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GALOIS CONNECTIONS AND FORMAL CONCEPT ANALYSIS

Dedicated to Professor Tadeusz Traczyk

1. Preliminaries

Let $(P, \leq_P)$ anf $(Q, \leq_Q)$ be two ordered sets. A pair $\varphi, \psi)$ of mappings $\varphi : P \to Q$ and $\psi : Q \to P$ is called a *Galois connection* between these two ordered sets [6] if:
(gc1) $x \leq_P y \Rightarrow \varphi(y) \leq_Q \varphi(x)$ for all $x, y \in P$;
(gc2) $u \leq_Q v \Rightarrow \psi(v) \leq_P \psi(u)$ for all $u, v \in Q$;
(gc3) $p \leq_P \psi(\varphi(p))$ and $q \leq_Q \varphi(\psi(q))$ for all $p \in P$, $q \in Q$.

A Galois connection can be also characterized by only one condition [2, 7]: the pair (φ, ψ) is a Galois connection iff
(gc4) $p \leq_P \psi(q) \Leftrightarrow q \leq_Q \varphi(p)$ for all $p \in P$, $q \in Q$.

Several observations of Galois connections have been made in [5, 6, 7] and a characterization of Galois connections between complete lattices appears in Shmuely [8]. In this paper we will interpret the Galois connections between ordered sets as formal concepts of a context. For the Galois connections between complete lattices a more efficient description than the Shmuely's characterization will be given by introducing Galois relations or Galois bonds. Before dealing with this some definitions and notations from formal concept analysis which was introduced by Wille [3, 10] are needed.

A (formal) *context* is a triple $\mathbb{K} = (G, M, I)$ consisting of two sets, G and M, and a binary relation $I \subseteq G \times M$ between G and M. Usually, the

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elements of G are called *objects*, those of M *attributes*. For $(g,m) \in I$ we write gIm and read it: the object g has the attribute m. A small context can convenient be represented by a cross-table whose rows are labeled with objects, columns with attributes, and in which there is a cross in row g and column m iff gIm. For a given context $\mathbb{K} = (G, M, I)$ we define for $A \subseteq G$ and $B \subseteq M$

$$A^I := \{m \in M \mid gIm \text{ for all } g \in A\} \text{ and } B^I := \{g \in G \mid gIm \text{ for all } m \in B\}.$$

Instead of $\{x\}^I$ we simply write x^I for $x \in G$ or $x \in M$. The two operators $A \mapsto A^I$ and $B \mapsto B^I$ form a Galois connection between the powerset lattices $\underline{\mathfrak{B}}(G)$ and $\underline{\mathfrak{B}}(M)$, and the two compositions $A \mapsto A^{II}$ and $B \mapsto B^{II}$ of these operators are dually isomorphic closure operators on G and M, respectively. For $A \subseteq G$, if $A = A^{II}$ then A is closed under the closure operator on G and is said to be an *extent* of the context, and a subset $B \subseteq M$ with $B = B^{II}$ is called an *intent* of the context. A pair (A, B) is said to be a (formal) *concept* with extent A and intent B of $\mathbb{K}$ iff $A \subseteq G$, $B \subseteq M$, $A^I = B$ and $B^I = A$. It is easy to see that each of A and B in a concept (A, B) is uniquely determined by the other. The system of all concepts of a context $\mathbb{K}$ can be hierachically ordered by

$$(A_1, B_1) \leq (A_2, B_2) :\Leftrightarrow A_1 \subseteq A_2 \ (\Leftrightarrow B_2 \subseteq B_1)$$

and forms a complete lattice denoted by $\underline{\mathfrak{B}}(\mathbb{K})$ or $\underline{\mathfrak{B}}(G, M, I)$ and called (formal) concept lattice of the context $\mathbb{K}$ in which infima and suprema can be described as follows:

$$\bigwedge_{t \in T} (A_t, B_t) = \Big(\bigcap_{t \in T} A_t, \Big(\bigcup_{t \in T} B_t\Big)^{II}\Big),$$

$$\bigvee_{t \in T} (A_t, B_t) = \Big(\Big(\bigcup_{t \in T} A_t\Big)^{II}, \bigcap_{t \in T} B_t\Big).$$

Moreover, every complete lattice $\mathbb{V} = (V, \leq)$ is isomorphic to the concept lattice of some suitable context, in particular, $\mathbb{V} \cong \underline{\mathfrak{B}}(V, V, \leq)$ [10].

2. Galois connections as concepts

In this section we interpet the Galois connections between ordered sets as concepts of a context. Let $(P, \leq_P)$ and $(Q, \leq_Q)$ be two ordered sets. We now construct a context $(P \times Q, Q \times P, I)$ by defining, for all $x, y \in P$ and $u, v \in Q$,

$$(x, v) I (u, y) :\Leftrightarrow (x \leq_P y \Leftrightarrow u \leq_Q v).$$

The relation I can be easy generated from the ordered sets. For an object

$(p,q) \in P \times Q$ of the context we have

$$(p,q)^I = ((q] \times [p)) \cup ((Q \setminus (q]) \times (P \setminus [p)))$$

and for an attribute $(q,p) \in Q \times P$ it is

$$(q,p)^I = ((p] \times [q)) \cup ((P \setminus (p]) \times (Q \setminus [q))),$$

where $(x]$ and $[y)$ denote the principal ideal generated by x and the principal filter generated by y in the ordered sets, respectively.

THEOREM 1. *Let $(P, \leq_P)$ and $(Q, \leq_Q)$ be ordered sets and let the context $(P \times Q, Q \times P, I)$ be constructed as above. Suppose that $\varphi : P \to Q$ and $\psi : Q \to P$ are mappings. Then the pair (φ, ψ) is a Galois connection between $(P, \leq_P)$ and $(Q, \leq_Q)$ iff (φ, ψ) is a concept of the context $(P \times Q, Q \times P, I)$. (Here a mapping $\Psi : X \to Y$ is also understood as a subset $\Psi = \{(x, \Psi(x)) |\ x \in X\} \subseteq X \times Y$.)*

Proof. Assume that the pair (φ, ψ) is a concept of the context $(P \times Q, Q \times P, I)$. Then we have

$$(p, \varphi(p)) I (q, \psi(q)) \text{ for all } p \in P, q \in Q,$$

that means, by the construction of I,

$$p \leq_P \psi(q)) \Leftrightarrow q \leq_Q \varphi(p) \text{ for all } p \in P, q \in Q,$$

but this equivalence is just the characterizing condition (gc4) for a Galois connection between $P, \leq_P)$ and $(Q, \leq_Q)$. Conversely, if (φ, ψ) is a Galois connection, then we have, by the construction of I,

$$(p, \varphi(p)) I (q, \psi(q)) \text{ for all } p \in P, q \in Q.$$

From this it follows that $\varphi \subseteq \psi^I$ and $\psi \subseteq \varphi^I$. Now, we show that (φ, ψ) is a concept of the context $P \times Q, Q \times P, I)$. Assuming $(p,q) \in \psi^I$, we have $(p,q) I (q, \psi(q))$, or equivalently, $p \leq_P \psi(q) \Leftrightarrow q \leq_Q q$. This implies $p \leq_P \psi(q)$ and therefore, by the condition (gc4), also $q \leq_Q \varphi(p)$. On the other hand, it follows from $(\varphi(p), \psi(p))) \in \psi$ that

$$(p,q) I (\varphi(p), \psi(\varphi(p))),$$

i.e.

$$p \leq_P \psi(\varphi(p)) \Leftrightarrow \varphi(p) \leq_Q q.$$

By the condition (gc3), we get $\varphi(p) \leq_Q q$. Therefore, $q = \varphi(p)$ is true. This shows $(p,q) = (p, \varphi(p)) \in \varphi$. So, we have $\varphi = \psi^I$. Analogously, one can prove $\psi = \varphi^I$. Thus, the pair (φ, ψ) is a concept of the context $(P \times Q, Q \times P, I)$. ∎

A similar manner of the interpretation in this Theorem appears in an author's earlier paper [3] in which the (fixed point free and) order-preserving self-mappings of an ordered set were characterized as concepts of a context. Using this characterization, a practicable algorithm for determining if a given finite ordered set has the fixed point property was given.

3. Contextual Galois connections

In this section we introduce, as a natural generalization of Galois connection between ordered sets, the contextual Galois connection between contexts, and then we show that every contextual Galois connection between two contexts can be uniquely extended to a Galois connection between the concept lattices of these two contexts. This implies immediately a result of Shmuely [8] that every Galois connection between ordered sets can be uniquely extended to a Galois connection between their Dedekind-MacNeille completions [1, 4] (the completions by cuts of the ordered sets).

Let $\mathbb{K}_1 = (G_1, M_1, I_1)$ and $\mathbb{K}_2 = (G_2, M_2, I_2)$ be contexts. A pair (ξ, η) of mappings $\xi : G_1 \to M_2$ and $\eta : G_2 \to M_1$ is said to be a *contextual Galois connection* between $\mathbb{K}_1$ and $\mathbb{K}_2$, if

$$gI_1\eta(h) \Leftrightarrow hI_2\xi(g) \text{ for all } g \in G_1, h \in G_2 .$$

Every ordered set $(P, \leq)$ corresponds, in a natural way, a context $(P, P, \leq)$. By the definition of contextual Galois connection, we have

COROLLARY 1. *A pair (φ, ψ) of mappings $\varphi : P \to Q$ and $\psi : Q \to P$ is a Galois connection between ordered sets $(P, \leq_P)$ and $(Q, \leq_Q)$ iff it is a contextual Galois connection between the contexts $(P, P, \leq_P)$ and $(Q, Q, \leq_Q)$.* ∎

One can also recognize, as the Galois connections between ordered sets, the contextual Galois connections between contexts in a suitable context. For this we call a context (G, M, I) *purified*, if the mappings $g \mapsto g^I (g \in G)$ and $m \mapsto m^I (m \in M)$ are injective. In non-purified contexts, there are different objects which have the same set of attributes, or there are different attributes which are possessed by the same set of objects.

THEOREM 2. *Let $\mathbb{K}_1 = (G_1, M_1, I_1)$ and $\mathbb{K}_2 = (G_2, M_2, I_2)$ be purified contexts. Let J be a relation between $G_1 \times M_2$ and $G_2 \times M_1$ defined by*

$$(g, n)J(h, m) :\Leftrightarrow (gI_1m \Leftrightarrow hI_2n) .$$

Then, for two mappings $\xi : G_1 \to M_2$ and $\eta : G_2 \to M_1$, the following are equivalent:

(1) *(ξ, η) is a contextual Galois connection between $\mathbb{K}_1$ and $\mathbb{K}_2$;*

(2) *(ξ, η) is a concept of the context $(G_1 \times M_2, G_2 \times M_1, J)$.*

Proof. The implication (2) $\Rightarrow$ (1) follows immediately from the definition of J. Now, we show (1) $\Rightarrow$ (2). Assume that (ξ, η) is a contextual Galois connection between $\mathbb{K}_1$ and $\mathbb{K}_2$. Then, by the definition of J, we get $\xi \subseteq \eta^J$ and $\eta \subseteq \xi^J$. If $(g,n) \in \eta^J$ then $(g,n)J(h,\eta(h))$ for all $h \in G_2$. On the other hand, $(g,\xi(g)) \in \xi \subseteq \eta^J$ implies $(g,\xi(g))J(h,\eta(h))$ for all $h \in G_2$. So we have, for all $h \in G_2$,

$$hI_2n \Leftrightarrow gI_1\eta(h) \Leftrightarrow hI_22\xi(g),$$

therefore $n^{I_2} = \xi(g)^{I_2}$. Since $\mathbb{K}_2$ is purified, $n = \xi(g)$ follows. Therefore, $(g,n) = (g,\xi(g)) \in \xi$. Thus, we have proved $\xi = \eta^J$. The other equation $\eta = \xi^J$ can be analogously shown. And this yields the statement (2). The proof is completed. ■

LEMMA 1. *A pair (ξ,η) of mappings $\xi : G_1 \to M_2$ and $\eta : G_2 \to M_1$ is a contextual Galois connection between (G_1,M_1,I_1) and (G_2,M_2,I_2) iff, for all subsets $A \subseteq G_1$ and $C \subseteq G_2$,*

$$A \subseteq \eta(C)^{I_1} \Leftrightarrow C \subseteq \xi(A)^{I_2},$$

where $\xi(A) := \{\xi(g) | g \in A\}$ and $\eta(C) := \{\eta(h) | h \in C\}$.

Proof. If the equivalence $A \subseteq \eta(C)^{I_1} \Leftrightarrow C \subseteq \xi(A)^{I_2}$ is true for all subsets $A \subseteq G_1$ and $C \subseteq G_2$, then it is also true for the special subsets $A = \{g\}$ $(g \in G_1)$ and $C = \{h\}$ $(h \in G_2)$. Therefore, we have $\{g\} \subseteq \eta(h)^{I_1} \Leftrightarrow \{h\} \subseteq \xi(g)^{I_2}$, this is equivalent to $gI_1\eta(h) \Leftrightarrow hI_2\xi(g)$. The last one is just the definition for the contextual Galois connection (ξ,η).

Conversely, let (ξ,η) be a contextual Galois connection between (G_1,M_1,I_1) and (G_2,M_2,I_2). Let $A \subseteq G_1$ and $C \subseteq G_2$ be subsets. For every $h \in C$, it follows obviously from $A \subseteq \eta(C)^{I_1}$ that $A \subseteq \eta(h)^{I_1}$, that means $gI_1\eta(h)$ for all $g \in A$. This implies $hI_2\xi(g)$ for all $g \in A$, and therefore $h \in \xi(A)^{I_2}$. Thus, the implication $A \subseteq \eta(C)^{I_1} \Rightarrow C \subseteq \xi(A)^{I_2}$ has been proved. The converse implication $C \subseteq \xi(A)^{I_2} \Rightarrow A \subseteq \eta(C)^{I_1}$ can be analogously shown.

Let $(P, \leq_P)$ be an ordered set. Then the Dedekind-MacNeille completion of it is the concept lattice $\mathfrak{B}(P,P,\leq_P)$, and, by $\iota(p) := ((p],[p))$, an order embedding ι of $(P,\leq_P)$ into $\mathfrak{B}(P,P,\leq_P)$ is defined [3]. Shmuely has proved that every Galois connection between two ordered sets $(P,\leq_P)$ and $(Q,\leq_Q)$ can be uniquely extended to a Galois connection between their Dedekind-MacNeille completions [8]. We will now show that the similar observation can be made for contextual Galois connections and get the Shmuely's result as a special case of the following Theorem 3. For this we need at first some notations and known results. Let (G,M,I) be a context. Define $\gamma(g) := (g^{II},g^{I})(g \in G)$ and $\mu(m) := (m^I,m^{II})(m \in M)$. Call $\gamma(g)$ the object-concept with respect to the object g and $\mu(m)$ the attribute-concept with respect to the attribute m. If the context (G,M,I) is purified, the mappings $g \mapsto \gamma(g)$ and $m \mapsto \mu(m)$ are injective and we can identity g with $\gamma(g)$ and

m with $\mu(m)$, respectively. Without proof, we recall the following Lemma that describes the characterization of Galois connections between complete lattices [7].

LEMMA 2. *Let $(V, \leq_V)$ and $(W, \leq_W)$ be complete lattices. For mappings $\varphi : V \to W$ and $\psi : W \to V$, the following statements are equivalent:*

(1) (φ, ψ) is a Galois connection between $(V, \leq_V)$ and $(W, \leq_W)$;

(2) $\varphi(\bigvee_{t\in T} x_t) = \bigwedge_{t \in T} \varphi(x_t)$ for $x_t \in V$ and $\bigvee\{x \in V | y \leq \varphi(x)\} = \psi(y)$ for $y \in W$;

(3) $\psi) \bigvee_{s\in S} y_s) = \bigwedge_{s\in S} \psi(y_s)$ for $y_s \in W$ and $\bigvee\{y \in W | x \leq \psi(y)\} = \varphi(x)$ for $x \in V$.

THEOREM 3. *Let $\mathbb{K}_1 = (G_1, M_1, I_1)$ and $\mathbb{K}_2 = (G_2, M_2, I_2)$ be two purified contexts. Then every contextual Galois connection between $\mathbb{K}_1$ and $\mathbb{K}_2$ can be uniquely extended to a Galois connection between the concept lattices $\underline{\mathfrak{B}}(\mathbb{K}_1)$ and $\underline{\mathfrak{B}}(\mathbb{K}_2)$.*

Proof. Suppose that (ξ, η) is a contextual Galois connection between $\mathbb{K}_1$ and $\mathbb{K}_2$. We define now two mappings $\varphi_\xi : \underline{\mathfrak{B}}(\mathbb{K}_1) \to \underline{\mathfrak{B}}(\mathbb{K}_2)$ and $\psi_n : \underline{\mathfrak{B}}(\mathbb{K}_2) \to \underline{\mathfrak{B}}(\mathbb{K}_1)$ by

$$\varphi_\xi(A, B) := (\xi(A)^{I_2}, \xi(A)^{I_2 I_2}) \text{ for } (A, B) \in \underline{\mathfrak{B}}(\mathbb{K}_1)$$

and

$$\psi_\eta(C, D) := (\eta(C)^{I_1}, \eta(C)^{I_1 I_1}) \text{ for } (C, D) \in \underline{\mathfrak{B}}(\mathbb{K}_2)\,.$$

For the showing that (φ_ξ, ψ_η) is a Galois connection between $\underline{\mathfrak{B}}(\mathbb{K}_1)$ and $\underline{\mathfrak{B}}(\mathbb{K}_2)$ it is sufficient to verify, for $(A, B) \in \underline{\mathfrak{B}}(\mathbb{K}_1)$ and $(C, D) \in \underline{\mathfrak{B}}(\mathbb{K}_2)$,

$$(A, B) \leq \psi_\eta(C, D) \Leftrightarrow (C, D) \leq \varphi_\xi(A, B)\,,$$

or equivalently,

$$A \subseteq \eta(C)^{I_1} \Leftrightarrow C \subseteq \xi(A)^{I_2}\,.$$

By Lemma 1, the last equivalence is true. Now, we show the Galois connection (φ_ξ, ψ_η) is an extension of (ξ, η). For $g \in G_1$ we have

$$\varphi_\xi(g^{I_1 I_1}, g^{I_1}) = (\xi(g^{I_1 I_1})^{I_2}, \xi(g^{I_1 I_1})^{I_2 I_2})\,.$$

Obviously, $\xi(g^{I_1 I_1})^{I_2} \subseteq \xi(g)^{I_2}$ follows from $\xi(g) \in \xi(g^{I_1 I_1})$. On the other hand, with $h \in \xi(g)^{I_2}$ we have $g \in \eta(h)^{I_1}$ and therefore $g^{I_1 I_1} \subseteq \eta(h)^{I_1}$, that yields, by Lemma 1, $h \in \xi(g^{I_1 I_1})^{I_2}$. Thus, $\xi(g)^{I_2} = \xi(g^{I_1 I_1})^{I_2}$, and that means

$$\varphi_\xi(g^{I_1 I_1}, g^{I_1}) = (\xi(g)^{I_2}, \xi(g)^{I_2 I_2}) \text{ with } \xi(g) \in M_2\,.$$

Similary, we can show for $h \in G_2$

$$\psi_\eta(h^{I_2 I_2}, h^{I_2}) = (\eta(h)^{I_1}, \eta(h)^{I_1 I_1}) \text{ with } \eta(h) \in M_1\,.$$

Therefore, identifying $g \in G_1$ with $\gamma_1(g) := (g^{I_1 I_1}, g^{I_1})$ and $h \in G_2$ with $\gamma_2(h) := (h^{I_2 I_2}, h^{I_2})$, it follows that (φ_ξ, ψ_η) is an extension of (ξ, η). The uniqueness of the extension can be shown by using Lemma 2 and the fact that

$$(A,B) = \bigvee_{g \in A} \gamma_1(g) \text{ for } (A,B) \in \underline{\mathfrak{B}}(\mathbb{K}_1),$$

$$(C,D) = \bigvee_{h \in C} \gamma_2(h) \text{ for } (C,D) \in \underline{\mathfrak{B}}(\mathbb{K}_2). \blacksquare$$

As a consequence of this Theorem we have the following

COROLLARY 2. *A Galois connection (φ, ψ) between $\underline{\mathfrak{B}}(G_1, M_1, I_1)$ and $\underline{\mathfrak{B}}(G_2, M_2, I_2)$ is an extension of some contextual Galois connection between $\mathbb{K}_1 = (G_1, M_1, I_1)$ and $\mathbb{K}_2 = (G_2, M_2, I_2)$ iff $\varphi(\psi)$ maps the object-concepts of $\mathbb{K}_1(\mathbb{K}_2)$ to the attribute-concepts of $\mathbb{K}_2(\mathbb{K}_1)$.*

Proof. In the proof of Theorem 3 we have seen that each of the two mappings φ_ξ and ψ_η of the unique extension (φ_ξ, ψ_η) of any contextual Galois connection (ξ, η) maps the object-concepts of the one context to the attribute-concepts of the other context. Suppoing that the mappings $\varphi : \underline{\mathfrak{B}}(\mathbb{K}_1) \to \underline{\mathfrak{B}}(\mathbb{K}_2)$ and $\psi : \underline{\mathfrak{B}}(\mathbb{K}_2) \to \underline{\mathfrak{B}}(\mathbb{K}_1)$ satisfy the sufficient conditions, we define now two mappings $\xi_\varphi : G_1 \to M_2$ and $\eta_\psi : G_2 \to M_1$ by

$$\xi_\varphi(g) := n \text{ if } \varphi(g^{I_1 I_1}, g^{I_1}) = (n^{I_2}, n^{I_2 I_2}),$$
$$\eta_\psi(h) := m \text{ if } \psi(h^{I_2 I_2}, h^{I_2}) = (m^{I_1}, m^{I_1 I_1}).$$

Then it is easy to verify that the pair (ξ_φ, η_ψ) is a contextual Galois connection between $\mathbb{K}_1$ and $\mathbb{K}_2$ and that (φ, ψ) is the unique extension of (ξ_φ, η_ψ). $\blacksquare$

4. Characterization of Galois connection between concept lattices

Galois connections between complete lattice were interpreted by G-ideals. We recall here some results from [8]. Let $(V, \leq)$ and $(W, \leq)$ be complete lattices. A subset $J \subseteq V \times W$ is said to be a *G-ideal* of the direct product $(V \times W, \leq)$, if

(gi1) $(a,b) \in J$ and $(x,y) \leq (a,b)$ implies $(x,y) \in J$;

(gi2) $\{(a_i, b_i) \mid i \in S\} \subseteq J \Rightarrow \left(\bigvee_{i \in S} a_i, \bigwedge_{i \in S} b_i\right), \left(\bigwedge_{i \in S} a_i, \bigvee_{i \in S} b_i\right) \in J$.

It is easy to see that (gi2) yields $(0_V, 1_W), (1_V, 0_W) \in J$. In [8] it was shown that the set of all G-ideals of $(V \times W, \leq)$, ordered by the set inclusion, forms a complete lattice which is isomorphic to the complete lattice of all Galois connections between $(V, \leq)$ and $(W, \leq)$. The relationships between G-ideals and Galois connections can be described as follows: For a Galois connection

(φ, ψ) the subset

$$J := \{(a,b) | \, b \leq \varphi(a)\} \subseteq V \times W$$

is a G-ideal; and for a G-ideal $J \subseteq V \times W$ a Galois connection (φ, ψ) will be defined by

$$\varphi(a) := \bigvee \{b | \, (a,b) \in J\} \quad \text{and } \psi(b) := \bigvee \{a | \, (a,b) \in J\}\,.$$

In order to judge whether a subset $J \subseteq V \times W$ is a G-ideal we should check that, firstly, J is indeed an order ideal of direct product and that, secondly, every subset of J is *compatible* to the $\bigvee - \bigwedge$ and $\bigwedge - \bigvee$ construction. This is obviously not easy to do even for finite lattices. In this section we will give a new interpretation of Galois connections between complete lattices by considering concept lattices. This interpretation follows by introducing Galois bonds and Galois relations between contexts which should be conveniently treated on the context plane, especially for the finite case.

Let $\mathbb{K}_1 = (G_1, M_1, I_1)$ and $\mathbb{K}_2 = (G_2, M_2, I_2)$ be contexts. A pair (I_{12}, I_{21}) of relations $I_{12} \subseteq G_1 \times M_2$ and $I_{21} \subseteq G_2 \times M_1$ is said to be a *weak Galois bond* between $\mathbb{K}_1$ and $\mathbb{K}_2$, if for all $g \in G_1$, $h \in G_2$ the condition

$$h^{I_{21}} \subseteq g^{I_1} \Leftrightarrow g^{I_{12}} \subseteq h^{I_2}$$

is satisfied. The word "bond" has appeared in a Wille's paper on complete subdirect product construction of concept lattices in which a bond **from** $\mathbb{K}_1$ **to** $\mathbb{K}_2$ is defined as a subset $J \subseteq G_1 \times M_2$ such that g^J is an intent of $\mathbb{K}_2$ $(g \in G_1)$ and n^J is an extent of $\mathbb{K}_1$ $(n \in M_2)$; i.e. the extents of the context (G_1, M_2, J) are extents of (G_1, M_1, I_1) and the intents of the context (G_1, M_2, J) are intents of (G_2, M_2, I_2) [11]. We call here the pair (I_{12}, I_{21}) weak Galois bond between $\mathbb{K}_1$ and $\mathbb{K}_2$, because the Galois connections between $\underline{\mathfrak{B}}(\mathbb{K}_1)$ and $\underline{\mathfrak{B}}(\mathbb{K}_2)$ can be characterized by such pairs.

THEOREM 4. *Let* $\mathbb{K}_1 = (G_1, M_1, I_1)$ *and* $\mathbb{K}_2 = (G_2, M_2, I_2)$ *be contexts.*

(1) *Every weak Galois bond* (I_{12}, I_{21}) *between* $\mathbb{K}_1$ *and* $\mathbb{K}_2$ *induces a Galois connection* $(\varphi_{I_{12}}, \psi_{I_{21}})$ *between* $\underline{\mathfrak{B}}(\mathbb{K}_1)$ *and* $\underline{\mathfrak{B}}(\mathbb{K}_2)$ *by defining*

$$\varphi_{I_{12}} : \underline{\mathfrak{B}}(\mathbb{K}_1) \to \underline{\mathfrak{B}}(\mathbb{K}_2), \quad (A, B) \mapsto \bigwedge_{g \in A} (g^{I_{12}I_2}, g^{I_{12}I_2I_2});$$

$$\varphi_{I_{21}} : \underline{\mathfrak{B}}(\mathbb{K}_2) \to \underline{\mathfrak{B}}(\mathbb{K}_1), \quad (C, D) \mapsto \bigwedge_{h \in C} (h^{I_{21}I_1}, h^{I_{21}I_1I_1});$$

(2) *For every Galois connection* (φ, ψ) *between* $\underline{\mathfrak{B}}(\mathbb{K}_1)$ *and* $\underline{\mathfrak{B}}(\mathbb{K}_2)$ *there exists a weak Galois bond* (I_φ, I_ψ) *between* $\mathbb{K}_1$ *and* $\mathbb{K}_2$ *with*

$$\varphi(A, B) = \bigwedge_{g \in A} (g^{I_\varphi I_2}, g^{I_\varphi I_2 I_2}) \textit{ for } (A, B) \in \underline{\mathfrak{B}}(\mathbb{K}_1)\,,$$

$$\psi(C,D) = \bigwedge_{h\in C} (h^{I_\psi I_1}, h^{I_\psi I_1 I_1}) \text{ for } (C,D) \in \underline{\mathfrak{B}}(\mathbb{K}_2).$$

Proof. (1) Let (I_{12}, I_{21}) be a weak Galois bond between $\mathbb{K}_1$ and $\mathbb{K}_2$. To prove that $(\varphi_{I_{12}}, \psi_{I_{21}})$ is a Galois connection between $\underline{\mathfrak{B}}(\mathbb{K}_1)$ and $\underline{\mathfrak{B}}(\mathbb{K}_2)$ it is enough to show that for all $(A,B) \in \underline{\mathfrak{B}}(\mathbb{K}_1)$ and $(C,D) \in \underline{\mathfrak{B}}(\mathbb{K}_2)$

$$(A,B) \leq \psi_{I_{21}}(C,D) \Leftrightarrow (C < D) \leq \varphi_{I_{12}}(A,B),$$

or equivalently,

$$A \subseteq \bigcap_{h\in C} h^{I_{21}I_1} \Leftrightarrow C \subseteq \bigcap_{g\in A} g^{I_{12}I_2}.$$

Assuming $A \subseteq \bigcap_{h\in C} h^{I_{21}I_1}$, it follows from $h_1 \in C$ that $A \subseteq h_1^{I_{21}I_1}$. This implies $h_1^{I_{21}} \subseteq A^{I_1} = \bigcap_{g\in A} g^{I_1}$. Therefore, we have $h_1^{I_{21}} \subseteq g^{I_1}$ for all $g \in A$, and this yields $g^{I_{12}} \subseteq h_1^{I_2}$ for all $g \in A$. So we get $h_1 \in h_1^{I_2 I_2} \subseteq \bigcap_{g\in A} g^{I_{12}I_2}$. That shows the implication $A \subseteq \bigcap_{h\in C} h^{I_{21}I_1} \Rightarrow C \subseteq \bigcap_{g\in A} g^{I_{12}I_2}$. The converse of this implication can be analogously verified.

(2) Let (φ, ψ) be a Galois connection between $\underline{\mathfrak{B}}(\mathbb{K}_1)$ and $\underline{\mathfrak{B}}(\mathbb{K}_2)$. We construct two relations $I_\varphi \subseteq G_1 \times M_2$ and $I_\psi \subseteq G_2 \times M_2$ by defining

$$g^{I_\varphi} := \{n \in M_2 \mid \varphi(g^{I_1 I_1}, g^{I_1}) \leq (n^{I_2}, n^{I_2 I_2})\} \quad \text{for } g \in G_1$$

and

$$h^{I_\psi} := \{m \in M_1 \mid \psi(h^{I_2 I_2}, h^{I_2}) \leq (m^{I_1}, m^{I_1 I_1})\} \quad \text{for } h \in G_2,$$

i.e. g^{I_φ} is the intent of the concept $\varphi(g^{I_1 I_1}, g^{I_1}) \in \underline{\mathfrak{B}}(\mathbb{K}_2)$ and h^{I_ψ} is the intent of the concept $\psi(h^{I_2 I_2}, h^{I_2}) \in \underline{\mathfrak{B}}(\mathbb{K}_1)$. Then, we have

$$\begin{aligned} h^{I_\varphi} \subseteq g^{I_1} &\Leftrightarrow (g^{I_1 I_1}, g^{I_1}) \leq (h^{I_\psi I_1}, h^{I_\psi}) = \psi(h^{I_2 I_2}, h^{I_2}) \\ &\Leftrightarrow (h^{I_2 I_2}, h^{I_2}) \leq \varphi(g^{I_1 I_1}, g^{I_1}) = (g^{I_\varphi I_2}, g^{I_\varphi}) \\ &\Leftrightarrow g^{I_\varphi} \subseteq h^{I_2}. \end{aligned}$$

So, the pair (I_φ, I_ψ) is a weak Galois bond between $\subseteq_1$ and $\subseteq_2$. Moreover, it follows that for $(A,B) \in \underline{\mathfrak{B}}(\mathbb{K}_1)$ and $(C,D) \in \underline{\mathfrak{B}}(\mathbb{K}_2)$

$$\begin{aligned} \varphi(A,B) &= \varphi\Big(\bigvee_{g\in A} (g^{I_1 I_1}, g^{I_1})\Big) = \bigwedge_{g\in A} \varphi(g^{I_1 I_1}, g^{I_1}) \\ &= \bigwedge_{g\in A} (g^{I_\varphi I_2}, g^{\varphi}) = \bigwedge_{g\in A} (g^{\varphi I_2}, g^{\varphi I_2 I_2}), \\ \psi(C,D) &= \psi\Big(\bigvee_{h\in C} (h^{I_2 I_2}, h^{I_2})\Big) = \bigwedge_{h\in C} \psi(h^{I_2 I_2}, h^{I_2}) \\ &= \bigwedge_{h\in C} (h^{I_\psi I_1}, h^{\psi}) = \bigwedge_{h\in C} (h^{\psi I_1}, h^{\psi I_1 I_1}). \end{aligned}$$

■

By $WGB = WGB(\mathbb{K}_1, \mathbb{K}_2)$ we denote the set of all weak Galois bonds between $\mathbb{K}_1$ and $\mathbb{K}_2$, and by $GC = GC(\underline{\mathfrak{B}}(\mathbb{K}_1), \underline{\mathfrak{B}}(\mathbb{K}_2))$ the set of all Galois connections between $\underline{\mathfrak{B}}(\mathbb{K}_1)$ and $\underline{\mathfrak{B}}(\mathbb{K}_2)$. Two order relations can be defined on the sets WGD and GC, respectively, as follows:

$$(I_{12}, I_{21}) \le (I^*_{12}, I^*_{21}) :\Leftrightarrow\ I_{12} \subseteq I^*_{12} \quad \text{and} \quad I_{21} \subseteq I^*_{21}\,,$$
$$(\varphi_1, \psi_1) \le (\varphi_2, \psi_2) :\Leftrightarrow\ \varphi_1 \le \varphi_2 \quad \text{and} \quad \psi_1 \le \psi_2\,.$$

It is well-known that the set GC is a complete lattice with respect to the order relation defioned above. For the set WGB we have the following Theorem.

THEOREM 5. *The set WGB is a complete lattice with respect to the order relation defined above.*

Proof. The pair $(\emptyset, \emptyset) \in WGB$ is obviously the smallest weak Galois bond. By the definition, it is easy to show that the supremum of a subset $\{I^i_{12}, I^i_{21})\mid i \in T\} \subseteq WGB$ is given by

$$\bigvee\{(I^i_{12}, I^i_{21})\mid i \in T\} = \Big(\bigcup_{i\in T} I^i_{12}, \bigcup_{i\in T} I^i_{21}\Big)\,. \qquad \blacksquare$$

In Theorem 4 the relationship between the complete lattices WGB and GC is established. In fact, we can define two mappings between them as follows:

$$\Gamma : WGB \to GC, \qquad (I_{12}, I_{21}) \mapsto (\varphi_{I_{12}}, \psi_{I_{21}})\,;$$
$$\Sigma : GC \to WGB, \qquad (\varphi, \psi) \mapsto (I_\varphi, I_\psi)\,.$$

From Theorem 4 we can see that $\Gamma(\Sigma(\varphi,\psi)) = \Gamma(I_\varphi, I_\psi) = (\varphi_{I_\varphi}, \psi_{I_\psi}) = (\varphi, \psi)$ for $(\varphi, \psi) \in GC$. That means that the mapping Γ is surjective, i.e. $\Gamma(WGB) = GC$.

THEOREM 6. *The pair (Γ, Σ) is a Galois connection between the complete lattices WGB and GC.*

Proof. What has to be proved is that for any $(I_{12}, I_{21}) \in WGB$ and any $(\varphi, \psi) \in GC$

$$(I_{12}, I_{21}) \le \Sigma(\varphi, \psi) \Leftrightarrow (\varphi, \psi) \le \Gamma(I_{12}, I_{21})\,,$$

or equivalently,

$$I_{12} \subseteq I_\varphi, I_{21} \subseteq I_\psi \Leftrightarrow \varphi \le \varphi_{I_{12}}, \psi \le \psi_{I_{21}}\,.$$

We show now $I_{12} \subseteq I_\varphi \Leftrightarrow \varphi \le \varphi_{I_{12}}$. Supposing $I_{12} \subseteq I_\varphi$, we get $g^{I_{12}} \subseteq g^{I_\varphi}$ and therefore $g^{I_{12}I_2} \supseteq g^{I_\varphi I_2}$ for all $g \in G_1$. Then, for all any $(A, B) \in \underline{\mathfrak{B}}(\mathbb{K}_1)$, it follows, by Theorem 4, that

$$\varphi(A,B) = \bigwedge_{g\in A}(g^{I_\varphi I_2}, g^{I_\varphi I_2 I_2}) \leq \bigwedge_{g\in A}(g^{I_{12}I_2}, g^{I_{12}I_2 I_2}) = \varphi_{I_{12}}(A,B)\,.$$

That means $\varphi \leq \varphi_{I_{12}}$. Conversely, with $\varphi \leq \varphi_{I_{12}}$ we have for any $g \in G_1$, $n \in M_2$, by Theorem 4, that

$$\begin{aligned} gI_{12}n &\Rightarrow n \in g^{I_{12}} \Rightarrow g^{I_{12}I_2} \subseteq n^{I_2} \Rightarrow \bigcap_{x\in g^{I_1 I_1}} x^{I_{12}I_2} \subseteq n^{I_2} \\ &\Rightarrow \varphi_{I_{12}}(g^{I_1 I_1}, g^{I_1}) \leq (n^{I_2}, n^{I_2 I_2}) \\ &\Rightarrow \varphi(g^{I_1 I_1}, g^{I_1}) \leq (n^{I_2}, n^{I_2 I_2}) \\ &\Rightarrow n \in g^{I_\varphi} \Rightarrow gI_\varphi n\,. \end{aligned}$$

That shows $I_{12} \subseteq I_\varphi$. The equivalence $I_{21} \subseteq I_\psi \Leftrightarrow \psi \leq \psi_{I_{21}}$ can be similary verified. ∎

Since (Γ, Σ) is a Galois connection between WGB and GC, the two compositions $\Gamma\Sigma$ and $\Sigma\Gamma$ are closure operators on GC and WGB, respectively, and then the coresponding closure systems $\Gamma(WGB) = GC$ and $\Sigma(GC)$ are dual isomorphic lattices. We call now the elements from $\Sigma(GC)$ the *Galois bonds* between $\mathbb{K}_1$ and $\mathbb{K}_2$, that are those weak Galois bonds which are images of the mapping Σ. Denote the set of all Galois bonds between $\mathbb{K}_1$ and $\mathbb{K}_2$ by $GB = GB(\mathbb{K}_1, \mathbb{K}_2)$. Then we have

COROLLARY 3. *The set GB, ordered by the restriction of the order relation of WGB on it, is a complete lattice dual isomorphic to the coomplete lattice GC. The restriction of Γ on GB, $\Gamma|_{GB} : GB \to GC$, and the mapping $\Sigma : GC \to GB$ are the corresponding dual isomorphisms. The suprema in GB can be described as follows: for any subset $\{(I^i_{12}, I^i_{21}) \mid i \in T\} \subseteq GB$, the supremum $(I^*_{12}, I^*_{21}) = \bigvee\{(I^i_{12}, I^i_{21}) \mid i \in T\}$ is given by*

$$g^*_{12} := \Big(\bigcup_{i\in T} g^{I^i_{12}}\Big)^{I_2 I_2} \quad \textit{for all } g \in G_1$$

and

$$h^*_{21} := \Big(\bigcup_{i\in T} h^{I^i_{21}}\Big)^{I_1 I_1} \quad \textit{for all } h \in G_2\,. \qquad ∎$$

Now, we give some equivalent conditions for Galois bonds.

LEMMA 3. *Let (I_{12}, I_{21}) be a weak Galois between contexts $\mathbb{K}_1 = (G_1, M_1, I_1)$ and $\mathbb{K}_2 = (G_2, M_2, I_2)$. Then, (I_{12}, I_{21}) is a Galois bond iff the following conditions are satisfied:*

(1) *For every $g \in G_1$ the subset $g^{I_{12}} \subseteq M_2$ is an intent of $\mathbb{K}_2$, and $g_1^{I_1} \subseteq g_2^{I_1}$ implies $g_1^{I_{12}} \supseteq g_2^{I_{12}}$ for all $g_1, g_2 \in G_1$;*

(2) *For every* $h \in G_2$ *the subset* $h^{I_{21}} \subseteq M_1$ *is an intent of* $\mathbb{K}_1$, *and* $h_1^{I_2} \subseteq h_2^{I_2}$ *implies* $h_1^{I_{21}} \supseteq h_2^{I_{21}}$ *for all* $h_1, h_2 \in G_2$.

Proof. Supposing that (I_{12}, I_{21}) is a Galois bond, i.e. there is a Galois connection $(\varphi, \psi) \in GC$ such that $(I_{12}, I_{21}) = \Sigma(\varphi, \psi) = (I_\varphi, I_\psi)$. Then, from the proof of Theorem 4 we can see that $g^{I_{12}} = g^{I_\varphi} (g \in G_1)$ is the intent of the concept $\varphi(g^{I_1 I_1}, g^{I_1})$. Moreover, as (φ, ψ) is a Galois connection, it follows from $g_1^{I_1} \subseteq g_2^{I_1} (g_1, g_2 \in G_1)$, or equivalently, $(g_2^{I_1 I_1}, g_2^{I_2}) \leq (g_1^{I_1 I_1}, g_1^{I_1})$ that $\varphi(g_1^{I_1 I_1}, g_1^{I_1}) \leq \varphi(g_2^{I_1 I_1}, g_2^{I_1})$, and this implies immediately $g_2^{I_{12}} = g_2^{I_\varphi} \subseteq g_1^{I_\varphi} = g_1^{I_{12}}$. So, (1) has been proved. Similary, (2) can be shown.

Conversely, assume that the weak Galois bond (I_{12}, I_{21}) satisfies the conditions (1) and (2). Consider the Galois bond $(I_{\varphi I_{12}}, I_{\psi I_{21}}) = \Sigma(\Gamma)(I_{12}, I_{21})$. By Theorem 4, $g^{I_\varphi I_{12}} (g \in G_1)$ is the intent of the concept $\varphi_{I_{12}}(g^{I_1 I_1}, g^{I_1})$, that means

$$g^{I_\varphi I_{12}} = \left(\bigcup \{ g^{I_{12} I_2 I_2} | \, g_1 \in g^{I_1 I_1} \} \right)^{I_2 I_2} = \left(\bigcup \{ g_1^{I_{12}} | \, g_1 \in g^{I_1 I_1} \} \right)^{I_2 I_2}$$
$$= g^{I_{12} I_2 I_2} = g^{I_{12}} .$$

So, $I_{12} = I_{\varphi_{I_{12}}}$ has been proved. Analogously, we can show $I_{21} = I_{\psi_{I_{21}}}$. Thus, (I_{12}, I_{21}) is the image of the Galois connection $(\varphi_{I_{12}}, \psi_{I_{21}})$ under Σ, i.e. a Galois bond. ∎

It is known that each mapping (φ or ψ) in a Galois connection (φ, ψ) can be uniquely determined by the other. In some case one considers only one mapping from them and speaks about Galois mappings and tensor products of complete lattices [8, 9]. The similar situation appears also for Galois bonds between contexts.

LEMMA 4. *For two contexts* $\mathbb{K}_1 = (G_1, M_1, I_1)$ *and* $\mathbb{K}_2 = (G_2, M_2, I_2)$ *and for two relations* $I_{12} \subseteq G_1 \times M_2$ *and* $I_{21} \subseteq G_2 \times M_1$ *the following statements are equivalent*:

(1) *The pair* (I_{12}, I_{21}) *is a Galois bond between* $\mathbb{K}_1$ *and* $\mathbb{K}_2$;

(2) *For all* $g_1, g_2, g \in G_1$, *the subset* $g^{I_{12}} \subseteq M_2$ *is an intent of* $\mathbb{K}_2$ *and* $g_1^{I_1} \subseteq g_2^{I_1}$ *implies* $g_1^{I_{12}} \supseteq g_2^{I_{12}}$, *and for every* $h \in G_2$, *the subset* $\{g \in G_1 | \, g^{I_{12}} \subseteq h^{I_2}\}$ *is an extent of* $\mathbb{K}_1$ *and* $h^{I_{21}} = \{g \in G_1 | \, g^{I_{12}} \subseteq h^{I_2}\}^{I_1}$;

(3) *For all* $h_1, h_2, h_3 \in G_2$, *the subset* $h^{I_{12}} \subseteq M_1$ *is an intent of* $\mathbb{K}_1$ *and* $h_1^{I_2} \subseteq h_2^{I_2}$ *implies* $h_1^{I_{21}} \supseteq h_2^{I_{21}}$, *and for every* $g \in G_1$, *the subset* $\{h \in G_2 | \, h^{I_{21}} \subseteq g^{I_1}\}$ *is an extent of* $\mathbb{K}_2$ *and* $g^{I_{12}} = \{h \in G_2 | \, h^{I_{21}} \subseteq g^{I_1}\}^{I_2}$.

Proof. The proof for (1) $\Leftrightarrow$ (3) is dual to the proof for (1) $\Leftrightarrow$ (2). We show here only (1) $\Leftrightarrow$ (2).

(1) $\Leftrightarrow$ (2): Let (I_{12}, I_{21}) be a Galois bond between $\mathbb{K}_1$ and $\mathbb{K}_2$. Then, by Lemma 3, the subset $g^{I_{12}} (g \in G_1)$ is certainly an intent of $\mathbb{K}_2$ and we have, for all $g_1, g_2 \in G_1$, $g_1^{I_1} \subseteq g_2^{I_1} \Rightarrow g_1^{I_{12}} \supseteq g_2^{I_{12}}$. Since the pair (I_{12}, I_{21})

is a Galois bond we get, by Corollary 3, $(I_{12}, I_{21}) = \Sigma(\Gamma(I_{12}, I_{21})$. That means, by Theorem 4, that the subset $h^{I_{21}} \subseteq M_1$ $(h \in G_2)$ is the intent of the concept $(A_1, B_1) := \psi_{I_{21}}(h^{I_2 I_2}, h^{I_2}) \in \underline{\mathfrak{B}}(\mathbb{K}_1)$. From Lemma 2 follows

$$\begin{aligned}(A_1, B_1) &= \bigvee\{(A, B) \in \underline{\mathfrak{B}}(\mathbb{K}_1) \mid (h^{I_2 I_2}, h^{I_2}) \leq \varphi_{I_{12}}(A, B)\} \\ &= \bigvee\{(A, B) \in \underline{\mathfrak{B}}(\mathbb{K}_1) \mid \bigcup_{g \in A} g^{I_{12}} \subseteq h^{I_2}\}.\end{aligned}$$

So, we get

$$\begin{aligned}A_1 &= \Big(\bigcup\{A \mid A^{I_1 I_1} = A \subseteq G_1, \bigcup_{g \in A} g^{I_{12}} \subseteq h^{I_2}\}\Big)^{I_1 I_1} \\ &= \Big(\bigcup\Big\{\Big(\bigcup_{g \in A} g^{I_1 I_1}\Big)^{I_1 I_1} \mid A^{I_1 I_1} = A \subseteq G_1, \bigcup_{g \in A} g^{I_{12}} \subseteq h^{I_2}\Big\}\Big)^{I_1 I_1} \\ &= \Big(\bigcup\Big\{\bigcup_{g \in A} g^{I_1 I_1} \mid A^{I_1 I_1} = A \subseteq G_1, \bigcup_{g \in A} g^{I_{12}} \subseteq h^{I_2}\Big\}\Big)^{I_1 I_1} \\ &= \Big(\bigcup\{g^{I_1 I_1} \mid g \in G_1, g^{I_{12}} \subseteq h^{I_2}\}\Big)^{I_1 I_1} \\ &= \{g \in G_1 \mid g^{I_{12}} \subseteq h^{I_2}\}^{I_1 I_1}.\end{aligned}$$

From this it follows that $h^{I_{21}} = B_1 = A_1^{I_1} = \{g \in G_1 \mid g^{I_{12}} \subseteq h^{I_2}\}^{I_1}$. In order to show that $\{g \in G_1 \mid g^{I_{12}} \subseteq h^{I_2}\}$ is an extent of $\mathbb{K}_1$ it is sufficient to prove $A_2 := \{g \in G_1 \mid g^{I_{12}} \subseteq h^{I_2}\} \supseteq \{g \in G_1 \mid g^{I_{12}} \subseteq h^{I_{12}} \subseteq h^{I_2}\}^{I_1 I_1} =: A_3$. For this we have

$$\begin{aligned}x \in A_3 &\Rightarrow h^{I_{21}} = \{g \in G_1 \mid g^{I_{12}} \subseteq h^{I_2}\}^{I_1} \subseteq x^{I_1} \\ &\Rightarrow x^{I_{12}} \subseteq h^{I_2} \\ &\Rightarrow x \in \{g \in G_1 \mid g^{I_{12}} \subseteq h^{I_2}\}.\end{aligned}$$

That shows $A_2 \supseteq A_3$.

(2) $\Rightarrow$ (1): Assume that the pair (I_{12}, I_{21}) satisfies the conditions in (2). By Lemma 3, we need to show that, for all $h_1, h_2, h \in G_2$ and $g \in G_1$,

(i) $h_1^{I_2} \subseteq h_2^{I_2} \Rightarrow h_1^{I_{21}} \supseteq h_2^{I_{21}}$;

(ii) the subset $h^{I_{21}} \subseteq M_1$ is an intent of $\mathbb{K}_1$;

(iii) $g^{I_{12}} \subseteq h^{I_2} \Leftrightarrow h^{I_{21}} \subseteq g^{I_1}$.

Since $y^{I_{21}} = \{g \in G_1 \mid g^{I_{21}} \subseteq y^{I_2}\}^{I_1}$ for any $y \in G_2$ the statements (i) and (ii) are obviously true. For (iii) we need to prove

$$g^{I_{12}} \subseteq h^{I_2} \Leftrightarrow \{x \in G_1 \mid x^{I_{12}} \subseteq h^{I_2}\}^{I_1} \subseteq g^{I_1}.$$

If $g^{I_{12}} \subseteq h^{I_2}$ then $g \in \{x \in G_1 \mid x^{I_{12}} \subseteq h^{I_2}\}$ and this implies $g^{I_1} \supseteq \{x \in G_1 \mid x^{I_{12}} \subseteq h^{I_2}\}^{I_1}$. On the other hand, $\{x \in G_1 \mid x^{I_{12}} \subseteq x^{I_2}\}^{I_1} \subseteq g^{I_1}$ yields

$g \in g^{I_1 I_1} \subseteq \{x \in G_1 | x^{I_{12}} \subseteq h^{I_2}\}^{I_1 I_1} = \{x \in G_1 | x^{I_{12}} \subseteq h^{I_2}\}$. That shows $g^{I_{12}} \subseteq H^{I_2}$. So, the proof is completed. ∎

Lemma 4 says that the two relations I_{12} and I_{21} in a Galois bond (I_{12}, I_{21}) between $\mathbb{K}_1$ and $\mathbb{K}_2$ are uniquely determined each other. For this reason we can also consider only one relation to describe the Galois connections between $\underline{\mathfrak{B}}(\mathbb{K}_1)$ and $\underline{\mathfrak{B}}(\mathbb{K}_2)$. For contexts $\mathbb{K}_1 = (G_1, M_1, I_1)$ and $\mathbb{K}_2 = (G_2, M_2, I_2)$ we call a relation $I_{12} \subseteq G_1 \times M_2$ a *Galois relation* between $\mathbb{K}_1$ and $\mathbb{K}_2$, if the following three conditions are satisfied:
(grl) For every $g \in G_1$ the subset $g^{I_{12}} \subseteq M_2$ is an intent of $\mathbb{K}_2$;
(gr2) For all $g_1, g_2 \in G_1$ the implication $g_1^{I_1} \subseteq g_2^{I_1} \Rightarrow g_1^{I_{12}} \supseteq g_2^{I_{12}}$ is true;
(gr3) For every $h \in G_2$ the subset $\{g \in G_1 | g^{I_{12}} \subseteq h^{I_2}\} \subseteq G_1$ is an extent of $\mathbb{K}_1$.
It is clear that a relation $I_{12} \subseteq G_1 \times M_2$ is a Galois realtion between $\mathbb{K}_1$ and $\mathbb{K}_2$ iff there is an unique relation $I_{21} \subseteq G_2 \times M_1$ such that the pair (I_{12}, I_{21}) is a Galois bond between $\mathbb{K}_1$ and $\mathbb{K}_2$.

COROLLARY 4. *For contexts $\mathbb{K}_1 = (G_1, M_1, I_1)$ and $\mathbb{K}_2 = (G_2, M_2, I_2)$ the set of all Galois relations between $\mathbb{K}_1$ and $\mathbb{K}_2$, ordered by the set inclusion, is a complete lattice dual isomorphic to the complete lattice of all Galois connections between $\underline{\mathfrak{B}}(\mathbb{K}_1)$ and $\underline{\mathfrak{B}}(\mathbb{K}_2)$. A dual isomorphism can be defined by $I_{12} \mapsto (I_{12}, I_{21}) \mapsto (\varphi_{I_{12}}, \psi_{I_{21}})$.* ∎

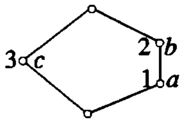
It is easy to see that the conditions (grl), (gr2) and (gr3) can be conventiely checked on the context plane. So we have a more efficient description of Galois connections between complete lattices than the description by G-ideals.

Finally, we will given an example. We take two contexts $\mathbb{K}_1$ and $\mathbb{K}_2$ and consider the corresponding concepts lattices $\underline{\mathfrak{B}}(\mathbb{K}_1) \cong N_5$ and $\underline{\mathfrak{B}}(\mathbb{K}_2) \cong M_3$ (see fig. 1). The extents of $\mathbb{K}_1$ are subsets $\emptyset$, $\{3\}$, $\{1\}$, $\{1,2\}$ and $\{1,2,3\}$ of G_1, the intents of $\mathbb{K}_2$ are subsets $\emptyset$, $\{d\}$, $\{e\}$, $\{f\}$ and $\{d,e,f\}$ of M_2. Firstly, we choose arbitrarily an intent of $\mathbb{K}_2$ as $3^{I_{12}}$, say $3^{I_{12}} := \{d\}$. Since here the intent $3^{I_1} = \{c\}$ of $\mathbb{K}_1$ is incomparable with the intents $1^{I_1} = \{a,b\}$ and $2^{I_1} = \{b\}$, we can arbitrarily choose $2^{I_{12}} := \{d,e,f\}$. Because of $2^{I_1} \subseteq 1^{I_1}$ and the conditions (gr2) and (gr3) we can only take $\{d,e,f\}$, $\{e\}$ or $\{f\}$ as $1^{I_{12}}$. If we choose $1^{I_{12}} := \{e\}$, then we get a Galois relation $I_{12} \subseteq G_1 \times M_2$ between $\mathbb{K}_1$ and $\mathbb{K}_2$ which stand as a contect (G_1, M_2, I_{12}) in fig. 2 right above. The context (G_2, M_1, I_{21}) standing in fig. 2 left below describes the unique relation $I_{21} \subseteq G_2 \times M_1$ such that the pair $I_{12}, I_{21})$ is a Galois bond between $\mathbb{K}_1$ and $\mathbb{K}_2$. The other 40 Galois relations between $\mathbb{K}_1$ and $\mathbb{K}_2$ are in fig. 3, and the complete lattice of all 41 Galois realtions between $\mathbb{K}_1$ and $\mathbb{K}_2$ is drawn as a nested line diagram [12] in fig. 4.

EXAMPLE. $G_1 = \{1,2,3\}$, $M_1 = \{a,b,c\}$, $G_2 = \{4,5,6\}$, $M_2 = \{d,e,f\}$.

	a	b	c
1	×	×	
2		×	
3			×

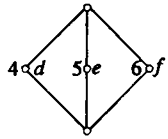
$\mathbb{K}_1 = (G_1, M_1, I_1)$



$\underline{\mathfrak{B}}(\mathbb{K}_1) \cong N_5$

	d	e	f
4	×		
5		×	
6			×

$\mathbb{K}_2 = (G_2, M_2, I_2)$



$\underline{\mathfrak{B}}(\mathbb{K}_2) \cong M_3$

Figure 1: Contexts and their concept lattices.

	a	b	c	d	e	f
1	×	×			×	
2		×		×	×	×
3			×	×		
4			×	×		
5	×	×			×	
6	×	×	×			×

Figure 2: Context $\mathbb{K}_1$ (left above) and $\mathbb{K}_2$ (right below) from fig. 1, and a Galois bond (I_{12}, I_{21}) between $\mathbb{K}_1$ and $\mathbb{K}_2$.

×	×	×
×	×	×
×	×	×

	×	
×	×	×

	×	
	×	
×		

	×	
	×	

		×
		×
		×

×		
×	×	×
×	×	×

		×
		×
×	×	×

		×
		×
×		

		×
		×
	×	

		×
		×

	×	
×	×	×
×	×	×

		×
×	×	×

×		

	×	

		×

		×
×	×	×
×	×	×

×	×	×

×	×	×
×	×	×
	×	

×	×	×
×	×	×
		×

×	×	×
×	×	×

×	×	×
×	×	×

×	×	×
×	×	×
×		

×		
×	×	×
	×	

×		
×	×	×
		×

×		
×		

×		
×		
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Figure 3: The 40 Galois realtions between $\mathbb{K}_1$ and $\mathbb{K}_2$ from fig. 1.

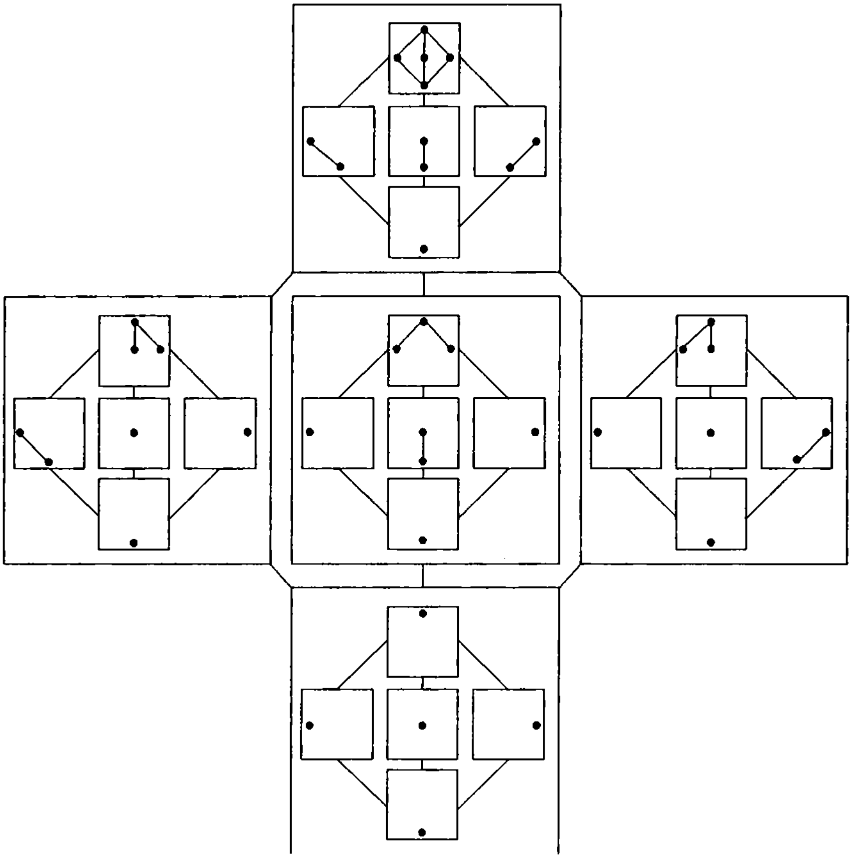

Figure 4: The complete lattice of all 41 Galois realtions between the contexts $\mathbb{K}_1$ and $\mathbb{K}_2$ from fig. 1.

References

[1] R. Dedekind, (1872) *Stetigkeit und irrationale Zahlen*, 7. Auflage, Vieweg, Braunschweig, 1969.

[2] M. Erné, *Einführung in die Ordnungstheorie*, B. I. Wissenschaftsverlag, Mannheim, Wien, Zürich (1982).

[3] B. Ganter, R. Wille, *Formale Begriffsanalyse*, B. I. Wissenschaftsverlag, Mannheim, Wien, Zürich (to appear).

[4] H. M. MacNeille, *Partially ordered sets*, Trans. Amer. Math. Soc., 42, (1937) 416–460.

[5] D. G. Mowat, *A Galois Problem for Mappings*, Thesis in University of Waterloo (1968).

[6] O. Ore, *Galois connexions*, Trans. Amer. Math. Soc. 55 (1944), 493–515.

[7] G. Pickert, *Bemerkungen über Galois-Verbindungen*, Archiv. Math. 3 (1952), 285–289.

[8] Z. Shmuely, *The structure of Galois connections*, Pacific J. Math. 54 (1974), 209–225.

[9] Z. Shmuely, *The tensor product of distributive lattices*, Algebra Universalis 9 (1979), 281–296.

[10] R. Wille, *Restructuring lattice theory: an approach based on hierarchies of concept*, in Ordered Sets (I. Rival ed.), Reidel, Dordrecht-Boston, (1982) 445–470.

[11] R. Wille, *Subdirect product construction of concept lattices*, Discrete Math. 63 (1987), 305–313.

[12] R. Wille, *Lattices in data analysis: how to draw them with a computer*, in Algorithms and Order (I. Rival ed.), Kluwer, Dordrecht-Boston (1989), 33–58.

[13] W. Xia, *Fixed point property and formal concept analysis*, Order 9 (1992), 255–264.

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