DEMONSTRATIO MATHEMATICA
Vol. XXVII No 3–4 1994

Peter Burmeister, Maciej Mączyński*

ORTHOMODULAR (PARTIAL) ALGEBRAS AND THEIR REPRESENTATIONS

Dedicated to Professor Tadeusz Traczyk

1. Introduction

The theory of orthomodular ordered sets — or orthomodular posets — has been well developed and has found many applications (quantum logics, Hilbert spaces, generalized probability theory). In order to define orthomodular ordered sets one usually introduces an order relation; therefore, from an algebraic point of view, they are algebras with an additional relation. Thus direct applications of important algebraic theorems on varieties, free products and so on to orthomodular ordered sets become difficult — they do not belong to abstract algebras. Moreover, the usual morphisms between them — preserving the nullary constants, the unary operation and the binary relation — are not always adequate for the corresponding orthomodular algebras, which form with the usual homomorphisms between partial algebras a "nice" category, as we shall show in section 5. The aim of this paper is to show that we can formulate the theory of orthomodular ordered sets in the framework of partial algebras, where the general algebraic and model theoretic properties have been investigated in detail (see e.g. [B86] or [B93]). This enables us to apply the theory of partial algebras to these structures and consequently also to quantum logics. In particular we get an adequate concept of morphisms between orthomodular algebras

* The cooperation of the two authors has been supported by the Deutsche Akademische Austauschdienst (DAAD) in connection with an exchange program between the Politechnika Warszawska and the Technische Hochschule Darmstadt.

This paper has been presented at the Conference on Universal Algebra and its Applications, organized by the Institute of Mathematics of Warsaw University of Technology held at Jachranka, Poland, 8–13 June 1993.

(and therefore also between orthomodular ordered sets — however, in their original theory they would not be that easily describable).

We formulate an axiom system for orthomodular algebras which turn out to be equivalent to orthomodular ordered sets, and we discuss some examples, e.g. orthomodular algebras derived from Boolean algebras. Next we prove a representation theorem for orthomodular algebras with a full set of probability measures. We also investigate numerical orthomodular algebras and give their simple characterization. We compare the category of orthomodular posets with that of orthomodular algebras, prove the existence of free orthomodular algebras, and discuss some properties of the category of orthomodular algebras with homomorphisms as morphisms.

2. An axiom system for orthomodular (partial) algebras

Before formulating the axiom system we recall some basic definitions and terminology from the theory of partial algebras. Let $\underline{T}(X,\Sigma)$ be any term algebra of signature or type Σ on some set X, let $t, t_1, t_2 \in T(X,\Sigma)$ be any terms, and let $\underline{A}$ be any partial algebra of type Σ. Thus, in $\underline{A}$ there are defined some operations, among them there may be proper partial operations (the domain of which is not all of A^n — n being the arity of the operation — but only some proper subset of A^n). We recall that an *existence equation* $t_1 \stackrel{e}{=} t_2$ holds in the partial algebra $\underline{A}$, iff for every valuation $v : X \to A$ the induced — i.e., as usual, recursively defined, but now along the partial structure of $\underline{A}$ — interpretations $v^{\sim}(t_1)$ and $v^{\sim}(t_2)$ of t_1 and t_2 *exist* and are equal. It should be observed that as set X of variables from which the valuation starts one usually chooses the set of variables occurring freely in the formula, if not stated otherwise (compare the footnote 3 to axiom (A0) below).

In particular, the *term existence statement* $t \stackrel{e}{=} t$ *is satisfied* in $\underline{A}$ with respect to a valuation $v : X \to A$, iff the interpretation $v^{\sim}(t)$ exists. We shall abbreviate the term existence statement $t \stackrel{e}{=} t$ by $\exists\, t$, i.e.

$$\exists\, t :\Longleftrightarrow t \stackrel{e}{=} t\,.$$

DEFINITION. By an *orthomodular (partial) algebra*[1] we understand a partial algebra $\underline{A} := (A; \oplus, {}', 0)$ of type $(2,1,0)$ such that the following list of axioms is satisfied in $\underline{A}$ for all $x, y, z \in X$ for any given countably infinite set X of variables[2]:

[1] We shall in what follows mostly omit the adjective "partial" and only speak of "orthomodular algebras".

[2] We have omitted brackets, when it does not influence understandability. It might be useful to compare the following axioms and facts with the usual definition of orthomodular

(A0) $\exists\ 0$.[3]
(A1) $x'' \stackrel{e}{=} x$.
(A2) $x \oplus x' \stackrel{e}{=} 0'$. (Notation: $0' =: 1$).
(A3) $x \oplus 0 \stackrel{e}{=} x$.
(A4) $\exists\ (x \oplus y) \Rightarrow x \oplus y \stackrel{e}{=} y \oplus x$.
(A5) $\exists\ ((x \oplus y) \oplus z) \Rightarrow (x \oplus y) \oplus z \stackrel{e}{=} x \oplus (y \oplus z)$.
(A6) $\exists\ (x \oplus y) \wedge \exists\ (y' \oplus z) \Rightarrow \exists\ (x \oplus z)$.
(A7) $\exists\ (x \oplus y') \wedge \exists\ (x' \oplus y) \Rightarrow x \stackrel{e}{=} y$.
(A8) $\exists\ (x \oplus y) \wedge \exists\ (y \oplus z) \wedge \exists\ (x \oplus z) \Rightarrow \exists\ (x \oplus (y \oplus z))$.[4]
(A9) $\exists\ (x \oplus y') \Rightarrow x \oplus (x \oplus y')' \stackrel{e}{=} y$.

An orthomodular algebra shall be called *rich*, if the following additional *richness axiom* holds:

$$(R) \exists\ (x_1 \oplus y_1) \wedge \exists\ (x_1 \oplus y_2) \wedge \exists\ (x_2 \oplus y_1) \wedge \exists\ (x_2 \oplus y_2) \Rightarrow$$
$$\Rightarrow (\exists z)(\exists\ (x_1 \oplus z) \wedge \exists\ (x_2 \oplus z) \wedge \exists\ (y_1 \oplus z') \wedge \exists\ (y_2 \oplus z')).$$

Observe that the operation "$\oplus$" is commutative by (A4) and associative by (A5) (in the sense of "strong equality" or "Kleene equality"). Directly from the above axioms we obtain moreover the following simple observations.

FACTS. *It follows that in each orthomodular algebra $\underline{A}$*

(i) *the unary operation $'$ is always a total bijection (see (A1)); a' is called the orthocomplement of $a \in A$;*
(ii) *for each $a \in A$, $a \oplus a'$ always exists with the constant value $0' = 1$ (see (A2));*

---

ordered sets — as given e.g. in Beran [Be85], pages 144ff and 152 or in Kalmbach [K83]. There they are defined as (bounded) orthocomplemented (partially) ordered sets $(P; \leq, ', 0, 1)$, i.e. ordered sets satisfying for all $a, b \in P$ that $a \leq b$ implies $b' \leq a'$, that $a'' = a$, $0' = 1$, and $a \sqcup a' = 1$; in addition they have to satisfy that, whenever $a \leq b'$, then the supremum $a \sqcup b$ always exists, and finally, that $a \leq b$ always implies $a \sqcup (a \sqcup b')' = b$ (orthomodularity).

Observe moreover that, since the logical operator "or" will be denoted by "$\vee$", the supremum operation of an orthomodular algebra will be denoted by the symbol "$\sqcup$". And since the logical operator "and" will be denoted by "$\wedge$", the infimum operation of an orthomodular algebra will be denoted here by "$\sqcap$".

[3] Observe that in connection with this axiom only valuations starting from the empty set of variables are considered. Exactly this fact guarantees that in each model, say $\underline{A}$, of (A0) the constant $0^{\underline{A}}$ really exists, and that in particular all models of (A0) are non-empty.

[4] For the effect of this axiom see Lemma 2. After a preprint of this note had appeared, S. Pulmannová has shown in [Pu93] that this axiom is the crucial one for orthomodular algebras, which has to be omitted in order to generalize our axiomatization to difference algebras and orthoalgebras, while all other axioms can be kept — and for orthoalgebras a weaker one has to be added.

(iii) *the constant* 0 *always exists (see* $(A0)$[5]*), and, for each element* $a \in A$, $a \oplus 0$ *always exists and yields* a *as value (see* $(A3)$*)*;

(iv) *the operation* $\oplus$ *is commutative (see* $(A4)$*) and associative (see* $(A5)$*), whenever it exists*[6]*; we shall make use of these properties in what follows without explicitly mentioning them.*

Moreover we have:

(v) *If one defines a relation "$\leq$" on an arbitrary orthomodular algebra* $\underline{A}$ *by*

$$a \leq b \text{ iff } a \oplus b' \text{ exists,}^{7}$$

then one easily realizes that from $(A2)$ *there follows reflexivity, that* $(A6)$ *implies transitivity, and that* $(A7)$ *means asymmetry of the relation "$\leq$", i.e. in any orthomodular algeba* $\underline{A}$ *the relation "$\leq$" defined above is always a (partial) order relation on* A*; moreover, the axiom* $(A3)$ *— together with* $(A1)$ *— implies that* 1 $(= 0')$ *is the greatest element of this order relation and* 0 *is its least element*;

(vi) *The axioms* $(A0)$ *through* $(A9)$ *are existence equations or "existentially conditioned existence equations*[8]*", and therefore they define a so-called "ECE-variety*[9]*", which we shall denote by* OMA, *the ECE-variety of all othomodular algebras, while we shall denote the axiomatic subclass of* OMA *of all rich orthomodular algebras by* OMAR.

(vii) *The fact that* OMA *is an ECE-variety is equivalent to saying that* OMA *is closed with respect to the formation of reduced products (therefore in particular w.r.t. direct products), (closed) subalgebras (i.e. relative subalgebras on closed subsets) and closed homomorphic images*[10]*. The*

---

[5] Note that by (A2) or (A3) it would only follow that the constant 0 is always defined in each *non-empty* orthomodular algebra, since the axioms (A1) through (A9) allow an empty model.

[6] Actually, in order to conclude the usual form of associativity, one would also have expected the axiom

$$\text{(A5')} \qquad \exists\ (x \oplus (y \oplus z)) \Rightarrow (x \oplus y) \oplus z \stackrel{e}{=} x \oplus (y \oplus z).$$

However, this follows from (A4) and (A5) (we argue here semantically):
Let $a, b, c \in A$ for some orthomodular partial algebra $\underline{A}$, and let $a \oplus (b \oplus c)$ exist, then, by (A4), also $(c \oplus b) \oplus a$ exists, and therefore one has by (A5) and (A4) that $c \oplus (b \oplus a)$ exists, and that

$$(a \oplus b) \oplus c = c \oplus (b \oplus a) = (c \oplus b) \oplus a = a \oplus (b \oplus c).$$

[7] Observe that by the commutativity of $\oplus$ and by (A1) this is also equivalent to $b' \leq a'$.

[8] They are briefly called ECE-equations.

[9] Compare [B82], see also [B86], section 8.

[10] Observe that a mapping $f : A \to B$ is a *homomorphism* between the orthomodular algebras $\underline{A} = (A; \oplus, ', 0)$ and $\underline{B} = (B; \oplus, ', 0)$, if $f(0) = 0$, if, for all $u, v \in A$, one has

axiomatic structure of OMAR *is more complicated than that of* OMA, *since (R) contains an existential quantifier — different from the one hidden in existence equations.*

(viii) *Since* OMA *is closed in particular with respect to isomorphic copies, subalgebras and direct products of orthomodular algebras, it forms an epireflective subcategory of the category of all partial algebras of similarity type* $(2,1,0)$ *with homomorphisms as morphisms*[11]. *Therefore, for each partial algebra* $\underline{P}$ *of type* $(2,1,0)$ *there exists an* OMA*-universal* OMA*-solution, say* $\underline{F}(\underline{P}, \mathrm{OMA})$, *and the* OMA*-universal homomorphism, say* $r_{\underline{P},\mathrm{OMA}} : \underline{P} \to \underline{F}(\underline{P}, \mathrm{OMA})$ *is an epimorphism*[12]. *However,* $r_{\underline{P},\mathrm{OMA}}$ *is injective — i.e. a monomorphism — only in those cases, where* $\underline{P}$ *allows an injective homomorphism into at least one orthomodular algebra. Therefore, in particular, for each set* X *the* OMA*-free* OMA*-algebra* $\underline{F}(X, \mathrm{OMA})$ OMA*-freely generated by* X *exists — and* $r_{X,\mathrm{OMA}}$ *is injective. Its structure will be characterized in section 5 of this note.*

The meaning of axioms (A8) and (A9) as well as that of the richness axiom (R) will be discussed later, when we shall have more information about the induced order relation in an orthomodular algebra.

We now prove some further properties about orthomodular algebras:

LEMMA 1. *Each orthomodular algebra is non-empty. And an orthomodular algebra is total iff it has exactly one element.*

Proof. Since by axiom (A0) the constant 0 always exists, each orthomodular algebra is non-empty. Let $\underline{A}$ be any total orthomodular algebra; then, for any two elements a and b from A, $a \oplus b'$ and $a' \oplus b$ exist, and therefore one gets $a = b$ by (A7). Since, by (A3), $0 \oplus 0$ $(= 0)$ always exists, each one-element orthomodular algebra is total. ■

LEMMA 2. *If, in an orthomodular algebra* $\underline{A}$ *with the related order relation* $\leq$, $a \oplus b$ *exists, then* $a \oplus b = a \sqcup b$, *i.e. then the supremum* $a \sqcup b$ *in* $(A; \leq)$ *exists and is equal to* $a \oplus b$. *Moreover,* $(a \oplus b)' = a' \sqcap b'$ *(de Morgan's law), i.e. the infimum* $a' \sqcap b'$ *of* a' *and* b' *then exists, too, and is equal to* $(a \oplus b)'$.

$f(u') = f(u)'$ and:

If $u \oplus v$ exists in $\underline{A}$, then $f(u) \oplus f(v)$ exists in $\underline{B}$ and one has $f(u \oplus v) = f(u) \oplus f(v)$.

f is a *closed* homomorphism, iff, in addition, the existence of $f(u) \oplus f(v)$ in $\underline{B}$ always implies that $u \oplus v$ exists in $\underline{A}$. $\underline{B}$ is said to be a *closed homomorphic image* of $\underline{A}$, if there exists a closed and surjective homomorphism from $\underline{A}$ onto $\underline{B}$.

[11] Cf. [B86], subsection 5.11, or originally J.Schmidt [Sch66].

[12] In this category the fact of being an epimorphism means that the image set of the carrier of the source algebra generates the target algebra.

In particular one always has $a \sqcup a' = 1$.

Proof. Let us first observe that (A9) and (A4) imply that one always has

$$\text{if } (a \oplus b) \text{ exists, then } (a \leq a \oplus b) \text{ and } (b \leq a \oplus b).$$

Assume, now, that $a \leq z$ and $b \leq z$. We have to show that $a \oplus b \leq z$, i.e. that $(a \oplus b) \oplus z'$ exists: However, we have by axiom (A8) that if $a \oplus b$, $a \oplus z'$ and $b \oplus z'$ exist, then $a \oplus (b \oplus z')$ $(= (a \oplus b) \oplus z'$ by (A5)) exists, i.e. we get $a \oplus b \leq z$. This shows that $a \oplus b$ is the supremum of a and b in $(A; \leq)$. In connection with axiom (A2) this implies in particular that for all $a \in A$ one has $a \sqcup a' = 0' = 1$.

Concerning the second part of the lemma, observe that we now have $a, b \leq a \sqcup b = a \oplus b$, and therefore, observing footnote 7, $(a \oplus b)' \leq a'$ and $(a \oplus b)' \leq b'$. Now assume that, for some $d \in A$, we have $d \leq a'$ and $d \leq b'$. Then both $d \oplus a$ and $d \oplus b$ — as well as $a \oplus b$ — exist. Hence axiom (A8) implies that $d \oplus (a \oplus b)$ exists. Therefore, $d \leq (a \oplus b)'$, showing that $(a \oplus b)'$ is indeed the greatest lower bound of a' and b'. ∎

LEMMA 3. $a \oplus a$ *or* $a \oplus 1$ *exist in an orthomodular algebra* $\underline{A}$, *iff* $a = 0$; *in particular, in any at least two-element orthomodular algebra one always has* $a \neq a'$, *and the infimum* $a \sqcap a'$ *exists and is equal to* 0, *the least element of* $(A; \leq)$.

Moreover, if $a \oplus b$ *exists, then also the infimum* $a \sqcap b$ *exists and is equal to* 0: $a \sqcap b = 0$.

Proof. $0 \oplus 0$ and $0 \oplus 1$ exist according to (A3) and (A2). If $a \oplus 1$ exists, then $a \oplus 0'$ exists, i.e. $a \leq 0$. However, 0 is the least element of $(A; \leq)$, and we get $a = 0$.

If $a \oplus a$ exists, then this means that $a \oplus (a')'$ exists, i.e. $a \leq a'$, and therefore $a \sqcup a' = a'$. However, since $a \oplus a' = 1$, we have $a \sqcup a' = 1$ by Lemma 2, hence $a' = 1 = 0'$, i.e. $a = 0'' = 0$ by (A1).

By the second statement in Lemma 2 we have $0 = 1' = (a \oplus a') = a' \sqcap a$.

If, finally, $a \oplus b$ exists, then, say, $b \leq a'$; and $a \sqcap a' = 0$ therefore implies $a \sqcap b = 0$. ∎

LEMMA 4. *If* $a \leq b$ *in an orthomodular algebra, then* $b = a \sqcup (a \sqcup b')'$. *In particular* $a \oplus b = 1$ *always implies* $b = a'$.

Proof. Since $a \leq b$, $a \oplus b'$ exists, and consequently $(a \oplus b')'$ also exists. From Lemma 2 there follows that $(a \oplus b')' = (a \sqcup b')'$. Now $a \leq a \sqcup b'$, and this implies that $a \oplus (a \oplus b')'$ exists. By Lemma 2 we get $a \oplus (a \oplus b')' = a \oplus (a \sqcup b')' = a \sqcup (a \sqcup b')'$. By axiom (A9) we infer that $b = a \oplus (a \oplus b')' = a \sqcup (a \sqcup b')'$.

Finally, if $a, b \in A$ with $a \oplus b = 1$, then $a \leq b'$. Hence

$$b' = a \sqcup (a \sqcup b'')' = a \sqcup (a \sqcup b)' = a \sqcup (a \oplus b)' = a \sqcup 1' = a \sqcup 0 = a.$$

Therefore, $b = (b')' = a'$. ∎

Lemma 4 shows that our partially ordered set $(A; \leq)$ corresponding to an orthomodular algebra $(A; \oplus, ', 0)$ is indeed orthomodular[13]. Hence every orthomodular (partial) algebra defines an orthomodular (partially) ordered set. It turns out that both notions are equivalent. This is shown in the following theorem.

THEOREM 1. *Let $(A; \oplus, ', 0)$ be an orthomodular algebra. If we define*

$$a \leq b \text{ iff } \exists\, a \oplus b',$$

then $(A; \leq, ', 0, 1)$ is an orthomodular (partially) ordered set.

Conversely, if $(A; \leq, ', 0, 1)$ is an orthomodular (partially) ordered set, and if we define the partial operation $\oplus$ by

$$a \oplus b = c, \text{ whenever } a \leq b' \text{ and } a \sqcup b = c,$$

then $(A; \oplus, ', 0)$ is an orthomodular algebra.

Moreover, going back and forth with these constructions starting from either of the two kinds of structure always yields back the original one.

Proof. The first part of the theorem has been proved above. In order to prove the second part it suffices to observe that the axioms (A1) through (A9) are implied by the properties of orthomodular (partially) ordered sets. The verification of these axioms is quite obvious and therefore omitted.

The last statement also easily follows from what has been shown so far (e.g. from Lemmas 2 and 4) as well as from the definitions of the transitions. ∎

3. Examples, Boolean algebras

It is well known that the order relation induced in an orthomodular lattice[14], in particular in a Boolean algebra, always yields an orthomodular ordered set, and since here suprema always exist, it is easily realized that

---

[13] Compare footnote 2.

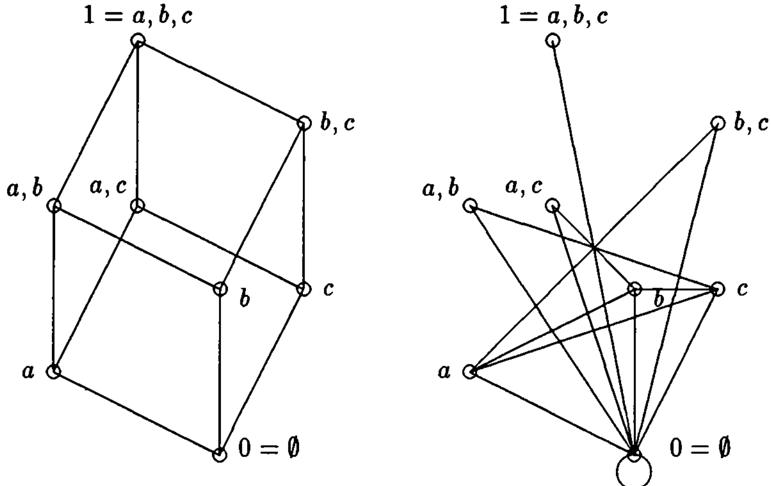
[14] Recall that according to Beran [Be85] — where $\leq$ designates the usual induced order relation with $a \leq b$ iff $a \sqcap b = a$ (iff $a \sqcup b = b$) — $(A; \sqcup, \sqcap, 0, 1, ')$ is an orthomodular lattice, iff it satisfies the axioms

- $(A; \sqcup, \sqcap)$ is a lattice.
- For every $a \in A$ one has $a \sqcup a' = 1$, $a \sqcap a' = 0$ and $a'' := (a')' = a$.
- If $a \leq b$, then $b' \leq a'$ for any $a, b \in A$.
- For any $a, b \in A$ one has that $a \leq b$ implies $a \sqcup (a' \sqcap b) = b$.

they also satisfy the richness axiom (R).

$\oplus$	$\emptyset$	a	b	c	a,b	a,c	b,c	a,b,c
$\emptyset$	$\emptyset$	a	b	c	a,b	a,c	b,c	a,b,c
a	a	–	a,b	a,c	–	–	a,b,c	–
b	b	a,b	–	b,c	–	a,b,c	–	–
c	c	a,c	b,c	–	a,b,c	–	–	–
a,b	a,b	–	–	a,b,c	–	–	–	–
a,c	a,c	–	a,b,c	–	–	–	–	–
b,c	b,c	a,b,c	–	–	–	–	–	–
a,b,c	a,b,c	–	–	–	–	–	–	–

Table 1: Composition table of $\oplus$ in $(\mathfrak{P}(\{a,b,c\}); \oplus, ', 0)$ (– : undefined)



a. The order relation b. The domain of $\oplus$

Figure 1: The Boolean algebra $\underline{B}_3$ as an orthomodular algebra

As a particular **example** we consider the Boolean algebra $\underline{B}_3$ with three atoms in its representation as the power set of the set $\{a, b, c\}$ (set theoretical brackets are omitted in the composition table[15] for $\oplus$ and in the corresponding figures[16]). In Figure 1a the usual (partial) order relation is represented, and in Figure 1b the pairs in dom $\oplus$ are connected by a line — the unary operation $'$ is just the set theoretical complementation.

The following lemma will be used in the proof of the next theorem, in which we want to characterize those orthomodular algebras belonging to Boolean algebras:

[15] See Table 1.

[16] See Figure 2.

LEMMA 5. *Let $\underline{A}$ be an orthomodular algebra, and assume that $a = c_1 \oplus c_2$, $b = c_2 \oplus c_3$, and that $c_1 \oplus c_3$ exists. Then the supremum $a \sqcup b$ exists in $(A; \leq)$, and we have $a \sqcup b = c_1 \oplus c_2 \oplus c_3$. — Moreover, the infimum $a \sqcap b$ exists in $(A; \leq)$, and we have $a \sqcap b = c_2$.*

Proof. Namely, from Axiom (A8) and the assumptions it follows that $c_1 \oplus (c_2 \oplus c_3)$ exists. From Lemma 2 we obtain $c_1 \oplus (c_2 \oplus c_3) = c_1 \oplus (c_2 \sqcup c_3) = c_1 \sqcup (c_2 \sqcup c_3) = (c_1 \sqcup c_2) \sqcup (c_2 \sqcup c_3) = a \sqcup b$. Hence $a \sqcup b$ exists, and $a \sqcup b = c_1 \oplus c_2 \oplus c_3$.

In order to realize the second statement, let $c_4 := (c_1 \oplus c_2 \oplus c_3)'$. Then $(c_1 \oplus c_2 \oplus c_3) \oplus c_4 = 1$. Obviously $c_2 \leq a$ and $c_2 \leq b$. Therefore, assume that, for some $d \in A$, we have $d \leq a$ and $d \leq b$. Then $d \oplus a' = (d \oplus c_4) \oplus c_3$ and $d \oplus b' = (d \oplus c_4) \oplus c_1$ exist. Since, by assumption, $c_1 \oplus c_3$ exists, we can infer the existence of $d \oplus c_4 \oplus c_1 \oplus c_3$, which implies $d \leq (c_1 \oplus c_3 \oplus c_4)' = c_2$. This shows that indeed c_2 is the infimum of a and b. ■

THEOREM 2. *Let $(A; \oplus, ', 0)$ be an orthomodular algebra, and let $\leq$ be the induced order relation and $1 := 0'$. Then $(A; \leq, ', 0, 1)$ is a Boolean algebra, if and only if it satisfies*

$$(*) \qquad (\forall x, y)(\exists z_1, z_2, z_3)((x \stackrel{e}{=} z_1 \oplus z_2) \wedge (y \stackrel{e}{=} z_2 \oplus z_3) \wedge \exists\, (z_1 \oplus z_3)).$$

Proof. It is obvious that for each Boolean algebra, say $\underline{B}$ the induced orthomodular algebra satisfies $(*)$ — choose $z_2 := x \sqcap y$, $z_1 := x \sqcap z_2'$, $z_3 := y \sqcap z_2'$.

Let now $(*)$ be satisfied, and consider $a, b \in A$. From Lemma 5 it follows that under the assumptions of Theorem 2 $a \sqcup b$ always exists. We shall show that $a \sqcap b$ also exists. Assume that $a = c_1 \oplus c_2$ and $b = c_2 \oplus c_3$ such that $c_1 \oplus c_3$ exists (in agreement with the assumptions of the theorem). Let $c_4 := (c_1 \oplus c_2 \oplus c_3)'$. We have by fact (ii) that $c_1 \oplus c_2 \oplus c_3 \oplus c_4 = 1$, hence $a' = (c_1 \oplus c_2)' = c_3 \oplus c_4$ and $b' = (c_2 \oplus c_3)' = c_1 \oplus c_4$ because of Lemma 4. Hence, by Lemma 5, $a' \sqcup b'$ exists, and we have $a' \sqcup b' = c_1 \oplus c_3 \oplus c_4$. Now, again by fact (ii) and Lemma 4, we have $(a' \sqcup b')' = (c_1 \oplus c_3 \oplus c_4)' = c_2$ By de Morgan's law[17] we infer that $a \sqcap b$ exists in $\underline{A}$ and is equal to $(a' \sqcup b')'$. Hence $(A; \leq, ', 0, 1)$ is an orthomodular lattice. Following the definition from Beran in [Be85] we say that two elements a and b of an orthomodular lattice commute, denoted by aCb, if $a = (a \sqcap b) \sqcup (a \sqcap b')$. It follows from the assumption of the theorem and Lemma 1 that in $(A; \leq, ', 0, 1)$ any two elements commute: In fact, let $a, b \in A$ and c_1, c_2, c_3 and c_4 be defined as above, i.e. we have $a = c_1 \oplus c_2$ and $b = c_2 \oplus c_3$. Then by Lemma 5 and the argumentation above we have $a \sqcup b = c_1 \oplus c_2 \oplus c_3$, $a \sqcap b = c_2$, and

[17] See Lemma 2

with a similar argumentation, since $b' = c_1 \oplus c_4$, that $a \sqcap b' = c_1$. Hence $(a \sqcap b) \sqcup (a \sqcap b') = c_2 \sqcup c_1 = c_1 \oplus c_2 = a$ and consequently aCb. It was shown by Foulis in [F62] that, if in an orthomodular lattice any two elements commute, then this lattice is distributive. Therefore $(A; \leq, ', 0, 1)$ is indeed a Boolean algebra. ■

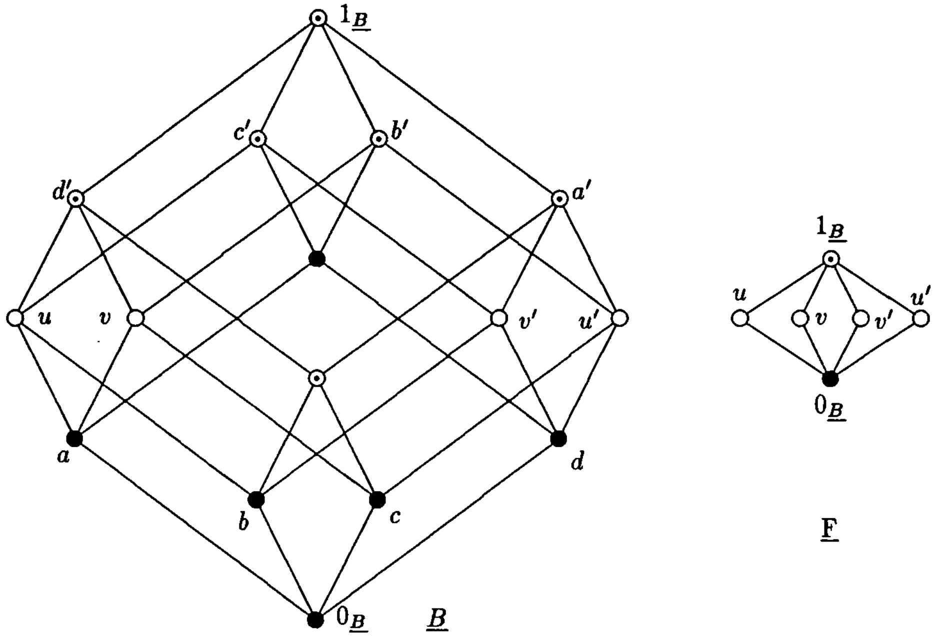


Figure 2: Orthomodular subalgebras of Boolean algebras need not be Boolean

The following example, given in Figure 2, shows that, in the language of orthomodular algebras, the axioms for Boolean algebras have to include existential formulas. Namely the orthomodular algebra shown on the right hand side, denoted by $\underline{F}$, is easily realized — via the labelling — to be an orthomodular subalgebra of the Boolean algebra $\underline{B}$, however, $\underline{F}$ is not Boolean, and therefore the class of orthomodular algebras derived from Boolean algebras is not closed with respect to (orthomodular) subalgebras.[18]

We conclude this section by observing that the example J_{18} by M. Janowitz[19], which we have depicted in Figure 3 by a Greechie diagram as well as by its order diagram, where the "ends" on the left and on the right have to

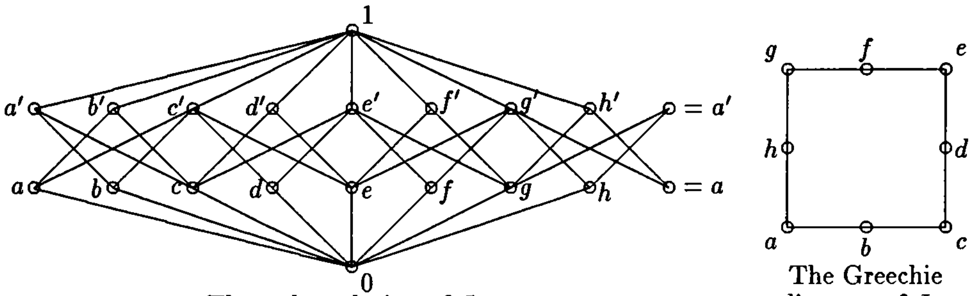
[18] Cf. e.g. [Co81], Theorem 2.8.

[19] See e.g. Beran [Be85], section IV.4, e.g. Fig. 39a. For "Greechie's First Theorem" quoted below see Theorem 49 in that section.

be identified, represents the order of an orthomodular algebra, which is not rich: If follows from "Greechie's First Theorem" that J_{18} corresponds to an orthomodular algebra, while an assignment like

$$x_1 \mapsto c,\ \ x_2 \mapsto g,\ \ y_1 \mapsto a \text{ and } y_2 \mapsto e$$

yields a situation, where the assumptions of (R) are satisfied, but not the conclusion. — Further examples of orthomodular algebras, which are not derived from orthomodular lattices, can be obtained from the *partial fields of sets*[20]



The order relation of J_{18} The Greechie diagram of J_{18}

Figure 3: The non-rich orthomodular algebra J_{18}

4. Probability measures on orthomodular algebras

Let $\underline{A} := (A; \oplus, ', 0)$ be an orthomodular partial algebra.

DEFINITION. A mapping

$$m : A \to [0,1]$$

is said to be a probability measure on $\underline{A}$, if

$$m(a \oplus b) = m(a) + m(b),$$

whenever $a \oplus b$ exists, and

$$m(a') = 1 - m(a); \qquad m(0) = 0.$$

DEFINITION. A set M of probability measures on $\underline{A}$ is said to be *full*, if the following condition (i) is satisfied:

[20] For a given set X a *partial field of sets* is a family $\mathfrak{M}$ of subsets of X satisfying: $\emptyset, X \in \mathfrak{M}$; $A \in \mathfrak{M}$ implies $X \setminus A \in \mathfrak{M}$; and $A, B \in \mathfrak{M}$ and $A \cap B = \emptyset$ imply $A \cup B \in \mathfrak{M}$.

And in order to get an orthomodular algebra on $\mathfrak{M}$ one defines as in [Go80] dom $\oplus :=$ $\{(A,B) \mid A, B \in \mathfrak{M},\ A \cap B = \emptyset\}$; $(A,B) \in$ dom $\oplus$ implies $A \oplus B := A \cup B$; $0 := \emptyset$; and $A \in \mathfrak{M}$ implies $A' := X \setminus A$.

(i) for all $a, b \in A$

$$(\text{for all } m \in M : m(a) + m(b) \leq 1) \text{ implies that } a \oplus b \text{ exists.}$$

M will be called *unital*, if it satisfies condition (ii) below:

(ii) for every $a \in A$ with $a \neq 0$ there exists $m \in M$ such that $m(a) = 1$.

Observe that in (i) the converse implication always holds. Namely, if $a \oplus b$ exists, then $m(a) + m(b) = m(a \oplus b)$, i.e. $m(a) + m(b) \leq 1$. Morover the orthomodularity axiom (A9) implies that

$$a \oplus b' \text{ exists, iff , for all } m \in M, \quad m(a) \leq m(b) = m(a) + m((a \oplus b')'),$$

(e.g. $a \leq b$ implies $m(a) \leq m(b)$). Consequently, property (i) above can be replaced by the following one (for all $a, b \in A$):

$$(\text{for all } m \in M : m(a) \leq m(b)) \text{ iff } a \leq b.$$

DEFINITION. Let S be a non-empty set, and let $L \subseteq [0,1]^S$ be a set of functions from S into $[0,1]$. L is said to be a *numerical orthomodular algebra*, if it is an orthomodular algebra with respect to the partial operation $\oplus$ defined by

$$f \oplus g := f + g \text{ iff } f + g \leq 1,$$

and the unary operation $'$ given by $f' := 1 - f$, with the constant $0 = 0^L$ being the function taking the value 0 for all $x \in S$.

There arises the question, when a set $L \subseteq [0,1]^S$ of functions is an orthomodular algebra. The answer is given in the following theorem:

THEOREM 3. *Let $L \subseteq [0,1]^S$, $S \neq \emptyset$, have the following properties:*

(i) $0 \in L$,

(ii) $f \in L \Rightarrow 1 - f \in L$,

(iii) *if $f_1, f_2, f_3 \in L$, and if $f_i + f_j \leq 1$ for $i \neq j$, then $f_1 + f_2 + f_3 \in L$.*

Then $\underline{L} := (L; \oplus, ', 0)$ is an orthomodular algebra with respect to the partial operation $f \oplus g := f + g$ (if $f + g \leq 1$), and the unary total operation $' : f \mapsto f' := 1 - f$.

Here $\leq$ denotes the order between real functions: $f \leq g$ iff $f(x) \leq g(x)$ for all $x \in S$, $+$ and $-$ denote the (componentwise) addition and subtraction of real functions, 1 denotes the function taking the real value 1 for all $x \in S$.

Proof. We could verify all the axioms (A1) through (A9), but we can also use the theorem of Mączyński and Traczyk in [MT73] to infer that $(L; \leq, ', 0, 1)$ is a partially ordered orthocomplemented and orthomodular set; and next, by Theorem 1, $(L; \leq, ', 0, 1)$ is equivalent to $(L, \leq, ', 0)$ with $f \oplus g := f \sqcup g = f + g$ for $f \leq g'$, i.e. for $f + g \leq 1$. Hence Theorem 3 holds. ∎

We also have the following **Representation Theorem:**

THEOREM 4. *Let* $(A;\oplus,',0)$ *be an orthomodular algebra, and let* M *be a full set of probability measures on* A. *For every* $a \in A$, *let* $\bar{a} : M \to [0,1]$ *be a function defined by* $\bar{a}(m) := m(a)$ *for all* $m \in M$. *Then* $(\{\bar{a} \mid a \in A\};\oplus,',\bar{0}) =: \underline{\bar{A}}$ *is a numerical orthomodular algebra isomorphic to* $(A;\oplus,',0)$.

Proof. In order to show that $\underline{\bar{A}} = (\bar{A};\oplus,',\bar{0})$ is a numerical orthomodular algebra, we verify the properties (i) – (iii) of Theorem 3. Since $m(0) = 0$ for all $m \in M$, the function $\bar{0}$ with $\bar{0}(m) = m(0) = 0$, belongs to $\bar{A}$. Now, if $\bar{a} \in \bar{A}$, then $1 - \bar{a}$ belongs to $\bar{A}$, since $(1 - \bar{a})(m) = 1 - \bar{a}(m) = 1 - m(a) = m(a')$ for all $m \in M$. Therefore $1 - \bar{a} = \overline{a'} \in \bar{A}$.

Condition (iii) also holds. Namely, let $\overline{a_1}, \overline{a_2}, \overline{a_3} \in \bar{A}$ with $\overline{a_i} + \overline{a_j} \leq 1$ for $i \neq j$. Then, for all $m \in M$, and for all $i \neq j$, $\overline{a_i}(m) + \overline{a_j}(m) = m(a_i) + m(a_j) \leq 1$. Since M is full, $a_i \oplus a_j$ exists. Hence we get $(\overline{a_1} + \overline{a_2} + \overline{a_3})(m) = \overline{a_1}(m) + \overline{a_2}(m) + \overline{a_3}(m) = m(a_1) + m(a_2) + m(a_3) = m(a_1 \oplus a_2 \oplus a_3) = (\overline{a_1 \oplus a_2 \oplus a_3})(m)$ for all $m \in M$. Hence (iii) holds. In the proof we have used the fact that $m(a_1 \oplus a_2 \oplus a_3) = m(a_1) + m(a_2) + m(a_3)$. However, this follows easily from axioms (A5) and (A8). Hence $\underline{\bar{A}} = (\bar{A};\oplus,',\bar{0})$ is an orthomodular algebra.

The map $\varphi : a \mapsto \bar{a}$ is clearly an isomorphism, since $\varphi(a \oplus b) = \overline{a \oplus b} = \bar{a} + \bar{b} = \varphi(a) + \varphi(b)$; $\varphi(a') = \overline{a'} = (\varphi(a))'$, $\varphi(0) = 0$. It is one-to-one and onto, since for $\overline{a_1} = \overline{a_2}$ one has $\overline{a_1}(m) = \overline{a_2}(m)$ for all $m \in M$. Hence $m(a_1) = m(a_2)$ for all $m \in M$, and this implies $a_1 \leq a_2$ and $a_2 \leq a_1$ (M is full), i.e. $a_1 = a_2$. Therefore φ is one-to-one. Since we deal with partial algebras, we still have to show that the homomorphism φ is closed, i.e. that $a \oplus b$ exists, whenever $\bar{a} + \bar{b}$ exists (the latter meaning that for all $m \in M$ $\bar{a}(m) + \bar{b}(m) = m(a) + m(b) \leq 1$). However, this is just guaranteed by the assumption on M to be a full set of probability measures (see condition (i) above defining fullness). This ends the proof of Theorem 4. ∎

For *rich* orthomodular algebras the representation theorem can be simplified. First we have the following lemma.

LEMMA 6. *Let* $\underline{L} = (L;\oplus,',0)$ *be a numerical orthomodular algebra. Then* $\underline{L}$ *is rich (i.e. it satisfies axiom (R)), iff the following condition holds:*

(∗) *If* $f_1, f_2, g_1, g_2 \in L$ *and* $\max\{f_1, f_2\} \leq \min\{g_1, g_2\}$,
then there exists $h \in L$ *such that* $\max\{f_1, f_2\} \leq h \leq \min\{g_1, g_2\}$.

Here $\min\{f, g\}$ *denotes the function defined by* $h(x) = \min\{f(x), g(x)\}$ *for every* $x \in S$ *(analogously for* $\max\{f, g\}$*).*

P r o o f. Assume that (R) holds for $x_1 = f_1$, $x_2 = f_2$, $y_1 = 1 - g_1$, $y_2 = 1 - g_2$. Then $x_1 \oplus y_1$ exists, iff $f_1 \leq g_1$; $x_1 \oplus y_2$ exists, iff $f_1 \leq g_2$; $x_2 \oplus y_1$ exists, iff $f_2 \leq g_1$; $x_2 \oplus y_2$ exists, iff $f_2 \leq g_2$.

This means that $\max\{f_1, f_2\} \leq \min\{g_1, g_2\}$. By (R) there is $z \in L$ such that $x_1 \leq z'$, $x_2 \leq z'$, $y_1 \leq z$, and $y_2 \leq z$; $x_1 \leq z'$, $x_2 \leq z'$, $z' \leq y_1'$, and $z' \leq y_2'$. This means that $\max\{f_1, f_2\} \leq h \leq \min\{g_1, g_2\}$ for $h = z'$. Hence condition $(*)$ holds, and therefore we have "(R) $\Rightarrow$ $(*)$".

The converse implication "$(*) \Rightarrow$ (R)" is also true, since in the above proof all the implications are in fact equivalences. Therefore Lemma 3.1 holds. ■

We now have the following theorem.

THEOREM 5. *Let $L \subseteq [0,1]^S$ be a set of functions with the following properties:*

1°. $1 \in L$.

2°. $(\forall f \in L)(f \neq 0 \Rightarrow (\exists \alpha \in S)(f(\alpha) > \frac{1}{2}))$.

3°. $f, g \in L,\ f + g \leq 1 \Rightarrow f + g \in L$.

4°. $f, g \in L,\ f \leq g \Rightarrow g - f \in L$.

5°. *The property $(*)$ from Lemma 3.1 holds.*

Then $(L; \oplus, ', 0)$ is a rich orthomodular algebra.

Observe that in this theorem condition (iii) of Theorem 3 is replaced by the weaker condition 3°. Condition (iii) involves three elements, condition 3° only two. However, in this case we have to assume the additional property $(*)$. D.Strojewski has tried in [S85] to prove this theorem without the assumption $(*)$, but his proof contains an error. His method of proof, however, can be used to prove Theorem 5.

P r o o f. It suffices to show that conditions 1°–5° imply conditions (i)–(iii) of Theorem 3. By first taking $f = g = 1$ in 4° we obtain $1 - 1 = 0 \in L$, thus (i) holds. Next taking in 4° $g = 1$, we obtain $f \in L \Rightarrow 1 - f \in L$, therefore (ii) holds. To show that (iii) holds, we first show that $f, g \in L$, $f + g \leq 1$ imply that $f + g = f \sqcup g$ in the (partially) ordered set $(L; \leq)$. Assume that, for some $h \in L$, $f \leq h$ and $g \leq h$. We have $\max\{f, g\} \leq \min\{f + g, h\}$; so, by 5°, we obtain that there exists $h_1 \in L$ such that $\max\{f, g\} \leq h_1 \leq \min\{f + g, h\}$. Hence $f, g \leq h_1 \leq f + g$. This implies $1 + h_1 - f - g \leq 1$, and consequently $(1 - f) + (h_1 - g) \leq 1$. Now, by 4° and 3°, we obtain $(1 - f) + (h_1 - g) \in L$. If we define $h_2 := 1 - ((1 - f) + (h_1 - g))$, then $h_2 \in L$ and $0 \leq h_2 \leq f, g$. But this gives $h_2 \leq \min\{f, g\} \leq \min\{f, 1 - f\} \leq \frac{1}{2}$, so, by 2°, $h_2 = 0$. This implies that $h_1 = f + g$. Since $h_1 \leq h$, it follows that $f + g \leq h$. Hence $f \sqcup g = f + g$. Now, in order to show that (iii) holds, let $f_1, f_2, f_3 \in L$ with $f_i + f_j \leq 1$ for $i \neq j$. Then $f_1 \sqcup f_2$ exists, and $f_1 + f_2 = f_1 \sqcup f_2$. Since

$f_1, f_2 \le 1 - f_3$, we obtain $f_1 \sqcup f_2 \le 1 - f_3$, i.e. $(f_1 \sqcup f_2) + f_3 \le 1$. By 3° this implies $(f_1 \sqcup f_2) + f_3 \in L$, that is $f_1 + f_2 + f_3 \in L$. Hence (iii) holds. This ends the proof of Theorem 5. ∎

We now obtain the following *representation theorem for rich orthomodular algebras with a full set of probability measures.*

THEOREM 6. *Every rich orthomodular algebra with a full and unital set of probability measures is isomorphic to a numerical orthomodular algebra of functions $L \subseteq [0,1]^S$ satisfying the properties 1° - 5°, where $f \oplus g = f + g$ and $f' = 1 - f$.*

Proof. Immediate from Theorems 4 and 5. In particular, the assumption of unitalness of the set of probability measures yields property 2° of Theorem 5. ∎

Observe that the property (∗) from Lemma 3.1 is satisfied, when e.g. $(L, \le)$ is a semilattice (upper or lower), so it holds in every lattice.

This property (∗) can be given some probabilistic interpretation: We call the members of L questions, the members of S are called states. For each $f \in L$, $f(x)$ is interpreted as the probability for the question f of being true in the state x. The property (∗) can be interpreted as follows:

If the probability of one pair of questions is always (i.e. in all states) less than the probability of another pair of questions, then there is a question with probability between these two pairs. This means that the fact that $\max\{f, g\} \le \min\{u, v\}$ can be experimentally verified by one question. Therefore, although by no means all questions are pairwise verifiable, still there are some pairs, which are verifiable with respect to some other pairs. This is a reasonable assumption to be made about quantum logic, which is a partially ordered orthocomplemented set with some regularity assumption.

6. On the category of orthomodular algebras, free objects Comparison of categories

In Theorem 1 we have shown that there exists — in a natural way — a bijection between the class of all orthomodular ordered sets and the class of all orthomodular algebras. This yields an embedding — in the sense of category theory — from the category with the class of all orthomodular algebras as class of objects and with the class of all homomorphisms — in the "weak sense" between partial algebras[21] — as class of morphisms, into the category with the class of all orthomodular ordered sets as class of objects and the class of all order preserving, 0-preserving and orthocomplementation

[21] See Footnote 10.

preserving mappings between these objects as morphisms: however, this is not a so-called full embedding, as the following lemma shows:

LEMMA 7. *Let* $\underline{A} = (A; \oplus, ', 0)$ *and* $\underline{B} = (B; \oplus, ', 0)$ *be orthomodular algebras, and let* $\underline{A}' = (A; 0, '; \leq)$ *and* $\underline{B}' = (B; 0, '; \leq)$ *be the corresponding orthomodular ordered sets. Moreover, let* $\psi : A \to B$ *be any mapping. Then we have the following:*

(i) *If* ψ *is a homomorphism from* $\underline{A}$ *into* $\underline{B}$, *then* ψ *is an order preserving mapping (naturally also preserving* 0 *and the orthocomplementation).*

(ii) *If* ψ *preserves the order relation* $\leq$, *the orthocomplementation* $'$ *and* 0, *then* ψ *need not necessarily be a homomorphism between orthomodular algebras.*

Proof. Ad (i): By assumption ψ preserves 0 and the orthocomplementation. Assume $a, b \in A$ such that $a \leq b$. Then $a \oplus b'$ exists, since one has the relationship

$$a \leq b \text{ if and only if } a \oplus b' \text{ exists},$$

and therefore one has

$$\psi(a \oplus b') = \psi(a) \oplus \psi(b') = \psi(a) \oplus \psi(b)'.$$

This implies that $\psi(a) \leq \psi(b)$, and therefore ψ is also order preserving.

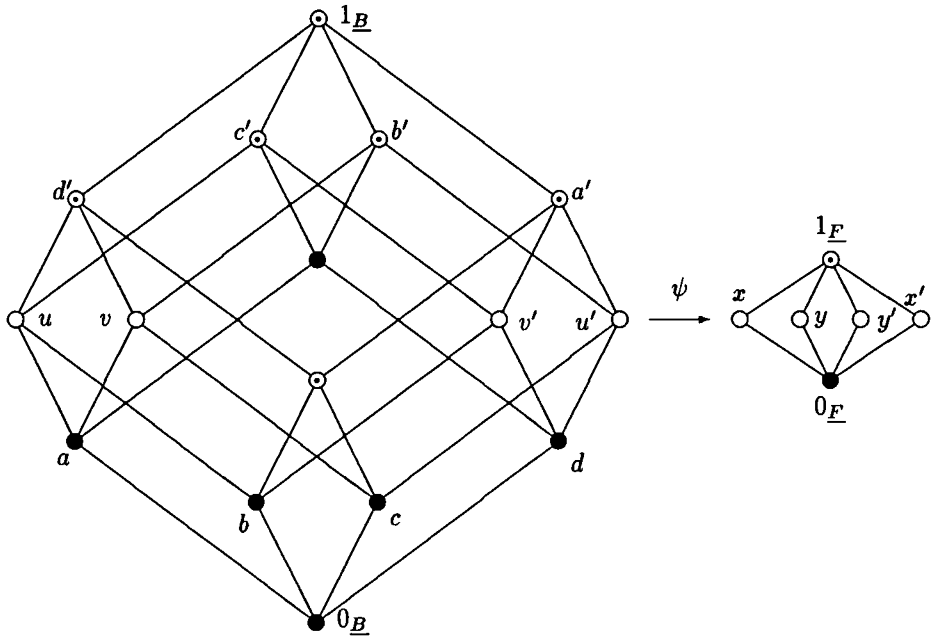


Figure 4: Order and complementation preserving, but no homomorphism

Ad (ii): In order to realize this consider Figure 4, where ψ is a mapping from the Boolean lattice $\underline{B}$ with four atoms a, b, c, d onto the orthomodular ordered set $\underline{F}$, which corresponds to the free orthomodular algebra $\underline{F}(\{x,y\},\mathrm{OMA})$, and which maps all points depicted by a black circle to $0_{\underline{F}}$, all those depicted by two circles to $1_{\underline{F}}$, and moreover u to x, v to y and therefore u' to x' and v' to y'. It is easy to check that ψ preserves 0, the order relation and the orthocomplementation $'$. Moreover, in $\underline{B}$ we have $a \leq c'$, and therefore $a \oplus c$ exists with value v. However, $\psi(a) = \psi(c) = 0_{\underline{F}}$, and therefore

$$0_{\underline{F}} = \psi(a) \oplus \psi(c) < \psi(a \oplus b) = y.$$

Thus ψ does not preserve the partial operation $\oplus$. ∎

Free objects

The category of all orthomodular algebras with their homomorphisms is, however, a "very nice" category. First let us observe that, since by definition OMA is an ECE-variety, it has, for each set X, a (relatively) free algebra $\underline{F}(X,\mathrm{OMA})$ with the OMA-free generating set X[22], and this has the following relatively simple structure:

THEOREM 7. *Let X be any set of variables, then the* OMA*-free* OMA*-algebra $\underline{F}(X,\mathrm{OMA})$,* OMA*-freely generated by X can be described as follows*[23]

$$F(X,\mathrm{OMA}) = X \dot{\cup} X^* \dot{\cup} \{0,1\},$$

dom$\oplus$

$$:= \{(0,0),(0,1),(1,0)\} \cup \bigcup_{x\in X} \{(x,0),(0,x),(x^*,0),(0,x^*),(x,x^*),(x^*,x)\},$$

and

$$0 \oplus 0 := 0;\ \ 0 \oplus 1 := 1 \oplus 0 := 1;\ \ 0 \oplus y := y \oplus 0 := y \textit{ for all } y \in X \cup X^*;$$

$$x \oplus x^* := x^* \oplus x := 1 \textit{ for all } x \in X.$$

$$0' := 1;\ \ 1' := 0;\ \ x' := x^*,\ \ (x^*)' := x \textit{ for all } x \in X.$$

0 is the least (and 1 the greatest) element.

Moreover, each OMA*-free* OMA*-algebra $\underline{F}(X,\mathrm{OMA})$ is rich, and therefore — for every set X — it is also an* OMAR*-free* OMAR*-algebra over X:*

$$\underline{F}(X,\mathrm{OMAR}) = \underline{F}(X,\mathrm{OMA}).$$

[22] This means that every mapping from X into any orthomodular algebra, say $\underline{A}$, has an extension to a homomorphism from $\underline{F}(X,\mathrm{OMA})$ into $\underline{A}$.

[23] Here $X^* := \{x^* \mid x \in X\}$ designates a set disjoint from and in one-to-one correspondence to X. Compare Figure 5.

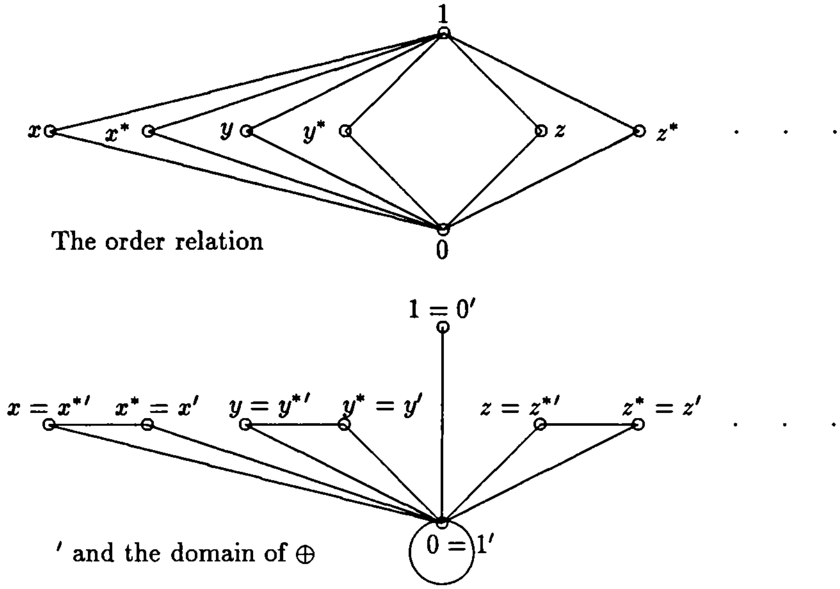


Figure 5: OMA-free OMA-algebra on X: $\underline{F}(\{x, y, z, \ldots\}, \mathrm{OMA})$

Proof. Because of the axioms (A0) through (A9) a free OMA-algebra has to contain $F_X := \{0\} \cup \{0'\} \cup X \cup X^*$, and the axioms do not imply any identifications, and they do not enforce the existence of any further element in order to make it an object from OMA. It is easy to realize that the indicated structure really yields an orthomodular algebra on $F(X, \mathrm{OMA})$ — as a matter of fact one gets an orthomodular algebra induced by the orthomodular lattice often denoted by MOX. Moreover, since this orthomodular algebra contains exactly the elements and the structure enforced by the axioms and the fact that it has to be generated by X, it really has to be (isomorphic to) the OMA-free OMA-algebra on X. It is also quite easy to realize that every mapping from X to any orthomodular algebra $\underline{B}$ is always extendable to a homomorphism from $\underline{F}(X, \mathrm{OMA})$ into $\underline{B}$. — One can also easily see that the free OMA-algebra on one free generator has exactly four elements, and that $\underline{F}(X, \mathrm{OMA})$ is the coproduct in the category OMA[24] of the family $\{\underline{F}(\{x\}, \mathrm{OMA}) \mid x \in X\}$.

It is quite obvious, too, that $\underline{F}(X, \mathrm{OMA})$ — as defined in the theorem — is rich. Namely the structure of $\underline{F}(X, \mathrm{OMA})$ as indicated in Figure 1 is so "poor" that, whenever one tries to realize the premise of (R) by four elements a_1, a_2, b_1, b_2 from $F(X, \mathrm{OMA})$, then either they have to form a subset of $\{0, 0'\}$ or there has to exist $x \in X$ such that they form a subset of

[24] See the next subsection (on category theoretical constructions).

the three-element set $\{x, x^*, 0\}$. In both cases an element $z \in F(X, \mathrm{OMA})$ can easily be found such that z also satisfies the conclusion of (R). E.g. assume that $a_1 = 0$, $a_2 = x$, $b_1 = 0$, $b_2 = x^*$; then one has to choose $z := x^*$. The other cases can be treated in a similar way.

Since OMAR is a subclass of OMA, and since therefore every OMA-free partial algebra over X is also OMAR-freely generated by X, this shows that $\underline{F}(X, \mathrm{OMA}) = \underline{F}(X, \mathrm{OMAR})$. ∎

We know from Lemma 1 that the only total algebras in OMA as well as in OMAR are the one-element orthomodular algebras, and both classes contain partial algebras with more than one element. However, in an *existence equationally defined class* (a so-called *E-variety*) $\mathfrak{K}$ of partial algebras every member of $\mathfrak{K}$ with at least two elements is fully embeddable into a total $\mathfrak{K}$-algebra[25]. Therefore, neither OMA nor OMAR can be defined by existence equations only; yet, only E-varieties can be said "to be determined by their free (partial) algebras".

Let us recall, in addition, that in an axiomatic class, say $\mathfrak{K}$, of partial algebras $\mathfrak{K}$-free $\mathfrak{K}$-algebras — whenever they exist — carry the *weakest structure* allowed by the axioms. Namely, every mapping from X into any $\mathfrak{K}$-algebra has to have a homomorphic extension; and therefore the structure of $\underline{F}(X, \mathfrak{K})$ must not be too rich. Since the precise image set of a homomorphism need not be a closed subset of the target algebra, the free partial algebras only give you a measure of what has to be generated at least by a subset of a given $\mathfrak{K}$-algebra, but in general it does not give full information about the generated subset and its structure.

In particular, already in connection with an ECE-variety $\mathfrak{K}$ of partial algebras and a given set X of generators, one may consider for every X-generated relative subalgebra, say $\underline{P}$, of the term algbra over X its *$\mathfrak{K}$-universal solution* $\underline{F}(\underline{P}, \mathfrak{K})$. One gets in this way a usually infinite set, say $\mathcal{F}_{X,\mathfrak{K}}$, of partial algebras, which are non-isomorphic over the identity mapping id_X of X such that this set can be considered in some way as another substitute for the $\mathfrak{K}$-free $\mathfrak{K}$-algebra on X from the total case: Namely one then has that for every $\mathfrak{K}$-algebra, say $\underline{K}$ and for every mapping $f : X \to K$ there exists exactly one partial algebra, say $\underline{F}_f$ in $\mathcal{F}_{X,\mathfrak{K}}$ such that f extends to a *closed* homomorphism from $\underline{F}_f$ onto the subalgebra of $\underline{K}$ generated by the image set $f(X)$. However, while it is in many cases not too difficult to provide a description of the single partial algebra $\underline{F}(X, \mathfrak{K})$, it is usually a very hard task to get a description of the set $\mathcal{F}_{X,\mathfrak{K}}$ — of which $\underline{F}(X, \mathfrak{K})$ is in some sense the smallest element. It might be an interesting — but very likely also very hard — project to determine $\mathcal{F}_{X,\mathrm{OMA}}$ (at least for finite sets X).

[25] See [B73].

Some category theoretical constructions in OMA

We want to conclude this note by briefly discussing some of the category theoretical properties of the category OMA of all orthomodular algebras as objects and all homomorphisms as morphisms. It is well known that reflective subcategories of complete and cocomplete categories are themselves complete and cocomplete[26]. Since OMA is a full and epireflective subcategory[27] of the complete and cocomplete category **Alg**$(2,1,0)$ of all partial algebras of type $(2,1,0)$ with homomorphisms as morphisms, it is therefore itself complete and cocomplete. In what follows we only want to discuss some of the most common constructions in this category, although in this connection the facts also mainly follow from the general theory.

Since OMA is closed with respect to direct products, which are the product objects in **Alg**$(2,1,0)$, the products in OMA are the usual direct products (in which the partial sum is defined componentwise, whenever it is defined in all the components) with the canonical projections as projection morphisms.

If $f, g : \underline{A} \to \underline{B}$ are homomorphisms, then the subset $A_{f,g} = \{a \in A \mid f(a) = g(a)\}$ is a closed subset of $\underline{A}$ and therefore the carrier set of the equalizer $(\underline{A}_{f,g}, id_{A_{f,g}})$ of f and g, where $id_{A_{f,g}}$ is the identity embedding of $\underline{A}_{f,g}$ into $\underline{A}$, which is a closed homomorphism.

Obviously the total one-element algebra of type $(2,1,0)$ is a terminal object of the category OMA.

It is known from the theory of orthomodular ordered sets or can easily be realized directly, that, for any family $\mathfrak{F}$ of orthomodular algebras, the "disjoint union of the algebras with identification of all zeros and all ones, respectively" yields the coproduct object for the family $\mathfrak{F}$, and the "canonical injections" are in this category really (closed and) injective homomorphisms.

The two-element OMA-free OMA-algebra $\underline{F}(\emptyset, \text{OMA})$, OMA-freely generated by the empy set represents the initial object — as usual in categories with free objects.

The description of coequalizers is a little more involved, since OMA is not closed w.r.t. homomorphic images in general, but one can only guarantee closedness w.r.t. closed homomorphic images. However, the construction of coequalizers can be described in general as follows:

Let $f, g : \underline{A} \to \underline{B}$ be any two homomorphisms between the orthomodular algebras $\underline{A}$ and $\underline{B}$, let $\Theta_{f,g}$ be the congruence relation on $\underline{B}$ generated by the set $\{(f(a), g(a)) \mid a \in A\}$, let $nat_{\Theta_{f,g}} : \underline{B} \to \underline{B}/\Theta_{f,g}$ be the quotient homomorphism, and let $r_{\underline{B}/\Theta_{f,g},\text{OMA}} : \underline{B}/\Theta_{f,g} \to \underline{F}(\underline{B}/\Theta_{f,g}, \text{OMA})$ be the

[26] Cf. e.g. [HS73], section 36.

[27] Compare e.g. J. Schmidt [Sch66].

OMA-universal OMA-solution of this quotient algebra. Then $r_{\underline{B}/\Theta_{f,g},\mathrm{OMA}} \circ nat_{\Theta_{f,g}} : \underline{B} \to \underline{F}(\underline{B}/\Theta_{f,g}, \mathrm{OMA})$ represents a coequalizer of f and g. In order to realize this, one should observe that $(nat_{\Theta_{f,g}}, \underline{B}/\Theta_{f,g})$ is a coequalizer of f and g in $\mathbf{Alg}(2,1,0)$.

Since $nat_{\Theta_{f,g}}$ need not be a closed homomorphism, one cannot say more without deeper investigations, which we did not carry through so far. In this connection we want to add some remarks concerning the closedness of OMA w.r.t. closed homomorphic images. This means that it can be guaranteed that, for any closed conguence relation[28], say Θ, on some orthomodular algebra $\underline{A}$, every partial algebra, say $\underline{B}$, isomorphic to the quotient algebra $\underline{A}/\Theta$ is again an orthomodular algebra. However, this does not say that $\underline{B}$ cannot be an orthomodular algebra, if Θ is not closed. E.g. the one-element total algebra, say $\underline{T}$, of type $(2,1,0)$ is orthomodular, and it is a homomorphic image of any other orthomodular algebra, say $\underline{A}$. However, the corresponding surjective homomorphism — and therefore its kernel — is closed, iff $\underline{A}$ has only one element, too.

For those, who do not know partial algebra theory so well, we add that in any partial algebra, say $\underline{A}$, (of finitary type) there exists a largest closed congruence relation, say Θ_c, and that the ideal gererated by Θ_c in the congruence lattice of $\underline{A}$ only consists of closed congruence relations. Θ_c is identical with the largest congruence relation $A \times A$ of $\underline{A}$, iff each fundamental operation of $\underline{A}$ is either total or empty.

Observe that, in any orthomodular algebra, say $\underline{A}$, with more than two elements, no element, say a, different from 0 can be identified with 0 by a closed congruence relation, since $0 \oplus 1$ exists, while $a \oplus 1$ does not exist. For a similar reason, a cannot be identified with 1 by a closed congruence, if $a \neq 1$ — since then $a' \oplus 0'$ does not exist, while $1' \oplus 0'$ exists.

References

[Be85] L. Beran, *Orthomodular Lattices — Algebraic Approach*, D. Reidel Publ. Company, 1985.

[B73] P. Burmeister, *An embedding theorem for partial algebras and the free completion of a free partial algebra within a primitive class*, Algebra Universalis 3 (1973), 271–279.

[28] A congruence relation, say Θ of some partial algebra $\underline{A}$ is said to be *closed*, iff the induced natural projection $nat_\Theta : \underline{A} \to \underline{A}/\Theta$ is a closed homomorphism. For a general intrinsic characterization cf. e.g. [B86], subsection 2.5; for an orthomodular algebra $\underline{A}$ a congruence relation Θ is closed, iff in addition to the usual compatibility with the fundamental operations one has that $(a,a'), (b,b') \in \Theta$ and the existence of $a \oplus b$ always imply the existence of $a' \oplus b'$.

[B82] P. Burmeister, *Partial algebras — survey of a unifying approach towards a two-valued model theory for partial algebras*, Algebra Universalis 15 (1982), 306–358.

[B86] P. Burmeister, *A Model Theoretic Oriented Approach to Partial Algebras. Introduction to Theory and Application of Partial Algebras — Part I*, Mathematical Research Vol. 32, Akademie-Verlag, Berlin, 1986.

[B93] P. Burmeister, *Partial Algebras — An Introductory Survey*, In: Algebras and Orders: Proceedings of the NATO Advanced Study Institute and Séminaire des Mathématiques, Montreal, Canada (Eds.: G. Sabidussi, I. Rosenberg), Kluwer Publ. Co., 1993.

[Co81] P. M. Cohn, Universal Algebra, D.Reidel Publishing Co., 2nd ed., 1981 (1st ed. 1965.)

[F62] D. J. Foulis, *A note on orthomodular lattices*, Portugaliae Math. 21 (1962), 65–72.

[Go80] R. Godowski, *Commutativity in orthomodular posets*, Reports on Mathematical Physics 18 (1980), 347–351.

[HS73] H. Herrlich, G. E. Strecker, *Category Theory — An Introduction*, Allyn and Bacon Inc., 1973.

[K83] G. Kalmbach, *Orthomodular Lattices*, Academic Press, 1983.

[MT73] M. Mączyński, T. Traczyk, *A characterization of orthomodular partially ordered sets admitting a full set of states*, Bull. Polon. Acad. Ser. Sci. Math. Astr. Phys. 21 (1973), 3–9.

[M73] M. Mączyński, *On some numerical characterization of Boolean algebras*, Colloq. Math. 27 (1973), 207–210.

[Pu93] S. Pulmannová, *A remark on orthomodular partial algebras*, Manuscript, 1993.

[Sch66] J. Schmidt, *A general existence theorem on partial algebras and its special cases*, Colloq. Math. 14 (1966), 73–87.

[S85] D. Strojewski, *Numerical representations of orthomodular lattices and Boolean algebras with infinite operations*, Bull. Polon. Acad. Ser. Sci. Math. Astr. Phys. 33 (1985), 341–348.

Peter Burmeister
FACHBEREICH MATHEMATIK
ARBEITSGRUPPE ALLGEMEINE ALGEBRA
TECHNISCHE HOCHSCHULE DARMSTADT
Schloßgartenstr. 7
D-64289 DARMSTADT, GERMANY;

Maciej Mączyński
DEPARTMENT OF MATHEMATICS
WARSAW UNIVERSITY OF TECHNOLOGY
Plac Politechniki 1
PL-00-661 WARSZAWA, POLAND

Received September 15, 1993.