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# ON AN ALGEBRAIC EQUIVALENT OF TETRAHEDRAL QUADRUPLE SYSTEMS

*Dedicated to Professor Tadeusz Traczyk*

## 1. Introduction and preliminaries

In [13] a class of quadruple systems called tetrahedral quadruple systems (TQSs) was defined. TQSs represent a generalization of Mendelsohn triple systems different from generalizations in [9], [11]. A TQS of order $v$ is a pair $(S,T)$, where $S$ is a finite set of $v$ elements and $T$ is a family of directed quadruples $\langle abcd\rangle, a,b,c,d$ distinct elements of $S$, such that every ordered triple of distinct elements of $S$ belongs to exactly one directed quadruple from $T$. A directed quadruple $\langle abcd\rangle$ is the following set of 12 ordered triples

$$\langle abcd\rangle = \{(abc),(bca),(cab),(adb),(dba),(bad),\\ (acd),(cda),(dac),(bdc),(dcb),(cbd)\}\,.$$

It was proved in [13] that TDSs are equivalent to generalized idempotent alternating symmetric (GIAS) 3-quasigroups, their properties were investigated and some parts of the spectrum of TQSs determined. In [5] further investigation of TQSs was carried on and it was proved that the spectrum of TQSs consists of all $v$ such that $v \equiv 1,2,4,5,8,10 \pmod{12}$. In [12] Mendelsohn triple systems derived from TQSs were considered.

The sequence $x_m, x_{m+1}, \ldots, x_n$ is denoted by $\{x_i\}_{i=m}^{n}$ or by $x_m^n$. If $m > n$, then $x_m^n$ will be considered empty.

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An $n$-ary groupoid ($n$-groupoid) $(Q; f)$ is called an $n$-quasigroup if the equation $f(a_1^{i-1}, x, a_{i+1}^n) = b$ has a unique solution $x$ for every $a_1^n, b \in Q$ and every $i \in \{1, \dots, n\}$.

By $S_n$ we denote the symmetric group of degree $n$ and by $A_n$ its alternating subgroup.

If $(Q; f)$ is an $n$-quasigroup and $\sigma \in S_{n+1}$, then the $n$-quasigroup $(Q; f^\sigma)$ defined by

$$f^\sigma(\{x_{\sigma(i)}\}_{i=1}^n) = x_{\sigma(n+1)} \Leftrightarrow f(x_1^n) = x_{n+1}$$

is called a $\sigma$-conjugate (or simply conjugate) of $f$. The set of all $\sigma$ such that $f = f^\sigma$ is a subgroup of $S_{n+1}$.

An $n$-quasigroup $(Q; f)$ is called

a) totally symmetric (TS) if $f = f^\sigma$ for all $\sigma \in S_{n+1}$,

b) alternating symmetric (AS) iff $f = f^\sigma$ for all $\sigma \in A_{n+1}$. AS-$n$-quasigroups were introduced and investigated in [10].

An $n$-groupoid $(Q; f)$ is called alternating symmetric iff for every permutation $\sigma \in A_{n+1}$

$$f(\{x_{\sigma(i)}\}_{i=1}^n) = x_{\sigma(n+1)} \Leftrightarrow f(x_1^n) = x_{n+1}.$$

It is not difficult to see that every AS-$n$-groupoid is necessarily an AS-$n$-quasigroup.

If $(Q; f)$ is an $n$-quasigroup and $\sigma \in S_{n+1}, \sigma^{-1}(n+1) = k$, then $f = f^\sigma$ iff for all $x_1^n \in Q$

$$f(\{x_{\sigma(i)}\}_{i=1}^{k-1}, f(x_1^n), \{x_{\sigma(i)}\}_{i=k+1}^n) = x_{\sigma(n+1)}.$$

Hence AS-$n$-quasigroups can be defined as $n$-quasigroups satisfying a system of identities.

It is easy to see that an $n$-quasigroup $(Q; f)$ is an AS-$n$-quasigroup iff $f = f^\sigma$ for every $\sigma \in \Gamma$, where $\Gamma$ is a generating set of the group $A_{n+1}$.

This implies that for $n = 3$ we have the following.

A 3-groupoid $(Q; f)$ is AS iff the following identities are satisfied

$$\begin{cases} f(x, y, z) = f(y, z, x), \\ f(y, f(x, y, z), z) = x. \end{cases}$$

A 3-groupoid $(Q; f)$ is called generalized idempotent (GI) iff for all $x, y \in S$

$$f(x, y, y) = f(y, x, y) = f(y, y, x) = x.$$

An AS-3-groupoid which is GI is called a GIAS-3-groupoid.

So, a 3-groupoid $(Q; f)$ is a GIAS-3-groupoid iff it satisfies the following identities

$$\begin{cases} f(x,y,z) = f(y,z,x), \\ f(y, f(x,y,z), z) = x, \\ f(x,y,y) = x. \end{cases}$$

Hence the class of all GIAS-3-groupoids is a variety.

In [13] it is proved that finite GIAS-3-groupoids are equivalent to TQSs.

If $(S,T)$ is a TQS of order $v$, and $f$ is defined for distinct elements $x,y,z,u \in S$ by

$$f(x,y,z) = u \Leftrightarrow \langle xyzu \rangle \in T \tag{1}$$

and

$$f(x,y,y) = f(y,x,y) = f(y,y,x) = x,$$

then $(S,f)$ is GIAS-3-groupoid of order $v$. Conversely, if $(S,f)$ is a GIAS-3-groupoid of order $v$, then by (1) a TQS $(S,T)$ of order $v$ is defined.

Hence GIAS-3-groupoids coordinatize TQSs. Since GITS-3-quasigroups are equivalent to Steiner quadruple systems, and every GITS-3-quasigroup is also a GIAS-3-quasigroup, it follows that TQSs represent a generalization of Steiner quadruple systems. Coordinatization of Steiner systems and their corresponding algebras were considered in [3], [4], [6], [7], [8].

## 2. The algebra of GIAS-3-groupoids

THEOREM 1. *Let $\mathcal{U} = (Q; f)$ be a GIAS-3-groupoid and let $C(\mathcal{U})$ be the congruence lattice of $\mathcal{U}$. Then*

*a) If $\theta \in C(\mathcal{U})$, then each $\theta$-class is a subalgebra of $\mathcal{U}$,*

*b) $\mathcal{U}$ has permutable congruences,*

*c) $\mathcal{U}$ has regular congruences,*

*d) $\mathcal{U}$ has uniform congruences,*

*e) $\mathcal{U}$ has coherent congruences.*

Proof. a) Obvious.

b) Follows from Mal'cev's theorem (a variety has permutable congruences iff it has a ternary polynomial $f(x,y,z)$ such that $f(x,y,y) = f(y,y,x) = x$.

c) Let $[a]\theta$, $\theta \in C(\mathcal{U})$, be a $\theta$-class. If $x \equiv y(\theta)$ then $f(x,y,a) \equiv f(y,y,a)(\theta)$, hence $a \equiv f(x,y,a)(\theta)$. Conversely, if $a \equiv f(x,y,a)(\theta)$, then $f(a,x,a) \equiv f(f(x,y,a),x,a)(\theta)$ and since $\mathcal{U}$ is AS $f(f(x,y,a),x,a) = y$, hence $x \equiv y(\theta)$. We have proved that for all $x, y \in Q, x \equiv y(\theta)$ iff $a \equiv f(x,y,a)(\theta)$, so one $\theta$-class defines the whole congruence.

d) Let $\theta \in C(\mathcal{U})$, $a, b \in Q$, $a \equiv b(\theta)$. The mapping $\varphi : [a]\theta \to [b]\theta$ defined by $\varphi(x) = f(x,a,b)$ is a bijection. $\varphi$ is obviously 1-1, and if $y \in [b]\theta$, then $x = f(y,b,a) \in [a]\theta$ is such that $\varphi(x) = f(f(y,b,a),a,b) = y$.

e) Let $\mathcal{B} = (B; f)$ be a subalgebra of $\mathcal{U}$ which contains a congruence class $C = [a]\theta$. If we assume that there exist elements $p \in Q \setminus B$, $q \in B \setminus C$, such that $p \equiv q(\theta)$, and if $r$ is an arbitrary element from $C$, then since the mapping $f : [r]\theta \to [q]\theta$ defined by $x \mapsto f(x, r, q)$ is a bijection, it follows that there exist an element $r_1 \in C$ such that $f(r_1, r, q) = p$. But, since $\mathcal{B}$ is a subalgebra, $p \in B$, which is a contradiction. Hence all elements congruent to an element of $B$ belong to $B$, i.e. a subalgebra which contains a congruence class must be a union of congruence classes.

We have proved that if a GIAS-3-groupoid has a nontrivial congruence, then that congruence is uniform and each congruence class is a subalgebra. Since factor algebra is also a GIAS-3-groupoid we have the following corollary.

COROLLARY 1. *A necessary condition that a finite GIAS-3-groupoid of order $v$ has nontrivial congruences, is that $v \equiv v_1 v_2$, where $v_1, v_2$ are integers greater than 1 such that $v_1, v_2 \equiv 1, 2, 4, 5, 8, 10 (mod 12)$.*

In [2] Fraser and Horn studied varietes $V$ with the property that for every $\mathcal{A}, \mathcal{B} \in V$ each congruence $\theta$ of $\mathcal{A} \times \mathcal{B}$ is a product congruence $\theta_1 \times \theta_2$. A variety $V$ of algebras has the Fraser–Horn property if for every $\mathcal{A}, \mathcal{B} \in V$ all congruences of $\mathcal{A} \times \mathcal{B}$ are product congruences. A congruence of a direct product which is not a product congruence is called skew.

THEOREM 2. *A variety of GIAS-3-groupoids does not have the Fraser–Horn property.*

Proof. In [8] it is proved that the variety which coordinatizes Steiner quadruple systems has a skew congruence. Since this variety is a subvariety of the variety $V$ of all GIAS-3-groupoids, it follows that $V$ does not have the Fraser–Horn property.

Using a theorem of Birkhoff ([1]) which states that if every algebra from a variety has permutable congruences and singleton subalgebras, then every finite algebra from that variety has a decomposition into a direct product of directly irreducible algebras which is unique up to isomorphism of the factors and up to their sequence, by Theorem 1 we get the next theorem.

THEOREM 3. *Each finite GIAS-3-groupoid has a decomposition into a direct product of directly irreducible factors which is unique up to isomorphism of the factors and up to their sequence.*

Now we shall consider normal subalgebras of GIAS-3-groupoids (a subalgebra is called normal if it is a congruence class) and establish some criterions for a subalgebra to be normal. By the preceding results a normal subalgebra of a GIAS-3-groupoid determines the whole congruence. If $\mathcal{B}$ is a normal subalgebra of a GIAS-3-groupoid $\mathcal{U}$, then by $x \equiv y(\mathcal{B})$ we denote that $x$ is congruent to $y$ in a congruence determined by $\mathcal{B}$.

THEOREM 4. *Let $\mathcal{U} = (Q; f)$ be a GIAS-3-groupoid, $\mathcal{B} = (B; f)$ a subalgebra in $\mathcal{U}$ and $\theta \in C(\mathcal{U})$. $\mathcal{B}$ is normal with respect to $\theta$ in $\mathcal{U}$ iff*

$$x \equiv y(\theta) \Leftrightarrow (\exists a \in B) f(a, x, y) \in B. \tag{2}$$

Proof. Let $\mathcal{B}$ be normal. If $x \equiv y(\theta)$, then $f(a, x, x) \equiv f(a, x, y)(\theta)$, hence $f(a, x, y) \in B$.

Conversely, if there is $a \in B$ such that $f(a, x, y) \in B$, then $f(a, x, a) \equiv f(f(a, x, y), x, a)(\theta)$. Since $f(f(a, x, y), x, a) = y, x \equiv y(\theta)$

If (2) is valid, then obviously $B$ is a congruence class.

We note that it is not difficult to see that

$$x \equiv y(\theta) \Leftrightarrow (\exists a \in B) f(a, x, y) \in B \Leftrightarrow (\forall a \in B) f(a, x, y) \in B.$$

THEOREM 5. *A subalgebra $\mathcal{B} = (B; f)$ of a GIAS-3-groupoid $\mathcal{U} = (Q; f)$, is normal iff for all $x_1^3, y_1^3 \in Q$ and all $a \in B f(a, x_i, y_i) \in B, i = 1, 2, 3$ imply $f(a, f(x_1^3 y_1^3 \in Q$ and all $a \in B f(a, x_i, y_i) \in B, i = 1, 2, 3$ imply $f(a, f(x_1^3), f(y_1^3)) \in B$.*

Proof. Let $\mathcal{B}$ be normal. If $f(a, x_1, y_i) \in B, i = 1, 2, 3$, then by Theorem 4 $x_i \equiv y_i(\mathcal{B})$, hence $f(x_1^3) \equiv f(y_1^3)(\mathcal{B})$.

Conversely, let the implication from the theorem be valid. Then the relation $\equiv$ defined by equivalence (2) is obviously reflexive. If there is $a \in B$, $f(a, x, y) = a_1 \in B$, then $f(a_1, y, x) = a \in B$, hence there is $a_1 \in B$ such that $f(a_1, y, x) \in B$ i.e. $\equiv$ is symmetric. If there exists $a \in B$ such that $f(a, x, y) \in B$ and $f(a, y, z) \in B$, then $f(a, f(x, y, y), f(y, z, y)) = f(a, x, z) \in B$, i.e. $\equiv$ is transitive, hence it is an equivalence relation.

From $f(a, x_i, y_i) \in B$, $i = 1, 2, 3$, we get $f(a, f(x_1^3), f(y_1^3)) \in B$, i.e. $f(x_1^3) \equiv f(y_1^3)(\mathcal{B})$ which means that $\equiv$ is a congruence.

THEOREM 6. *Every subalgebra of a finite GIAS-3-groupoid $\mathcal{U} = (Q; f)$ of order $|Q|/2$ is normal.*

Proof. Let $\mathcal{B} = (B; f)$ be a subalgebra of order $|Q|/2$, $P = Q \setminus B$, and let $\equiv$ be an equivalence relation on $Q$ having two equivalence classes $B$ and $P$.

First we shall prove that

$$x \equiv y \Leftrightarrow (\forall a \in B) f(a, x, y) \in B.$$

If $x \equiv y$, then $x, y \in B$ or $x, y \in P$. If $x, y \in B$, then obviously $(\forall a \in B) f(a, x, y) \in B$. If $x, y \in P$, then $(\forall a \in B)(\exists x_1 \in B) f(x_1, a, y) = x$, since the mapping $z \mapsto f(z,a,y)$ is a bijection of $B$ onto $P$. This implies $f(a,x,y) = x_1 \in B$.

Now let $(\forall a \in B) f(a,x,y) = b \in B$, and assume $x \not\equiv y$. Then $x \in B$, $y \in P$ (or $x \in P, y \in B$), and $f(a,x,y) = b$ implies $f(a,b,x) = y$, which, since $f(a,b,x) \in B$ and $y \in B$, is a contradiction.

It remains to prove that $\equiv$ is a congruence.

If $x_1^3 \in Q$, we shall determine to which equivalenbce class $f(x_1^3)$ belongs. If at least one of the elements $x_1^3$ belongs to $B$, say $x_1 \in B$, then if $x_2^3$ belong to the same class (i.e. $x_2 \equiv x_3$), we have

$$x_2 \equiv x_3 \Leftrightarrow (\forall a \in B) f(a,x_2,x_3) \in B,$$

and $f(x_1^3) \in B$, but if $x_2^3$ are not in the same class, then $f(x_1^3) \in P$.

If $x_1^3 \in P$, then the assumption that $f(x_1^3) = a \in B$ implies $f(a,x_2,x_1) = x_3 \in P$, hence $x_1 \not\equiv x_2$, which is a contradiction. So, in this case $f(x_1^3) \in P$.

If $x_i \equiv y_i, i = 1,2,3$, then from the preceding it follows that $f(x_1^3)$ and $f(y_1^3)$ belong to the same equivalence class, that is $\equiv$ is a congruence.

THEOREM 7. *If a finite GIAS-3-groupoid $\mathcal{U} = (Q; f)$ has a proper subalgebra $\mathcal{B} = (B; f)$ of order $b$, then $|Q| \geq 2b$.*

Proof. Since $B$ is a proper subset of $Q$, there is $p \in Q \setminus B$. If a mapping $\varphi$ is defined by $\varphi(x) = f(x,a,p)$, where $a \in B$ is fixed, then for all $x \in B$, $\varphi(x) \in Q \setminus B$ (since $\varphi(x) = f(x,a,p) = c \in B$ implies a contradiction $f(x,c,a) = p, x,c,a \in B$), hence $\varphi : B \to Q \setminus B$.

Since $f$ is a 3-quasigroup, $\varphi$ is 1-1.

THEOREM 8. *If $\theta$ is a congruence of a GIAS-3-groupoid $(Q; f)$ and $S$ and $T$ two congruence classes, then $S \cup T$ is a subalgebra od $(Q; f)$.*

Proof. For singleton congruence classes the theorem is obviously true. Let $a,b \in S$, $c,d \in T$. We have to prove that $f(a,b,c), f(b,a,c), f(a,c,d)$, $f(a,d,c) \in S \cup T$. Since $a \equiv b(\theta)$ $f(a,a,c) \equiv f(a,b,c)(\theta)$, hence $f(a,b,c) \in T$. Similarly for other cases.

COROLLARY 2. *If $\theta$ is a congruence of a finite GIAS-3-groupoid $(Q; f)$ having more then two congruence classes, then for all $a \in Q$*

$$4|[a]\theta| \geq |Q| .$$

Proof. Follows from Theorems 7 and 8.

THEOREM 9. *Let $(Q; f)$ be a GIAS-3-groupoid and $A, B, C$ its subalgebras such that $A \cup B$ and $A \cup C$ are also subalgebras. Then $(\exists a \in A)(\exists b \in B)(\exists c \in C) f(a,b,c) \in A \cup B \cup C$ iff at least one of the sets $A \cap B$, $A \cap C$ and $B \cap C$ is nonempty.*

Proof. Assume that $f(a,b,c) = a_1 \in A$. Then $f(a,a_1,b) = c$, and since $A \cup B$ is a subbalgebra $c \in A$ or $c \in B$, hence $A \cap C \neq \emptyset$ or $B \cap C \neq \emptyset$. Similarly, if any of the sets $A \cap B$, $A \cap C$ or $B \cap C$ is nonempty, say $A \cap B \neq \emptyset$, then for any $a_1 \in A \cap B$, $a \in A$, $c \in C$, $f(a,a_1,c) \in A \cup C \subseteq A \cup B \cup C$, since $A \cup C$ is a subalgebra. If $B \cap C \neq \emptyset$, then for any $b_1 \in B \cap C$, $a \in A$, $b \in B$, $f(a,b,b_1) \in A \cup B \subseteq A \cup B \cup C$. Analogously in the case when $A \cap C \neq \emptyset$.

A corollary of the preceding theorems is that the union of three congruence classes of a GIAS-3-groupoid is never a subalgebra.

THEOREM 10. *A complement $B$ of a subalgebra $A$ of a finite GIAST-3-groupoid is a subalgebra iff* $|A| = |B|$.

Proof. If $|A| = |B|$, then by Theorem 6 $A$ is normal, hence $B$ is a congruence class and by Theorem 1 it is a subbalgebra.

Let $B = Q \setminus A$ be a subalgebra and let $a \in A$, $b \in B$. If we assume that $|A| > |B|$, then the mapping $f : A \to B$ defined by $f : x \mapsto f(x,a,b)$ is $1-1$, which is a contradiction. Analogously if $|A| < |B|$.

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