DEMONSTRATIO MATHEMATICA
Vol. XXVII No 3–4 1994

K. Denecke

PRE-SOLID VARIETIES

Dedicated to Professor Tadeusz Traczyk

1. Introduction

An identity $t \approx t'$ of terms of any type τ is called a hyperidentity for a universal algebra $\mathcal{A} = (A; (f_i^A)_{i \in I})$ if $t \approx t'$ holds identically for every choice of n-ary term operations to represent n-ary operation symbols occuring in t and t' ([8]). Although the concept of a hyperidentity is very strong there are countable infinitely many semigroup varieties for which every identity is a hyperidentity (solid varieties of semigroups) ([3]). Since any projection defined on A is a term operation of $\mathcal{A}$, a hyperidentity must be satisfied at least for the projections. Therefore there are identities which cannot be hyperidentities. Substituting one of the binary projections for F in $F(x,y) \approx F(y,x)$ we see that the commutative law fails to be a hyperidentity in any nontrivial variety with a binary operation symbol. This observation suggests the idea to weaken the concept of a hyperidentity. The simplest way for weakeness could be to substitute only term operations different from projections. The set of all term functions of $\mathcal{A}$ which are different from projections can be regarded as the universe of an algebra whose fundamental operations describe the composition of functions, the so-called pre-iteratire algebra in the sense of I.A. Mal'cev ([6]). This motivates to denote these „weaker" hyperidentities as pre-hyperidentities. An algebra or a variety for which every identity is a pre-hyperidentity is called pre-solid. After developing the theory of pre-hyperidentities and pre-solid varieties we will apply the results on semigroups and determine the greatest pre-solid variety of commutative semigroups.

This paper has been presented at the Conference on Universal Algebra and its Applications, organized by the Institute of Mathematics of Warsaw University of Technology held at Jachranka, Poland, 8–13 June 1993.

The author is indebted to K. Głazek who recommended to consider pre-solid varieties and to Sh. Wismath for stimulating discussions during the workshop on „Universal Algebra and Category Theory" in Berkeley, July 1993.

2. Basic concepts and results

Hyperidentities can be characterized more precisely using the concept of a hypersubstitution. We fix a type $\tau = (n_i)_{i \in I}$, $n_i > 0$ for all $i \in I$, and operation symbols $(f_i)_{i \in I}$, where f_i is n_i-ary. Let $W_\tau(X)$ be the set of all terms of type τ over some fixed alphabet X, and let $Alg(\tau)$ be the class of all algebras of type τ.

A mapping

$$\sigma : \{f_i \mid i \in I\} \to W_\tau(X)$$

which assigns to every n_i-ary operation symbol f_i an n_i-ary term will be called a hypersubstitution of type τ. Applying a hypersubstitution σ to a term t we get a term $\sigma[t]$ which can be defined inductively by:

(i) $\widehat{\sigma}[x] := x$ for any variable x in the alphabet X, and

(ii) $\widehat{\sigma}[f_i(t_1, \dots, t_n)] := \sigma(f_i)(\widehat{\sigma}[t_1], \dots, \widehat{\sigma}[t_{n_i}])$.

It is clear that $\sigma(f_i)$ on the right hand side of (ii) must be interpreted as the operation induced by the term $\sigma(f_i)$ on the term algebra $\mathcal{W}_\tau(X)$. According to the ideas explained in the introduction we define a prehypersubstitution of type τ as a mapping

$$\sigma_p : \{f_i \mid i \in I\} \to W_\tau(X) \setminus X$$

which assigns to every operation symbol f_i an n_i-ary term which is different from a variable. (Note that we consider the first n_i variables $x_0, \dots, x_{n_i - 1}$ of the standard alphabet $X = \{x_0, \dots, x_{n_i-1}, \dots\}$ as n_i-ary terms).

The extension $\widehat{\sigma}_p[t]$ of a pre-hypersubstitution to a term t is defined inductively by rules corresponding to (i) and (ii).

If $t \approx t'$ is an equation, then we denote by $\Xi[t \approx t']$ the set

$$\{\widehat{\sigma}[t] \approx \widehat{\sigma}[t'] \mid \sigma : \{f_i | i \in I\} \to W_\tau(X)\}$$

and by $\Xi_p]t \approx t']$ the set

$$\{\widehat{\sigma}_p[t] \approx \widehat{\sigma}[t'] | \sigma_p : \{f_i | i \in I\} \to W_\tau(X) \setminus X\}.$$

If Σ is a set of equations, we use $\Xi[\Sigma]$ for the union of the sets $\Xi[t \approx t']$, for $t \approx t'$ in Σ. In the same way we define $\Xi_p[\Sigma]$.

Let $\mathcal{A} = (A; (f_i^A)_{i \in I})$ be an algebra in $Alg(\tau)$, let $K \subseteq Alg(\tau)$, let σ be a hypersubstitution, and let σ_p be a pre-hypersubstitution. Then we make the following definitions:

$$\sigma[\mathcal{A}] := (A; (\sigma(f_i)^{\mathcal{A}})_{i\in I}),$$

$$\sigma_p[\mathcal{A}] := (A; (\sigma_p(f_i)^{\mathcal{A}})_{i\in I}),$$

$$\Xi_p[\mathcal{A}] := \{\sigma_p[\mathcal{A}] \mid \sigma_p \text{ is a pre-hypersubstitution of type } \tau\},$$

$$\Xi[\mathcal{A}] := \{\sigma[\mathcal{A}] \mid \sigma \text{ is a hypersubstitution of type } \tau\},$$

$$\Xi[K] := \bigcup_{\mathcal{A}\in K} \Xi[\mathcal{A}]$$

$$\Xi_p[K] := \bigcup_{\mathcal{A}\in K} \Xi_p[\mathcal{A}].$$

In [2] it was proved that Ξ is a closure operator on sets of equations and on classes of algebras.

In the same way we get:

PROPOSITION 2.1: *Ξ_p is a closure operator on sets of equations Σ and on classes of algebras K of type τ, i.e.*

(i) $\Sigma \subseteq \Xi_p[\Sigma]$,
(ii) $\Sigma' \subseteq \Sigma \Rightarrow \Xi_p[\Sigma'] \subseteq \Xi_p[\sigma]$,
(iii) $\Xi_p[\Xi_p[\Sigma]] = \Xi_p[\Sigma]$,
(i') $K \subseteq \Xi_p[K]$,
(ii') $K' \subseteq K \Rightarrow \Xi_p[K'] \subseteq \Xi_p[K]$,
(iii') $\Xi_p[\Xi_p[K]] = \Xi_p[K]$. ■

Since every pre-hypersubstitution is a hypersubstitution we have

PROPOSITION 2.2: *Let K be a class of algebras of type τ and let Σ be a set of equations of type τ. Then*

(i) $\Xi_p[\Sigma] \subseteq \Xi[\Sigma]$ *and*
(ii) $\Xi_p[K] \subseteq \Xi[K]$. ■

Using hypersubstitutions and pre-hypersubstitutions we define hyperidentities and pre-hyperidentities in the following way:

DEFINITION 2.3 : Let $\mathcal{A} \in Alg(\tau)$ be an algebra of type τ. Then the identity $t \approx t'$, where t, t' are terms of type τ is a hyperidentity of type τ in $\mathcal{A}$ ($\mathcal{A}$ hypersatisfies $t \approx t'$) if $\hat{\sigma}[t] \approx \hat{\sigma}[t']$ are identities for every hypersubstitution σ. The identity $t \approx t'$ is a pre-hyperidentity of type τ in $\mathcal{A}$ ($\mathcal{A}$ pre-hypersatisfies $t \approx t'$ if $\hat{\sigma}_p[t] \approx \hat{\sigma}_p[t']$ are identities for every pre-hypersubstitution σ_p.

Clearly, every hyperidentity of type τ is a pre-hyperidentity of this type. In general, the converse is false.

Let K be a class of algebras of type τ. Then the identity $t \approx t'$ is a hyperidentity respectively a pre-hyperidentity in K if it is a hyperidentity (a pre-hyperidentity) in every algebra of K.

For a class K of algebras of type τ and for a set Σ of identities of this type we fix the following notations:
IdK — the class of all identities of K,
$HIdK$ — the class of all hyperidentities of K,
H_pIdK — the class of all pre-hyperidentities of K,
$Mod\Sigma = \{\mathcal{A} \in Alg(\tau) | \mathcal{A}$ satisfies $\Sigma\}$ — the variety defined by Σ,
$HMod\Sigma = \{\mathcal{A} \in Alg(\tau) | \mathcal{A}$ hypersatisfies $\Sigma\}$ — the hyperequational class defined by Σ,
$H_pMod\Sigma = \{\mathcal{A} \in Alg(\tau) | \mathcal{A}$ pre-hypersatisfies $\Sigma\}$ — the pre-hyperequational class defined by Σ,
$VarK = ModIdK$ — the variety generated by K,
$HVarK = HModHIdK = \{\mathcal{A} \in Alg(\tau) | \mathcal{A}$ hypersatisfies $HIdK\}$ — the hypervariety generated by K.

For these sets we get the following inclusions:

$$HIdK \subseteq H_pIdK, \qquad HMod\Sigma \subseteq H_pMod\Sigma.$$

By definition every hyperidentity or every pre-hyperidentity is an identity. Very natural there arises the problem to find algebras or varieties for which every identity is a hyperidentity or such that every identity is a pre-hyperidentity.

DEFINITION 2 .4: Let V be a variety of type τ. Then V is called solid if $\Xi[V] = V$. The variety V is called pre-solid if $\Xi_p[V] = V$.

For solid varieties in [2] the following propositions were proved:

THEOREM 2.5 ([2]): *Let $K \subseteq Alg(\tau)$ be a variety. Then the following conditions are equivalent*:

(i) *K is a hyperequational class*,
(ii) *K is solid*,
(iii) *$IdK \subseteq HIdK$, i.e. every identity of K is a hyperidentity*,
(iv) *$\Xi[IdK] = IdK$, i.e. IdK is closed under hypersubstitutions.* ■

For a given type τ by $\mathcal{L}(\tau)$ we denote the lattice of all varieties of this type and by $S(\tau)$ the set of all solid varieties of this type. Then we have the following results:

PROPOSITION 2.6 ([2], [5]):

(i) *The set $S(\tau)$ forms a sublattice of $\mathcal{L}(\tau)$*,
(ii) *If τ is a finite type then the lattice $S(\tau)$ is atomic. The unique atom is the variety RA_τ of all rectangular algebras of type τ. (RA_τ is the*

variety generated by all algebras of type τ whose fundamental operations are projections). ∎

Clearly, every solid variety is pre-solid. Now we are going to characterize pre-solid varieties.

3. The class of all pre-solid varieties of type τ

At first we will prove a theorem similar to Theorem 2.5 for pre-hyperequational classes.

THEOREM 3.1: *Let $K \subseteq Alg(\tau)$ be a variety. Then the following conditions are equivalent:*

(i) *K is a pre-hyperequational class,*

(ii) *K is pre-solid,*

(iii) *$IdK \subseteq H_pIdK$, i.e. every identity of K is a pre-hyperidentity,*

(iv) *$\Xi_p[IdK] = IdK$, i.e. IdK is closed under pre-hypersubstitutions.*

Proof. Let $\mathcal{A}$ be an algebra of K and let $t \approx t'$ be any pre-hyperidentity satisfied in $\mathcal{A}$. Then $\hat{\sigma}_p[t] \approx \hat{\sigma}_p[t'] \in Id\mathcal{A}$ for every pre-hypersubstitution σ_p, i.e. $\Xi_p[t \approx t'] \subseteq Id\mathcal{A}$. Applying the operator Ξ_p on the algebra $\mathcal{A}$ we get $t \approx t' \in Id\Xi_p[\mathcal{A}]$. Therefore we have $H_pId\mathcal{A} \subseteq Id\Xi_p[\mathcal{A}]$ for every $\mathcal{A} \in \mathcal{K}$. Conversely, $t \approx t' \in Id\Xi_p[\mathcal{A}]$ implies $\Xi_p[t \approx t'] \in Id\mathcal{A}$ and therefore $t \approx t' \in H_pId\mathcal{A}$. Altogether we have

$$H_pIdK = Id\Xi_p[K]. \quad (1)$$

Let Σ be a set of equations of type τ and let $\mathcal{A} \in H_pMod\Sigma$, i.e. Σ is pre-hypersatisfied in $\mathcal{A}$ and thus $\Sigma \subseteq H_pId\mathcal{A}$. Then $\Xi_p[\Sigma] \subseteq Id\mathcal{A}$ by Definition 2.3. This means $\mathcal{A} \in Mod\Xi_p[\Sigma]$. Conversely, $\mathcal{A} \in Mod\Xi_p[\Sigma]$ implies $\mathcal{A} \in H_pMod\Sigma$ and we get

$$H_pMod\Sigma = Mod\Xi_p[\Sigma]. \quad (2)$$

With $\Sigma = H_pIdK$ from (2) and (1) we obtain:

$$H_pModH_pIdK = Mod\Xi_p[H_pIdK] = ModId\Xi_p[K]$$

and therefore

$$H_pVarK = Var\Xi_p[K]. \quad (3)$$

Now, let K be a pre-hyperequational class, i.e. $K = H_pVarK$. Then by (3) we have $K = Var\Xi_p[K]$. Clearly, $\Xi_p[K] \subseteq K = Var\Xi_p[K]$. Together with the closure property (Proposition 1.2 (i')) we get $\Xi_p[K] = K$ and K is pre-solid.

This shows (i)$\Rightarrow$(ii).

Let K be pre-solid. By definition we have $\Xi_p[K] = K$ and further $IdK = Id\Xi_p[K] = H_pIdK$ by (1). Thus (iii) is satisfied.

From $IdK = H_pIdK$ by definition of a pre-hyperidentity it follows that IdK is closed under pre-hypersubstitutions. This shows: (iii)⇒(iv).

The equation $IdK = H_pIdK$ and the definition of a pre-hyperidentity show that IdK is closed under pre-hypersubstitutions, i.e. $\Xi_p[IdK] \subseteq IdK$. Together with $IdK \subseteq \Xi_p[IdK]$ we get that (iii)⇒(iv).

By definition of a pre-hyperidentity the equation $\Xi_p[IdK] = IdK$ implies $IdK = H_pIdK$ and further $K = VarK = ModIdK = Mod\Xi_p[IdK] = Mod\Xi_p[H_pIdK] = H_pModH_pIdK$ by (2). This means $K = H_pVarK$ and K is a pre-hyperequational class. ∎

Note that the equivalence of (i) and (ii) is a Birkhoff-type-characterization of pre-hyperequational classes. A variety is a pre-hyperequational class if and only if it is closed under the operator Ξ_p.

Let $S_p(\tau)$ be the class of all pre-solid varieties of type τ. Then we have:

THEOREM 3.2: *$S_p(\tau)$ forms a meet-subsemilattice of $\mathcal{L}(\tau)$ containing $S(\tau)$ as a sublattice.*

Proof. Let V_1 and V_2 be two pre-solid varieties of type τ. The inclusion $V_1 \cap V_2 \subseteq V_i$, shows $\Xi_p[V_1 \cap V_2] \subseteq \Xi_p[V_i] = V_i, (i = 1,2)$ and $\Xi_p[V_1 \cap V_2] = V_1 \cap V_2$. Since $V_1 \vee V_2$ agress with $V_1 \cap V_2$ by Theorem 3.1 the variety $V_1 \wedge V_2$ is pre-solid. Since every solid variety is pre-solid and since the solid varieties of type τ form a lattice the second property is clear.

4. Pre-solid varieties of semigroups

By $\mathcal{L}(S)$ we denote the lattice of all semigroup varietes. Now we will describe a bit more of the structure of all pre-solid varieties of semigroups. We start with the observation that for a variety of semigroups to be solid it must satisfy the associative law as a hyperidentity. In [9] semigroup varieties with this property are called hyperassociative. In [1] the hyperequational class defined by the associative law was determined.

Consider the following sets of identities:

$$I_1 := \{(x^{k_1}y^{k_2}\dots x^{k_{n-1}}y^{k_n})^{k_1}z^{k_2}\dots(x^{k_1}y^{k_2}\dots x^{k_{n-1}}y^{k_n})^{k_{n-1}}z^{k_n} \approx x^{k_1}(y^{k_1}z^{k_2}\dots y^{k_{n-1}}z^{k_n})^{k_2}\dots x^{k_{n-1}}(y^{k_1}z^{k_2}\dots y^{k_{n-1}}z^{k_n})^{k_n} \mid n \in \{2,4,6\} \text{ for } 1 \le k_1,\dots,k_n \le 3\}.$$

$$I_2 := \{(x^{k_1}(y^{k_1}z^{k_2}y^{k_3}\dots z^{k_{n-1}}y^{k_n})^{k_2}\dots(y^{k_1}z^{k_2}y^{k_3}\dots z^{k_{n-1}}y^{k_n})^{k_{n-1}}x^{k_n} \approx (x^{k_1}y^{k_2}x^{k_3}\dots y^{k_{n-1}}x^{k_n})^{k_1}z^{k_2}\dots(x^{k_1}y^{k_2}x^{k_3}\dots y^{k_{n-1}}x^{k_n})^{k_n} \mid n \in \{3,5\} \text{ for } 1 \le k_1,\dots,k_n \le 3\}.$$

We put

$$V_{HS} := Mod(I_1 \cup I_2 \cup \{x^2 \approx x^4\}).$$

Then for any class K of semigroups the following is equivalent:

(i) $K \subseteq V_{HS}$
(ii) $K \subseteq HMod(F(F(x,y),z) \approx F(x,F(y,z))$.

It is easy to check that for a variety of commutative semigroups the following proposition is true:

PROPOSITION 4.1: *Let V be a nontrivial variety of commutative semigroups. Then V is hyperassociative if and only if it fulfils the identity* $x^2 \approx x^4$. ∎

Since V_{HS} is a hyperequational class the following is obvious:

PROPOSITION 4.2 ([1]): *The variety V_{HS} is solid and for any variety V of solid semigroups,* $V \subseteq V_{HS}$.

Let $\mathcal{S}(V_{HS})$ be the lattice of all solid semigroup varieties. According to Proposition 2.6, every solid semigroup variety contains the variety RA_2. It is well-known that the variety RA_2 is equal to the variety RB of all rectangular bands which is defined by the identities $x(yz) \approx (xy)z$, $x^2 \approx x$, $xyz \approx xz$. Then we obtain:

PROPOSITION 4.3 ([7]): *RB is the least nontrivial element of* $\mathcal{S}(V_{HS})$. ∎

Of course, not every variety in the interval between RB and V_{HS} is solid. But for pre-solid varieties of semigroups we have:

PROPOSITION 4.4: *The variety V_{HS} is pre-solid and for any pre-solid variety V of semigroups,* $V \subseteq V_{HS}$. ∎

Proof. As a solid variety V_{HS} is pre-solid. Since V_{HS} is the hyperequational class generated by the associative law it is also the pre-hyperequational class generated by the associative law and thus the greatest pre-solid variety of semigroups. ∎

Let $S_p(V_{HS})$ be the class of all pre-solid semigroup varieties. We want to discuss the following question:

Are there pre-solid semigroup varieties in the interval between RB and V_{HS} which are not solid?

Attacking our question we prove:

LEMMA 4.5: *Let V be a variety of type $\tau = (2)$ such that $RB \subseteq V$. Then V is solid if and only if V is pre-solid.*

Proof. The "⇒" — direction is trivial. Let V be pre-solid. If $t \approx t' \in IdV$ and if σ_p is a pre-hypersubstitution then $\sigma_p[t] \approx \sigma_p[t'] \in IdV$. If σ is a hypersubstitution different from a pre-hypersubstitution then σ assigns to the binary fundamental operation one of the projections, i.e. $t \approx t'$ must be satisfied in a projection algebra and therefore it must be satisfied in RB. (RB is the variety generated by the projection algebras). Because of $RB \subseteq V$

we have $IdRB \supseteq IdV$, i.e. every identity in V is satisfied in a projection algebra. Therefore, for any hypersubstitution we have $\sigma[t] \approx \sigma[t'] \in IdV$ and by Theorem 3.1 the variety V is solid. ■

As a consequence of this result a pre-solid variety of semigroups which is not solid must be outside of the interval between RB and V_{HS}. Notice that the same argument works if we have an n-ary operation symbol, i.e. if $\tau = (n)$ for arbitrary $n \geq 2$. It fails to work in the general case.

Now we consider examples for pre-solid varieties. By Proposition 4.4 all these varieties must be hyperassociative. In [1] we determined all minimal hyperassociative semigroup varieties. They are generated by the following sets of identities:

1. $x(yz) \approx (xy)z, xy \approx yx, x^2 \approx x$, (semilattices),
2. $x(yz) \approx (xy)z, xy \approx x$, (right semigroups),
3. $x(yz) \approx (xy)z, xy \approx y$, (left semigroups),
4. $x(yz) \approx (xy)z, xy \approx zu$, (zero semigroups),
5. $x(yz) \approx (xy)z, xy \approx yx, x^2y \approx y$, (commutative groups of exponent 2).

THEOREM 4.6: *The variety of zero-semigroups is the only pre-solid atom in the lattice of all semigroup varieties.*

Proof. Substituting the binary term $t(x, y) = x^2$ in the commutative law we obtain $x^2 \approx y?2$ and by indempotency $x \approx y$, i.e. no nontrivial semilattice is presolid. Substituting $t(x, y) = x^2$ in $F(x, y) \approx y$ or $t(x, y) = y^2$ in $F(x, y) \approx x$ we get $x^2 \approx y$ or $y^2 \approx x$, respectively. Using the idempotency we have $x \approx y$, i.e. a nontrivial right or left semigroup cannot be pre-solid.

Substituting any binary term t for F in $F(x, y) \approx F(z, u)$ and using the identity $xy \approx zu$ we get an identity in a zero semigroup.

$F(x, F(x, y)) \approx y$ is not hypersatisfied in a commutative group of exponent 2 since for $t(x, y) = x^2$ we get $x^2 \approx y$ and therefore $x^2y \approx y^2$. Using the identity $x^2y \approx y$ we have $y^2 \approx y$, i.e. all semilattice identities would be saisfied in contradiction to the minimality of the variety of all semilattices. ■

Note that there is the following striking difference between hypersubstitutions and pre-hypersubstitutions. The operator Ξ_p is not permutable with the application of Birkhoff's derivation rules for identities as the example of semilattices shows. Using at first the identities in a semillattice we obtain only $t_1(x, y) = x, t_2(x, y) = y$, and $t_3(x, y) = xy$ as binary semigroup terms. Since we have to substitute only $t_3(x, y) = xy$ in the axioms the variety of all semilattices would be pre-solid. But the definition of a pre-hypersubstitution says that we have at first to substitute arbitrary binary terms of the type $\tau = (2)$. Substituting $t(x, y) = x^2$ in the commutative law we obtain $x^2 \approx y^2$

and then using the idempotency we have $x \approx y$. This means, no nontrivial semilattice can be pre-solid.

Since the commutative law cannot be a hyperidentity there is no nontrivial solid variety of commutative semigroups. By Proposition 4.1 and Proposition 4.2 a pre-solid variety of commutative semigroups must satisfy the identity $x^2 \approx x^4$. We ask for the greatest pre-solid variety of commutative semigroups. By Theorem 3.1 we have to determine the pre-hyperequational class generated by the associative and the commutative law.

THEOREM 4.7: *For every class K of semigroups the following propositions are equivalent:*

(i) $K \subseteq H_p Mod\{F(F(x,y),z) \approx F(x,F(y,z)), F(x,y) \approx F(y,x)\}$,

(ii) $K \subseteq Mod\{(xy)z \approx x(yz), xy \approx yx, xy^2 \approx x^2y, x^2 \approx y^2\}$.

Proof. (i)⇒(ii): Let $\mathcal{S} \in K$. Assume that (i) is satisfied. Substituting the binary term $t_1(x,y) = x^2 \neq x$ in the hyperidentity $F(x,y) \approx F(y,x)$ we get $x^2 \approx y^2$. With $t_2(x,y) = xy$ we obtain $xy \approx yx$ and $t_3(x,y) = xy^2$ leads to $xy^2 \approx x^2y$. The associative hyperidentity gives the associative identity. This shows that $\mathcal{S} \in Mod\{(xy)z \approx x(yz), xy \approx yx, xy^2 \approx x^2y, x^2 \approx y^2\}$.

(ii)⇒(i): Assume that (ii) is satisfied. By Proposition 4.1 $K \subseteq V_{HS}$ and the associative law is a pre-hyperidentity satisfied in S. We show that the commutative law is a pre-hyperidentity too. Firstly, from the identities $x^2 \approx y^2$ and $x^2y \approx xy^2$ we get the identity $x^2y \approx x^2y^2$ and then $x^2y \approx x^2$. Let now $t(x,y)$ be a binary term different from a variable. Substituting $t(x,y)$ in the pre-hypercommutative law and using the identities $xy \approx yx, x^2y \approx xy^2$, and $x^2y \approx x^2$ we have

a) $t(x,y) \approx xy \approx yx \approx t(y,x)$ or

b) $t(x,y) \approx x^2 \approx y^2 \approx t(y,x)$.

Therefore the commutative law is a pre-hyperidentity and $\underline{S} \in Mod\{F(F(x,y),z) \approx F(x,F(y,z)), F(x,y) \approx F(y,x)\}$. ∎

We set $V_{PC} := Mod\{(xy)z \approx x(yz), xy \approx yx, xy^2 \approx x^2y, x^2 \approx y^2\}$. By Theorem 3.1 V_{PC} is the greatest pre-solid variety of commutative semigroups.

References

[1] K. Denecke, J. Koppitz, *Hyperassociative semigroups*, preprint, 1993, to appear in Semigroup Forum.

[2] K. Denecke, D. Lau, R. Pöschel, D. Schweigert, *Hyperidentities, hyperequational classes and clone congruences*, Contributions to General Algebra 7, Verlag Hölder-Pichler-Tempsky, Wien 1991 — Verlag B.G. Teubner, Stuttgart (1991) 97–118.

[3] K. Denecke, S. L. Wismath, *Solid varieties of semigroups*, preprint, 1993, to appear in Semigroup Forum.
[4] T. Evans, *The lattice of semigroup varieties*, Semigroup Forum Vol. 2, No 1, (1971), 1–43.
[5] E. Graczyńska, D. Schweigert, *Hyperidentities of a given type*, Algebra Universalis 27 (1990), 305–318.
[6] I. A. Mal'cev, *Iterative Post's Algebras* (Russian). Novosibirsk 1976.
[7] R. Pöschel, M. Reichel, *Projection algebras and rectangular algebras*, General Algebra and Applications, Heldermann-Verlag, Berlin (1993), 180–194.
[8] W. Taylor, *Hyperidentitied and hypervarieties*, Aequationes Math. 23 (1981), 111—127.
[9] S. Wismath, *Hyperidentity bases for rectangular bands and other semigroup varieties*, to appear in Journal of the Australian Math. Society.

FACHBEREICH MATHEMATIK
UNIVERSITÄT POTSDAM
Am Neuen Palais 10
14 415 POTSDAM, GERMANY

Received October 22, 1993.