DEMONSTRATIO MATHEMATICA
Vol. XXVII No 3–4 1994

Moses Mogambery, Teo Sturm

ON κ-COMPACT ELEMENTS OF BOOLEAN ALGEBRAS

Dedicated to Professor Tadeusz Traczyk

0. Introduction

Two well-known nonequivalent notions of κ-compactness are considered in this paper (κ — an infinite cardinal). The central result is that the sets of all κ-compact / weakly κ-compact elements of a Boolean algebra are ideals of it. The paper is presented in two parts. The first part provides some useful results concerning κ-compactness, weak κ-compactness and the behaviour of the McNeille completion with respect to the compactness. In the second part these notions are considered specifically in the context of Boolean algebras. Finally, some open problems are presented.

I. The case of posets

1. Conventions. Throughout the text we will assume that P is a poset, and κ is an infinite cardinal. Let $X \subseteq P$. We will denote

(i) $\downarrow a = \{x \in P; x \leq a\}$ for $a \in P$.

(ii) $X^u = \{a \in P; x \leq a$ for all $x \in X\}$, the set of all upper bounds of X in P.

(iii) $X^l = \{b \in P; b \leq x$ for all $x \in X\}$, the set of all lower bounds of X in P.

(iv) $X^{ul} = (X^u)^l$.

(v) $C_\kappa(P)$ is the set of all κ-compact elements of P (see Definition 2(i)).

(vi) $W_\kappa(P)$ is the set of all weakly κ-compact elements of P (see Definition 2(iii)).

Mathematics Subject Classification (1991). 06A23, 06E99.

Keywords: algebraic lattice, Boolean algebra, compact element, MacNeille completion, ordered set.

(vii) $cf(\kappa)$ is the cofinality of κ.

2. Definition. Let $a \in P$.

(i) a is called a κ-compact element of P if for every $X \subseteq P$, whenever $a \in X^{ul}$ there exists a subset Y of X such that $|Y| < \kappa$ and $a \in Y^{ul}$.

(ii) X is called a *κ-directed* subset of P if $X \neq \emptyset$, and for each $Y \subseteq X$ with $|Y| < \kappa$ there is an $x \in X$ such that $y \leq x$ for all $y \in Y$. That is, $Y^u \cap X \neq \emptyset$.

(iii) a is called a *weakly κ-compact* element of P if for every κ-directed subset X of P, whenever $a \in X^{ul}$ there is an $x \in X$ such that $a \leq x$.

Weakly ω_0-compact elements are defined in [2], but are referred to as compact elements. The notion of κ-compactness is considered in [5] and [7]. If P is a complete lattice (which is a typical assumption in universal algebra), the relation $a \in X^{ul}$ is equivalent to $a \leq \bigvee X$. Since we shall consider κ-compact elements for Boolean algebras in general, which need not be complete, we have extended the usual definitions.

Before proceeding with the next Lemma, let us recall the following well-known fact: A poset P is said to be *join κ-complete* if for every $X \subseteq P$ such that $0 < |X| < \kappa$, there exists $\bigvee X$ in P.

3. Lemma. Let P be a poset. Then

(i) $C_\kappa(P) \subseteq W_\kappa(P)$.

(ii) If κ is a regular cardinal and P is a join κ-complete. semilattice, then we have $C_\kappa(P) = W_\kappa(P)$.

Proof. (i) Suppose $a \in C_\kappa(P)$ and X is a κ-directed subset of P such that $a \in X^{ul}$. Then by the definition of κ-compactness, there exists a subset Y of X such that $|Y| < \kappa$ and $a \in Y^{ul}$. Since X is κ-directed, there exists $x \in X$ such that $y \leq x$ for all $y \in Y$. Thus $x \in Y^u$ and because $a \in Y^{ul}$ we have that $a \leq x$. Hence $a \in W_\kappa(P)$.

(ii) Let $a \in W_\kappa(P)$ and assume that $a \in X^{ul}$ for some $X \subseteq P$. If $|X| < \kappa$ then the condition of κ-compactness is trivially satisfied for our X. Hence we assume that $\kappa \leq |X|$ and define

$$U = \left\{ \bigvee Y;\ Y \subseteq X \text{ and } 0 < |Y| < \kappa \right\}.$$

Then U is a κ-directed of P (κ is a regular cardinal), and $U^u = X^u$ (see also [5, Lemma 1.8]). Hence $a \in U^{ul}$. Since a is weakly κ-compact, there exists $y \in U$ such that $a \leq y$. As $y \in U$, we have that $y = \bigvee Y$ for some $Y \subseteq X$ with $0 < |Y| < \kappa$. Then $a \in \downarrow \bigvee Y = Y^{ul}$ and the κ-compactness of a follows.

4. Lemma. Suppose X is a subset of a poset P such that $\bigvee X$ exists in P.

(i) If $X \subseteq W_\kappa(P)$ and $|X| < \kappa$, then $\bigvee X \in W_\kappa(P)$.

(ii) If $X \subseteq C_\kappa(P)$ and $|X| < cf(\kappa)$, then $\bigvee X \in C_\kappa(P)$.

Proof. (i) Let $\bigvee X \in M^{ul}$ for some κ-directed subset M of P. Then for each $x \in X$ we have $x \in M^{ul}$. In consequence of the weak κ-compactness of x, there is an $f(x) \in M$ satisfying $x \leq f(x)$. Since $f[X] \overset{df}{=} \{f(x); x \in X\} \subseteq M$ and $|f[X]| \leq |X| < \kappa$, there exists $m \in M$ such that $m \in f[X]^u$. Then also $x \leq f(x) \leq m$ for each $x \in X$. Hence $\bigvee X \leq m$, and $\bigvee X \in W_\kappa(P)$ follows.

(ii) Let $\bigvee X \in M^{ul}$ for some subset M of P. Then for each $x \in X$ we have $x \in M^{ul}$. Since x is κ-compact, there exists $F(x) \subseteq M$ such that $|F(x)| < \kappa$ and $x \in (F(x))^{ul}$. Define

$$S = \bigcup \{F(x), x \in X\}.$$

Since $|X| < cf(\kappa)$ and $|F(x)| < \kappa$ for each $x \in X$, we have

$$|S| \leq \sum_{x \in X} |F(x)| < \kappa.$$

Moreover, $F(x)^{ul} \subseteq S^{ul}$ for each $x \in X$. Since $x \in F(x)^{ul}$ for every $x \in X$, we have $x \leq s$ for each $x \in X$ and each (fixed) $s \in S^u$. Hence $\bigvee X \leq s$ for every $s \in S^u$, that is, $\bigvee X \in S^{ul}$. The κ-compactness of $\bigvee X$ follows.

5. The MacNeille completion. Let P be a poset. Then the mapping $\nu_P : \exp P \to \exp P$ defined by

$$\nu_P(X) = X^{ul} \text{ for } X \subseteq P$$

is the well known MacNeille closure operator on $\exp P$. Therefore $N_P \overset{df}{=} \nu_P[\exp P]$ is a complete lattice itself. The mapping $n_P : P \to N_P$ defined by

$$n_P(x) = \nu_P(\{x\}) = \downarrow x \text{ for } x \in P$$

is called the *MacNeille completion* of P. In particular, the mapping n_P is an order embedding of P into N_P. We shall frequently omit the subscript P if P is fixed (in our proofs and remarks for instance).

Numerous properties of the MacNeille completion may be found in [1], [4] and [6]. We note the well-known theorem proved by V.I. Glivenko (1929) and M.H. Stone (1937): The MacNeille completion of a Boolean algebra is a complete Boolean algebra (see [1] and [6]).

6. Theorem. Let $n : P \to N$ be the MacNeille completion of P. Then for each element x of $P, x \in C_\kappa(P)$ iff $n(x) \in C_\kappa(N)$.

Proof. $\Rightarrow$: Suppose that $x \in C_\kappa(P)$ and $n(x) \leq \bigvee S$ in N for some $S \subseteq N$. Then $x \in (\bigcup S)^{ul}$. Since x is κ-compact in P, there is a $Y \subseteq \bigcup S$ satisfying $|Y| < \kappa$ and $x \in Y^{ul}$. For each $y \in Y$ there are elements of S

containing y; we pick one of them and denote it by s_y. Set $T \overset{\text{df}}{=} \{s_y; y \in Y\}$. Then $|T| \leq |Y| < \kappa, T \subseteq S$, and moreover, $x \in Y^{ul} \subseteq (\bigcup T)^{ul} = \bigvee_N T$, that is, $n(x) \leq \bigvee T$ in N. Now $n(x) \in C_\kappa(N)$ follows.

$\Leftarrow$: On the contrary, let $x \in P - C_\kappa(P)$. Then there exists $X \subseteq P$ such that $x \in X^{ul}$, and $x \notin Y^{ul}$ for any $Y \subseteq X$ with $|Y| < \kappa$. Then $n(x) \leq \bigvee n[X]$ in N although $n(x) \nleq \bigvee Z$ in N for any $Z \subseteq n[X]$ of cardinality less than κ. Hence $n(x) \in N - C_\kappa(N)$, and the implication follows.

7. Remark.

7.1 By the above theorem, the notion of κ-compactness is "well-behaved" with respect to the MacNeille completion. For a given P we have

$$n[C_\kappa(P)] = C_\kappa(N) \cap n[P],$$

that is, the κ-compactness is both preserved and reflected here.

7.2 The situation is rather different for weak κ-compactness. Take any $a \in W_\kappa(P) - C_\kappa(P)$. Then $n(a) \notin C_\kappa(N)$ by Theorem 6. If we assume that κ is a regular cardinal, then $C_\kappa(N) = W_\kappa(N)$ by Lemma 3(ii). Hence $n(a) \notin W_\kappa(N)$. Therefore the weak κ-compactness, for regular κ, is of a conditional character. In the case of Boolean algebras, it depends on the degree of the join completeness od a considered Boolean algebra.

7.3 In the case of a singular cardinal κ, we note only the following: Since a subset X of P is κ-directed iff it is κ^+-directed where κ^+ is the cardinal successor of κ (see [5, Lemma 1.9]), we have that $W_\kappa(Q) = W_{\kappa^+}(Q)$ for each poser Q. Especially,

$$C_\kappa(N) \subseteq W_\kappa(N) = W_{\kappa^+}(N) = C_{\kappa^+}(N)$$

since κ^+ is a regular cardinal. It characterizes $W_\kappa(N) \cap n[P]$ in terms of the well-behaved $C_{\kappa^+}(N) \cap n[P]$, namely,

$$W_\kappa(N) \cap n[P] = n[C_{\kappa^+}(P)].$$

However, we cannot say anything about $n(a)$ for $a \in W_\kappa(P) - C_\kappa(P)$ when κ is a singular cardinal.

II. The case of Boolean algebras

8. Lemma. Let M be a subset of a Boolean algebra B and let $a \in B$.

(i) $a \in M^{ul}$ iff $a = \bigvee(a \wedge M)$.

(ii) If a is a weakly κ-compact element of B and $M \subseteq\downarrow a$, then we have:

- (a) if M is κ-directed and if $\bigvee M$ exists, then $\bigvee M \in M$. (That is, M has a greatest element.)
- (b) If M is dually κ-directed and if $\bigwedge M$ exists, then $\bigwedge M \in M$. (That is, M has a least element.)

Proof. (i) $\Rightarrow$: Obviously a is an upper bound of $a \wedge M$. Let $b \in (a \wedge M)^u$. Then for each $m \in M$, we have

$$m = (m \wedge a) \vee (m \wedge a') \leq b \vee (m \wedge a') \leq b \vee a'.$$

Hence $b \vee a' \in M^u$. Since $a \in M^{ul}$, we have $a \leq b \vee a'$. Therefore $a = a \wedge (v \vee a') = a \wedge b$, that is, $a \leq b$. Now $a = \bigvee(a \wedge M)$ follows.

$\Leftarrow$: Take any $b \in M^u$. Then $b \in (a \wedge M)^u$ and hence $a \leq b$. Therefore $a \in M^{ul}$.

(ii.a) Denote $s = \bigvee M$, and define, for $x \in M$,

$$f(x) = a \wedge (s' \vee x).$$

We now claim that f is an order isomorphism of M onto $f[M]$: Clearly, f is an order preserving surjection of M onto $f[M]$. We shall show that f is order reflecting. Assume $x, y \in M$ and $f(x) \leq f(y)$. Then $f(y) = f(x) \vee f(y)$. That is,

$$f(y) = a \wedge (s' \vee y) = f(x) \vee f(y) = a \wedge (s' \vee x \vee y).$$

Also $a \vee (s' \vee y) = a \vee s'$ since $y \leq a$, and $a \vee (s' \vee x \vee y) = a \vee s'$ as $x, y \leq a$. This together with the distributivity of the lattice B yields $s' \vee y = s' \vee x \vee y$. Then $y = (s \wedge s') \vee (s \wedge y) = s \wedge (s' \vee x \vee y) = (s \wedge s') \vee (s \wedge (x \vee y)) = x \vee y$. That is, $x \leq y$ and hence f reflects the order.

Now the assumed existence of $\bigvee M$ yields

$$a = a \wedge (s' \vee s) = a \wedge \left(s' \vee \bigvee M\right) = a \wedge \bigvee(s' \vee M)$$
$$= \bigvee(a \wedge (s' \vee M)) = \bigvee f[M].$$

Moreover, $f[M]$ is κ-directed because it is isomorphic to the κ-directed poset M. Since a is weakly κ-compact, there exists $z \in M$ such that $a = f(z)$. Then $f(z)$ is the greatest element of $f[M]$. Hence z must be the greatest element of M. Especially, $\bigvee M = z \in M$.

(ii.b) Consider the set $a \wedge M'$ where $M' \overset{\mathrm{df}}{=} \{m'; m \in M\}$. Take $x, y \in M$. If $x \leq y$, then $a \wedge y' \leq a \wedge x'$. Conversely, let $a \wedge y' \leq a \wedge x'$. Then $a' \vee y = (a \wedge y')' \geq (a \wedge x')' = a' \vee x$. Therefore

$$y = a \wedge (a' \vee y)[\text{since } y \leq a] \geq a \wedge (a' \vee x) = x.$$

We have just proved that the map $x \mapsto a \wedge x'$ for $x \in M$, is a dual isomorphism of the poset M onto the poset $a \wedge M'$. Therefore, since M is dually κ-directed, $a \wedge M'$ is a κ-directed subset of $\downarrow a$. Also, since there exists the infimum $\bigwedge M$ of M in B, we have

$$\left(a' \vee \bigwedge M\right)' = a \wedge \bigvee M' = \bigvee(a \wedge M'),$$

that is, the supremum $\bigvee(a \wedge M')$ of $a \wedge M'$ does exist in B. Then by part (ii.a) of this lemma, the set $a \wedge M'$ has a greatest element. Since M and $a \wedge M'$ are antiisomorphic posets, M has a least element.

We are now in a position to prove the main theorem of this paper. But before that we require the following definition:

9. Definition.

(i) A subset S of a Boolean algebra B is said to be *hereditary* in B if for all $b \in B$ and all $s \in S, b \leq s$ implies that $b \in S$.

(ii) A set I called a *κ-regular ideal* in B if it is a hereditary subset of B and if $\bigvee X \in I$ for each $X \subseteq I$ with $|X| < \kappa$, provided $\bigvee X$ esists in B. (Observe that every κ-regular ideal is an ideal in B.)

10. Theorem. Suppose B is a Boolean algebra.

(i) $W_\kappa(B)$ is a κ-regular ideal od B.

(ii) $C_\kappa(B)$ is a $cf(\kappa)$-regular ideal of B.

Proof. (i) To show that $W_\kappa(B)$ is hereditary in B, we take $c \in W_\kappa(B)$ and let $a \leq c$. Also assume that M is a κ-directed subset of B such that $a \in M^{ul}$. Then $a \wedge M$ is a κ-directed subset of $\downarrow c$. Moreover, by Lemma 8(i), $a = \bigvee(a \wedge M)$. Therefore by Lemma 8(ii.a), $a \in (a \wedge M)$. Hence $a = a \wedge m$ for some $m \in M$, that is, $a \leq m$. Now $a \in W_\kappa(B)$ follows. That $W_\kappa(B)$ is a κ-regular ideal of B is a simple consequence of Lemma 4(i) and the definition of a κ-regular ideal given in 9(ii).

(ii) Again we first show the heredity of $C_\kappa(B)$. Let $c \in C_\kappa(B)$ and $a \leq c$. Take any $M \subseteq B$ satisfying $a \in M^{ul}$. Then $a = \bigvee(a \wedge M)$ by Lemma 8(i). Define

$$g(x) = c \wedge (a' \vee x) \textit{ for } x \in M.$$

Observe that g is an order isomorphism of M onto $g[M]$ (as shown for f in the proof of Lemma 8(ii.a)), and moreover, $c = \bigvee g[M]$. Since c is a κ-compact element of B, there exists $K \subseteq M$ such that $|K| < \kappa$ and $c \in g[K]^{ul}$. Hence by Lemma 8(i) we have $c = \bigvee(c \wedge g[K]) = \bigvee g[K]$. Therefore

$$a = a \wedge c = a \wedge \bigvee g[K] = \bigvee(a \wedge g[K]) = \bigvee\{a \wedge (a' \vee x)); x \in K\} = \bigvee(a \wedge K).$$

Thus, by Lemma 8(i) again, $a \in K^{ul}$. So $a \in C_\kappa(B)$ follows and the heredity of $C_\kappa(B)$ is established. The rest is a consequence of Lemma 4(ii) and the definition 9(ii) of the regularity of an ideal.

11. Remark. The situation is more or less clear in the case where $\kappa = \omega (= \omega_0)$. Using the usual terminology for ω-compact and ω-directed, we state the following properties of each Boolean algebra B: Obviously, $W_\omega(B) = C_\omega(B)$. Moreover, for every element a of B, a is compact in B iff the set $\downarrow a$ is finite iff a is the join of a finite set of atoms of B. The set of all compact elements of B is the ideal of B which is generated by the set of all atoms of B. Finally, B is an algebraic lattice iff it is isomorphic to the power set algebra of a set.

12. Theorem. Let B be a Boolean algebra and let $n : B \to N$ be its MacNeille completion.

(i) If $X \in C_\kappa(N)$, then $X \subseteq C_\kappa(B)$ and $X = Y^{ul}$ for some $Y \subseteq X$ with $|Y| < \kappa$.

(ii) If $X \subseteq C_\kappa(B)$ such that $|X| < cf(\kappa)$, then $X^{ul} \in C_\kappa(N)$.

Proof. (i) Let $X \in C_\kappa(N)$. Then for each $x \in X$ we have $n(x) \subseteq X^{ul} = X$. Hence $n(x) \in C_\kappa(N)$ by Theorem 10(ii). Now by Theorem 6, $x \in C_\kappa(B)$, and $X \subseteq C_\kappa(B)$ follows.

Since $X \in N$, we have $X = X^{ul}$. Therefore $X = \bigvee_N n|X|$. By the κ-compactness of X in N, there is a set $Y \subseteq X$ such that $|Y| < \kappa$ and $X = \bigvee_N n|Y| = Y^{ul}$.

(ii) If $X \subseteq C_\kappa(B)$ such that $|X| < cf(\kappa)$, then for each $x \in X, n(x) \in C_\kappa(N)$ by Theorem 6. Hence $n[X] \subseteq C_\kappa(N)$. By Lemma 4(ii), $\bigvee_N n[X] \in C_\kappa(N)$. Therefore $X^{ul} \in C_\kappa(N)$.

13. Corollary. Suppose B is a Boolean algebra and $n : B \to N$ is its MacNeille completion.

(i) Let κ be a regular cardinal. Then for each $X \in N, X \in C_\kappa(N)$ iff $X \subseteq C_\kappa(B)$ and $X = Y^{ul}$ for some $Y \subseteq X$ of cardinality less than κ.

(ii) Let v be the least regular cardinal less than κ. Then for each $X \in N, X \in W_\kappa(N)$ iff $X \subseteq C_v(B)$ and $X = Y^{ul}$ for some $Y \subseteq X$ of cardinality less than v.

Proof. (i) The necessary condition is obviously satisfied by the first part of the above theorem. For sufficiency we need only to note that $\kappa = cf(\kappa)$ for regular κ, and apply the second part of Theorem 12.

(ii) From Remark 7.3, we have $W_\kappa(N) = W_v(N) = C_v(B)$, and hence Corollary 13(ii) reduces to Corollary 13(i) since v is regular.

14. Remark. With respect to the set $W_\kappa(B) - C_\kappa(B)$, a natural question to ask is, whether the set is empty for each Boolean algebra B? Trivially, since ω_0 is regular and B is join ω_0-complete, we have Lemma 3(ii), that $W_{\omega_0}(B) = C_{\omega_0}(B)$. However, in general, this is not the case, as is shown in the corollary to the following example.

Suppose A is the finite-cofinite algebra over a set S, i.e.

$$A = \{X \subseteq S;\ |X| < \omega_0 \text{ or } |S - X| < \omega_0\}.$$

Then the following facts hold.

(i) For each $M \subseteq A, M^{ul} = \{X \in A;\ X \subseteq \bigcup M\}$. Therefore $\bigcup M \in A$ iff there exists $\bigvee_A M$; then $\bigcup M = \bigvee_A M$.

(ii) Every element of A is weakly ω_1-compact.

(iii) $X \in A$ is κ-compact in A iff $|X| < \kappa$.

Proof. (i) Obviously $M^u = \{Y \in A;\ \bigcup M \subseteq Y\}$. Therefore $\{X \in A;\ X \subseteq \bigcup M\} \subseteq M^{ul}$. Conversely, let Z be a set such that $Z \not\subseteq \bigcup M$. Take $a \in Z - \bigcup M$. Then $Z \not\subseteq S - \{a\} \in M^u$. Hence $Z \notin M^{ul}$. Now the reverse inclusion $M^{ul} \subseteq \{X \in A;\ X \subseteq \bigcup M\}$ follows since $X^{ul} \subseteq A$. The subsequent assertions are obvious.

(ii) Take any $T \in A$ and an ω_1-directed system $M \subseteq A$ such that $T \in M^{ul}$. Then by Lemma 8(i) and part (i) of this example, $T = \bigcup\{X \cap T;\ X \in M\}$. Hence for each $t \in T$ there exists $X_t \in M$ satisfying $t \in X_t$. Let us denote

$$W = \{X_t;\ t \in T\}.$$

If T is finite, then there exists $X \in M$ such that $\bigcup W \subseteq X$ by the ω_1-directedness of M. Then of course, $T \subseteq X$. So let T be infinite and take any denumerable $T_0 \subseteq T$. By the ω_1-directedness of M, there exists $X \in M$ such that $X_t \subseteq X$ for each $t \in T_0$. Then X is infinite. Hence both the sets $S - X$ and $T - X \subseteq S - X$ are finite. Therefore $\{X\} \cup \{X_t;\ t \in T - X\}$ is a finite subsystem of M, and as such has an upper bound $Y \in M$. Then $T \subseteq Y$ and the weak ω_1-compactness of T follows.

(iii) This fact is a direct consequence of (i): Take $Y \stackrel{\text{df}}{=} \{\{x\};\ x \in X\} \subseteq A$, and any proper subset Z of Y. Then $X \in Y^{ul}$ as $X = \bigcup Y$; nevertheless $X \notin Z^{ul}$ because $X \not\subseteq \bigcup Z$. Therefore X is not a κ-compact element for any infinite cardinal $\kappa \leq |X|$.

Now for some $V \subseteq A$ assume that $X \in V^{ul}$, that is, $X \subseteq \bigcup V$. Then for each $x \in X$ there is $W_x \in V$ such that $x \in W_x$. Then $U \stackrel{\text{df}}{=} \{W_x;\ x \in X\}$ is a subsystem of V of cardinality at most $|X|$, and $X \in U^{ul}$ because $X \subseteq \bigcup U$. Hence X is a $\max\{|X|^+, \omega_0\}$-compact element ($|X|^+$ is the cardinal successor of $|X|$). Especially, $X \in C_\kappa(A)$ for each infinite cardinal $\kappa > |X|$.

15. Corollary. For each uncountable cardinal κ there is a Boolean algebra B such that $W_\kappa(B) - C_\kappa(B) \neq 0$.

Proof. Let κ be an uncountable cardinal, and A_κ be the finite-cofinite algebra over κ. Obviously, $W_v(B) \subseteq W_\mu(B)$ and $C_v(B) \subseteq C_\mu(B)$ for any Boolean algebra B and any infinite cardinals $v \leq \mu$. This, together with (ii) and (iii) of Example 14 yields

$$W_\kappa(A_\kappa) - C_\kappa(A_\kappa) \supseteq W_{\omega_1}(A_\kappa) - C_\kappa(A_\kappa) = \{X \subseteq \kappa;\ |\kappa - X| < \omega_0\} \neq 0.$$

16. Open problems.

(i) In view of Theorem 10 we are prompted to question the nature of the quotient algebra $B/C_\kappa(B)$ (or $B/W_\kappa(B)$). Section 2.5.5 of [3], as well as subsequent volumes of the same publication, offers some rather nontrivial results about $\exp\omega/C_\omega(\exp\omega)$.

(ii) A Boolean algebra B is said to be *κ-algebraic* if

$$x = \bigvee (C_\kappa(B) \cap \downarrow x) \text{ for each } x \in B\,.$$

It is well-known that a Boolean algebra B is algebraic (= ω-algebraic) iff it is isomorphic to the power set algebra $P(S)$ of a set S. The problem is to find a characterization of κ-algebraic Boolean algebras.

Acknowledgement. The second author was partially supported by a South African FRD grant.

References

[1] G. Birkhoff: *Lattice Theory* (3rd Ed.). Amer. Math. Soc., Providence 1967.

[2] G. Gierz, K. H. Hofmann, K. Keimel, J. D. Lawson, M. Mislove, D. S. Scott: *A Compendium of Continuous Lattices*. Springer Verlag, Berlin, Heidelberg, New York 1980.

[3] S. Koppelberg: *Handbook of Boolean algebras*, Vol. 1. North-Holland, New York 1989.

[4] H. M. MacNeille: *Partially ordered sets*, Trans. Amer. Math. Soc. **42** (1937), 416–460.

[5] J. Ryšlinková: *On m-algebraic closures of n-compact elements*, Comment. Math. Univ. Carolin. **19** (1978), 743–754.

[6] L. A. Skornjakov: *Elements of Lattice Theory*. Hindustan Publishing Company, Delhi 1977.

[7] T. Sturm: *Closure operators which preserve the m-compactness*, Boll. Un. Mat. Ital. A(6), **1-B** (1982), 197–209.

Moses Mogambery

DEPARTMENT OF MATHEMATICS, SPRINGFIELD COLLEGE OF EDUCATION, PRIVATE BAG DORMERTON 4015, SOUTH AFRICA.

Teo Sturm

DEPARTMENT OF MATHEMATICS AND APPLIED MATHEMATICS, UNIVERSITY OF NATAL, DURBAN 4001, SOUTH AFRICA.

Received October 8, 1993.