = a$, since $a \boxminus a = 0$ by 1.1(ii). By 2.1 we have $-a = 0 \boxminus a$ as inverse, $a \uplus (-a) = 0$. Since $c \boxminus a = b$ means $a \uplus b = c$ and this is equivalent to $c - a = b$, it follows that the operations $\boxminus$ and $-$ coincide.

PROPOSITION 2.3. *Let* $(X; \boxminus, 0)$ *be an abelian RI-set. Assume that for* X *a partial binary operation* $\uplus$ *be defined as in* 2.1 *and for all* $a, b \in X$

(i) *if both* $a \boxminus b$ *and* $b \boxminus a$ *are defined then* $a = b$.

Then the relation $\leq$ *of* X *defined by*

(ii) $a \leq b$ *iff* $b \boxminus a$ *is defined*

is a partial order of X *with least element* 0.

Moreover, for $a, b, c \in X$ *the following hold*

(iii) $a \leq a \uplus b$, $b \leq a \uplus b$,

(iv) $a \leq b$ *implies* $a \uplus c \leq b \uplus c$,

whenever the appropriate $\uplus$ *operations are defined.*

P r o o f. (1) By 1.1(i), 1.1(ii) $0 \leq a, a \leq a$ for all $a \in X$. If $a \leq b$ and $b \leq a$ then, by the assumption (i) it holds $a = b$. The transitivity of $\leq$ follows from 1.1(v).

(2) Assume $a \uplus b = c$. Then $a = c \boxminus b$ and by 1.1(iii) $c \boxminus (c \boxminus b)$ is defined, which implies $c \boxminus b \leq c$. Therefore $a \leq a \uplus b$. Since $b \uplus a = a \uplus b$ holds, we also have $b \leq a \uplus b$.

(3) Assume $a \leq b$ and $a \uplus c$, $b \uplus c$ are defined. Then $b \boxminus a$ is defined and it follows from $b \boxminus (b \boxminus a) = a$ and the definition of $\uplus$ that $(b \boxminus a) \uplus a = b$. Therefore $b \uplus c = ((b \boxminus a) \uplus a) \uplus c = (a \uplus c) \uplus (b \boxminus a) \geq a \uplus c$, by (2).

An abelian RI-set with property 2.3(i) and the partial order defined by 2.3(ii) we call an abelian RI-poset.

PROPOSITION 2.4. *Let for an abelian RI-poset* $(X; \leq, \boxminus, 0)$ *for all* $a, b \in X$, $a \neq b$ *exactly one of the operations* $a \boxminus b, b \boxminus a$ *is defined. Then* X *is a linearly ordered RI-poset with the least element* 0.

Let us recall the notion of a D-poset P, introduced by Kôpka-Chovanec [4]. An equivalent definition was given by Navara-Pták [5]: P is a bounded poset with a partially defined binary difference operation $\ominus$ which satisfies $b \ominus a$ is defined iff $a \leq b$, $a \ominus 0 = a$ for all a and if $a \leq b \leq c$ then $c \ominus b \leq c \ominus a$ and $(c \ominus a) \ominus (c \ominus b) = b \ominus a$.

An example of an abelian RI-poset, which is not a D-poset, is a ring X of subsets of a set E, with E not an element of X, and $B \boxminus A = B - A$ is defined iff $A \subseteq B$ for $A, B \in X$. Here X contains the empty set, is closed under the formation of set theoretical difference $B - A$ and union $A \cup B$, for $A, B \in X$.

PROPOSITION 2.5. *Let* X *be an abelian RI-poset. Assume there exists* $1 \in X$ *such that all* $1 \boxminus a$, $a \in X$, *are defined. Then* X *is a D-poset.*

Proof. Assume $1 \boxminus a = b$, then $a \uplus b = 1$ and $a \leq 1$ for all $a \in X$. 1.1(i) assures that $0 \leq a$. Assume $a \leq b \leq c$. Then $c \boxminus b, b \boxminus a$ and $c \boxminus a$ are defined. By 1.3 (ii), $b \boxminus a = (c \boxminus a) \boxminus (c \boxminus b)$. Hence $c \boxminus b \leq c \boxminus a$ and $\boxminus$ satisfies all properties of $\ominus$ of a D-poset.

In the following proposition we take from [6] the definition of an *RI-semigroup* (relative inverse semigroup) $(X; \leq, \uplus, 0)$, which is a poset with smallest element 0 and $a \uplus 0 = a$, $\uplus$ is partially defined such that the commutative and associative law hold, whenever one side of these equations is defined, $a \leq b$ implies $a \uplus c \leq b \uplus c$, whenever $a \uplus c$, $b \uplus c$ are defined and

(1) for $a \leq b$ there exists a unique $c = b \boxminus a$ such that $a \uplus c = b$.

Examples of RI-semigroups are the positive cone of ordered abelian groups, or orthoalgebras $(X; \oplus, 0, 1)$ (see [1]), if $\leq$ on X is defined by: $a \leq b$ iff there exists $c \in X$ with $a \oplus c = b$. We then define $\ominus$ on X by $c = b \ominus a$, which makes an orthoalgebras also to a D-poset. Then X is an abelian RI-poset (RI-semigroup) replacing $\ominus$ by $\boxminus$ ($\oplus$ by $\uplus$).

An RI-semigroup X is, on the other hand, *an orthoalgebra iff there exists an element $1 \in X$ such that $1 \ominus a$ is defined for all $a \in X$ and $a \leq 1 \ominus a$ implies $a = 0$* (see [5] or [6]). For the reader, not familiar with the axioms for orthoalgebras we add, that $\oplus$ is for them partially defined, satisfies the commutative and associative law, in case the left side of these equations is defined, furthermore, if $p \oplus p$ is defined then $p = 0$, and for every $p \in X$ there exists a unique $q \in X$ with $p \oplus q = 1$.

PROPOSITION 2.6. *Let X be a set. The following conditions are equivalent*:

(i) $(X; \leq, \uplus, 0)$ *is an abelian RI-poset with $\uplus$ defined as in 2.1* (*and $\leq$ defined as in* 2.3).

(ii) $(X; \leq, \uplus, 0)$ *is an RI-semigroup, with $\ominus$ defined for $a \leq b$ by $b \ominus a = c$ iff $b = a \uplus c$.*

Proof. Assume (i). Then for $a \leq b$, $b \ominus a = c$ is defined and $a \uplus c = b$. This, together with the properties of an abelian RI-poset, shown in 2.2, proves (ii).

Assume (ii). From 2.3 we conclude that X is an abelian RI-poset if it is an abelian RI-set. 1.1(i),(ii) follows from $a \uplus 0 = a$ for $a \in X$. We have by the definition of $\ominus$ in (ii) that for $a \leq b$, $b = a \uplus (b \ominus a)$ which means $b \ominus (b \ominus a) = a$. Therefore 1.1(iii) holds. 1.1(iv) is an immediate consequence from the associative and commutative law for X in (ii). If $b \ominus a, c \ominus b$ are defined then $a \leq b \leq c$ and, by the transitivity of $\leq, c \ominus a$ is defined. This shows 1.1(v). In order to prove 1.1(vi), observe that if $c \ominus a = b = d \ominus a$ then $c = a \uplus b = d$.

PROPOSITION 2.7. *Let $(X, \ominus, 0)$ be an abelian RI-set such that $b \ominus a$ is defined for all $a, b \in X$. Let there exist $X_1, X_2 \subseteq X$ with the following properties*:

(i) $X_1 \cup X_2 = X$, $X_1 \cap X_2 = \{0\}$,

(ii) $a \ominus b \in X_1$ *iff* $b \ominus a \in X_2$,

(iii) *if $a \in X_1$ and $b \ominus a \in X_1$ then $b \in X_1$.*

Then the relation $\leq$ of X defined by $a \leq b$ iff $b \ominus a \in X_1$ is the linear order of X.

Proof. For all $a \in X$ we have $a \ominus a = 0 \in X_1$, hence $a \leq a$. If $a \leq b$ and $b \leq a$, then by (ii) $a \ominus b \in X_1 \cap X_2$ and hence $a \ominus b = 0$. It follows $a = b$ by 1.2(iii). If $a \leq b, b \leq c$ then $b \ominus a, c \ominus b \in X_1$ and (since all operations $\ominus$ are defined) by 1.3(ii) we have $(c \ominus a) \ominus (c \ominus b) = b \ominus a \in X_1$. Now by (iii) we obtain $c \ominus a \in X_1$ and thus $a \leq c$. Finally for any $a, b \in X$ we obtain $a \leq b$ or $b \leq a$ by (i).

PROPOSITION 2.8. *Let an abelian RI-poset* $(X;\leq,\uplus,0)$ *have the following properties for all* $a,b,c\in X$:

(i) *There exists* $1\in X$ *and for all* $a\in X$, $a'=1\uplus a$ *is defined.*
(ii) *If* $(1\uplus a)\uplus a$ *is defined then* $a=0$.
(iii) *If* $b\uplus a, b\uplus c$ *and* $(1\uplus a)\uplus c$ *are defined then* $(b\uplus a)\uplus c$ *is defined.*

Then $(X;\leq,',0,1)$ *is an orthomodular poset.*

Proof. X is a D-poset by the Proposition 2.5. In view of (ii) it follows that $a\leq 1\uplus a$ implies $a=0$. Thus by [5] or [6] we obtain that X with orthocomplementation $a'=1\uplus a$ is an orthoposet. Now by [6], Proposition 3.4 it suffices to show that if $a\leq b'$ then the smallest upper bound avb of elements a,b exists in X. Suppose $a\leq b'$. Then $b'\uplus a$ is defined and $b'\uplus a\leq b'$, $b'\uplus a\leq 1\uplus a=a'$. Hence $b'\uplus a$ is a lower bound of elements a',b'. On the other hand if $c\leq a'$, $c\leq b'$ then $b'\uplus c$, $(1\uplus a)\uplus c$ are defined and in view of (iii) also $(b'\uplus a)\uplus c$ is defined. It follows that $c\leq b'\uplus a$ and $b'\uplus a$ is the greatest lower bound of a' and b'. Finally by $a\vee b=(a'\wedge b')'=1\uplus(b'\uplus a)$ we obtain that $a\vee b$ exists in X.

Suppose now that $(X;\leq,',0,1)$ is an orthomodular poset. Let us put for $a,b\in X$ $\quad b\uplus a=b\wedge a'$ iff $a\leq b$. Then $(X;\leq,\uplus,0,1)$ is an abelian RI-poset satisfying conditions (i)–(iii) of the Proposition 2.8. Clearly, $1\uplus a$ is an orthocomplement of $a\in X$.

3. Boolean RI-sets

PROPOSITION 3.1. *Let* $(X;\uplus,0)$ *be an RI-set and the following conditions are satisfied*:

(i) $a\uplus b=0, b\uplus c=0$ *imply* $a\uplus c=0$,
(ii) *if* $b\uplus a=0=a\uplus b$ *then* $a=b$.

Then the relation $\leq$ *of* X *defined by* $a\leq b$ *iff* $a\uplus b=0$ *is a partial order.*

Proof. By 1.1(ii) $a\leq a$ holds. If $a\leq b$, $b\leq a$ then by the assumption (ii) we obtain $a=b$. If $a\leq b$, $b\leq c$ then $a\leq c$ by (i).

DEFINITION 3.2. Let X be an RI-set with $1\in X$. We call $(X;\uplus,0,1)$ a *Boolean RI-set* if the following conditions are satisfied:

(i) $a\uplus b=0, b\uplus c=0$ imply $a\uplus c=0$,
(ii) If $b\uplus a=0=a\uplus b$ then $a=b$,
(iii) $0\uplus a=0$, for all $a\in X$,
(iv) $a'=1\uplus a$ exists for all $a\in X$ and $1\uplus(1\uplus a)=a$.

In every Boolean RI-set X we can introduce a partial order as in 3.1 and then we call X a *Boolean RI-poset.*

An example of a Boolean RI-poset is a Boolean algebra X with the binary operation $\uplus$ defined for all $a, b \in X$ by $a \uplus b = a \wedge b$. An example of a Boolean RI-poset which is neither a Boolean algebra nor an orthoposet is a chain $0 < a < 1$ with the operation $\uplus$ defined as follows:

$$0 \uplus a = 0 \uplus 1 = a \uplus 1 = a \uplus a = 1 \uplus 1 = 0 \uplus 0 = 0,$$
$$1 \uplus a = a \uplus 0 = a, \quad 1 \uplus 0 = 1$$

PROPOSITION 3.3. *Let* $(X; \leq, \uplus, 0, 1)$ *be a Boolean RI-poset. Then for all* $a, b \in X$ *are satisfied*:

(i) $a \uplus b$ *is defined*,
(ii) $(a \uplus b) \uplus a = 0$,
(iii) $a' \uplus b' = b \uplus a$,
(iv) $0' = 1$.

Moreover, if for all $a \in X (1 \uplus a) \uplus a = 0$ *implies* $a = 1$, *then* $(X; \leq, ', 0, 1)$ *is a Boolean orthoposet* (*i.e.* $a \wedge b = 0 \Rightarrow a \leq b'$).

Proof. (i), (ii). Let $a, b \in X$. By 3.2(iii) and 1.1.(ii) we have $0 = 0 \uplus b = (a \uplus a) \uplus b$. Now by 1.1(iv) $(a \uplus b) \uplus a$ is defined and $(a \uplus b) \uplus a = (a \uplus a) \uplus b = 0$.

(iii) For $a, b \in X$ by 3.2(iv) and 1.1(iv) we have

$$a \uplus b = [1 \uplus (1 \uplus a)] \uplus [1 \uplus (1 \uplus b)] = (1 \uplus [1 \uplus (1 \uplus b)]) \uplus (1 \uplus a)$$
$$= (1 \uplus b) \uplus (1 \uplus a) = b' \uplus a'.$$

(iv) $1 = 1 \uplus (1 \uplus 1) = 1 \uplus 0 = 0'$ by 3.2(iv) and 1.1(ii).

For the rest of the proposition we have $0 \leq a \leq 1$ for all $a \in X$, since by 3.2(iii) $0 \uplus a = 0$ and $a \uplus 1 = [1 \uplus (1 \uplus a)] \uplus 1 = (1 \uplus 1) \uplus (1 \uplus a) = 0 \uplus a' = 0$ using 3.2(iv), 1.1(iv) and 3.2(iii). If $a \leq b$ then $a \uplus b = 0 = b' \uplus a'$ implies $b' \leq a'$. Now it is sufficient to show that 1 is the least upper bound of a and a'. Let $a \leq c, a' \leq c$ for $c \in X$. The $c' \leq a \leq c$ implies $c' \uplus c = 0$. We obtain $(1 \uplus c) \uplus c = 0$ which by the assumption implies $c = 1$. We conclude that the mapping $a \to a' = 1 \uplus a$ for all $a \in X$ is an orthocomplementation.

Now let $a, b \in X$ with $a \wedge b = 0$. Then by (ii) we have $(a \uplus b') \uplus a = 0$ which implies $a \uplus b' \leq a$. By (iii) $a \uplus b' = b \uplus a' \leq b$. Thus $a \uplus b' \leq a \wedge b = 0$ and by the definition of $\leq$ we have $a \leq b'$. It follows that X is a Boolean orthoposet.

It is known that a Boolean orthoposet need not be even orthomodular (see [7], Example 7). On the other hand, every Boolean ortholattice is a

Boolean algebra. The next proposition gives the condition under which a Boolean RI-poset is a Boolean algebra.

PROPOSITION 3.4. *Let* $(X; \leq, \uplus, ', 0, 1)$ *be a Boolean RI-poset. If the condition*

(i) $c \uplus b = 0$, $c \uplus (1 \uplus a) = 0$ *imply* $c \uplus (b \uplus a) = 0$ *for all* $a, b, c \in X$

holds, then $(X; \leq, ', 0, 1)$ *is a Boolean algebra.*

Proof. We show in (a) that X is an orthoposet, in (b) we use the assumption (i) to show that $b \uplus a$ is the greatest lower bound of b and a', which by (a) implies that X is an ortholattice. In (c) we show that X is orthomodular and every pair of elements in X commute. This proves that X is a Boolean algebra.

(a) Suppose $(1 \uplus a) \uplus a = 0$. By 1.1(ii) we have $(1 \uplus a) \uplus (1 \uplus a) = 0$ and by the assumption (i), $(1 \uplus a) \uplus (a \uplus a) = 0$. Hence $1 \uplus a = 0$. On the other hand by 3.3(iii) $a \uplus 1 = 1' \uplus a' = (1 \uplus 1) \uplus (1 \uplus a) = 0 \uplus 0 = 0$. Thus by 3.2(ii) $a = 1$ and by the Proposition 3.3, X is an orthoposet.

(b) By 3.3(ii) we have $(b \uplus a) \uplus b = 0$ and $b \uplus a \leq b$ for all $a, b \in X$. By 3.3(iii) we have $b \uplus a = a' \uplus b'$ which is less or equal to a' by 3.3(ii). Hence $b \uplus a$ is a lower bound of b, a'. The assumption (i) (in view of the definition of $\leq$) implies that $b \uplus a$ is the greatest lower bound $b \wedge a'$ of b and a'.

(c) Assume $a \leq b$ and $a' \wedge b = 0$, which implies $a \uplus b = 0$ and $a' \uplus b' = 0$. Therefore $b \uplus a = a' \uplus b' = 0 = a \uplus b$ and $a = b$ by 3.2(ii). Hence X is orthomodular (see [2]). Finally, we show that all $a, b \in X$ commute (aCb in the form of [2]), i.e. $a \wedge b = b \wedge (b' \vee a)$. It is $a \wedge b \leq b \wedge (b' \vee a)$. In order to show the reverse inequality, observe that $(b \uplus (b \uplus a)) \uplus a = (b \uplus a) \uplus (b \uplus a) = 0$ implies $b \uplus (b \uplus a) \leq a$. By (b) we also have $b \uplus (b \uplus a) \leq b$. Therefore $b \wedge (b' \vee a) = b \uplus ((b' \vee a)') = b \uplus (b \vee a') = b \uplus (b \uplus a) \leq a \wedge b$.

Note that a Boolean algebra $(X; \leq, ', 0, 1)$ with the binary operation $\uplus$ defined for all $a, b \in X$ by $a \uplus b = a \wedge b'$ is a Boolean RI-poset, satisfying the condition 3.4(i).

4. Subalgebras, products and homomorphisms

We define in the class of RI-sets the notions of subalgebras, products and homomorphisms. These definitions demonstrate how strong unifying is the notion of relative inverse (resp. the RI-set operation $\uplus$) for many algebraic structures (see examples in 1.1) which are basic namely in classical and noncompatible measure theory. Partial orderings, lattice operations, orthocomplementations and some other characteristic notions of those structures can be introduced by relative inverse operation $\uplus$.

DEFINITION 4.1. Let X be an RI-set. We call a subset $\emptyset \neq Y \subseteq X$ a *subalgebra* of X if for all $a.b \in Y$ with $b \uplus a$ defined in X it holds that $b \uplus a \in Y$. If $1 \in X$ then we claim that also $1 \in Y$.

Evidently for every subalgebra Y of a RI-set X we have $0 = a \uplus a \in Y$. If the RI-set X is an abelian RI-set then clearly every subalgebra Y of X (in the sense 4.1) is also an abelian RI-set. Since for all $a, b \in Y, b \uplus a$ exists in Y iff $b \uplus a$ exists in X, the restriction of the partial order of an abelian RI-poset X to a subalgebra Y of X is a partial order which makes Y an abelian RI-poset. In view of the Proposition 2.5, if X is a D-poset then Y is also a D-poset and by the Proposition 2.8 if X is an orthomodular poset then so is Y. Further it is easy to see that if X is a linearly ordered abelian RI-set then Y is also a linearly ordered abelian RI-poset (see Proposition 2.4). Finally from the propositions of the part 3. of this article it is clear that if RI-set X is a Boolean poset (Boolean orthoposet, Boolean algebra) then every subalgebra Y of X defined by 4.1 is a Boolean poset (Boolean orthoposet, Boolean algebra).

DEFINITION 4.2. For RI-sets $(X_i, \uplus_i, O_i), i \in I$ we define on the cartesian product $X = \prod_{i \in I} X_i$ the element $O = (O_i)_{i \in I}$ as zero element and the partial binary operation $\uplus$ by

$$(b_i)_{i \in I} \uplus (a_i)_{i \in I} = (c_i)_{i \in I} \text{ iff all } b_i \uplus_i a_i = c_i \text{ exist in } X_i, i \in I.$$

Then we call $(X; \uplus, 0)$ the *direct product of RI-sets* $X_i, i \in I$.

Evidently rules 1.1(i)–(v) hold then for the direct product $X = \prod_{i \in I} X_i$. If X_i, $i \in I$, are RI-posets (abelian or Boolean) $(a_i)_{i \in I} \leq (b_i)_{i \in I}$ iff $a_i \leq b_i$ in X_i for all $i \in I$ (see the Propositions 2.3 and 3.1). If all $(X_i)_{i \in I}$ have unit elements $1_i \in X_i$ then we put $1 = (1_i)_{i \in I}$ as the unit element in X. Now, it is clear that if all $X_i, i \in I$ are abelian RI-sets (abelian RI-posets, D-posets, orthomodular posets, Boolean RI-posets, Boolean orthoposets, Boolean algebras) then the same is their direct product $X = \prod_{i \in I} X_i$ defined by 4.2. (see the Propositions in part 2 and 3 of this article).

DEFINITION 4.3. Let $(X_1; \uplus_1, 0), (X_2; \uplus_2, 0)$ be RI-sets, $h : X_1 \to X_2$ be a map. We call h a *homomorphism* if for all $a, b \in X_1$ with $b \uplus_1 a$ defined in X_1 it holds that $h(b) \uplus_2 h(a)$ is defined in X_2 and $h(b \uplus_1 a) = h(b) \uplus_2 h(a)$.

From the definition of a homomorphism it is clear that for $0 \in X_1$ it holds $h(0) = h(a \uplus_1 a) = h(a) \uplus_2 h(a) = 0 \in X_2$. Moreover if X_1, X_2 are RI-posets (abelian or Boolean) then for all $a, b \in X_1$ with $a \leq b$ we have $h(a) \leq h(b)$ in X_2 (see Propositions 2.3, 3.1). The identity map and also the projection maps which map a direct product of RI-sets onto its factors are homomorphisms.

DEFINITION 4.4. Let X_1, X_2 be RI-sets, $h : X_1 \to X_2$ be a map. We call h an *isomorphism* if it is bijective and h and its inverse map are homomorphisms.

In view of Propositions of the part 2 and 3 of this article it is not difficult to show that if two RI-sets X_1, X_2 are isomorphic in the sense of 4.4 then if X_1 is an abelian RI-set (abelian RI-poset, D-poset, orthomodular poset, orthoposet, Boolean algebra) then so is X_2.

5. Concluding remarks

We add some remarks on homomorphisms and additive maps.

As we have shown in part 2 for an abelian RI-set X we can introduce a partially defined commutative semigroup operation $\uplus$ by $a \uplus b = c$ iff $c \ominus a = b$ exists in $X; a, b, c \in X$. Evidently for abelian RI-sets $(X_1; \ominus_1, O_1), (X_2; \ominus_2, O_2)$ a map $h : X_1 \to X_2$ is a homomorphism iff for all $a, b \in X_1$ with $a \uplus_1 b$ defined the operation $h(a) \uplus_2 h(b)$ is defined and $h(a \uplus_1 b) = h(a) \uplus_2 h(b)$. Moreover we have shown that $h(0) = 0$. A map $h : X_1 \to X_2$ with last two properties we call an *additive map*. Clearly, if X_1 and X_2 are abelian RI-posets then h is also monotone. For *σ-additivity* of h we claim an additional convergence property for ascending countable chains $(a_n)_{n \in N}$ of $X : a_n \uparrow a$ implies $h(a_n) \uparrow h(a)$ (see [3]).

Many cases of additive or σ-additive maps from rings or algebras of sets, quantum logics, Boolean algebras and D-posets into real numbers or some partially ordered spaces (partially ordered semigroups, lattice ordered groups, orthomodular posets) occur this way. For instance measures and probabilities in classical measure theory and observables, states and probabilities in noncompatible probability theory are examples. For an abelian RI-poset X which is a lattice, one may to introduce the notion of valuation from X into an abelian RI-poset Y.

Some of these questions will be discussed in a further paper of one of the authors.

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