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Sylvia Pulmannová

A REMARK ON ORTHOMODULAR PARTIAL ALGEBRAS

Dedicated to Professor Tadeusz Traczyk

1. Introduction

In [4], an axiom system for orthomodular partial algebras (OMA) is introduced, and it is shown that orthomodular partial algebras are equivalent to orthomodular partially ordered sets (orthomodular posets, OMP).

In this paper, we show that by weakening, resp. omitting one axiom in the axiom system for orthomodular partial algebras, we obtain axiom systems of partial algebras equivalent to orthoalgebras (OA), resp. difference partially ordered sets (difference posets, D-posets, DP). The free algebras in both latter cases exist and coincide with the free orthomodular algebra (see [4]).

We note that orthoalgebras have been found a useful tool into pursuit of quantum mechanical constructions (see, e.g., [6, 7, 8, 12, 13]). Difference posets have been introduced in [9] as a generalization of orthoalgebras (see also [10]). An important example of a D-poset is the set of all effects (i.e., s.a. operators A with $0 \leq A \leq I$ on a Hilbert space), which play an important role in unsharp quantum measurements ([2, 5]).

2. Difference posets, orthoalgebras, orthomodular posets

Let us first recall the definition of a difference poset (see [9]).

DEFINITION 2.1. Let J be a partially ordered set with a partial order $\leq$, greatest element 1, and with a partial binary operation $\ominus : J \times J \to J$, called a difference, such that, for $a, b \in J$, $b \ominus a$ is defined iff $a \leq b$, and the following three axioms hold for any $a, b, c \in J$:

(DPi) $a \leq b \;\Rightarrow\; b \ominus a \leq b$;
(DPii) $a \leq b \;\Rightarrow\; b \ominus (b \ominus a) = a$;
(DPiii) $a \leq b \leq c \;\Rightarrow\; c \ominus b \leq c \ominus a$ and $(c \ominus a) \ominus (c \ominus b) = b \ominus a$.

Then J is called a difference poset or a D-poset.

The following statements have been proved in [9]:

PROPOSITION 2.2. *Let a, b, c, d be elements of a D-poset J. Then*

(i) $1 \ominus 1$ *is the least element of* J, *denoted by* 0;
(ii) $a \ominus 0 = a$;
(iii) $a \ominus a = 0$;
(iv) $a \leq b \Rightarrow (b \ominus a = 0 \Longleftrightarrow b = a)$;
(v) $a \leq b \Rightarrow (b \ominus a = b \Longleftrightarrow a = 0)$;
(vi) $a \leq b \leq c \Rightarrow b \ominus a \leq c \ominus a$ *and* $(c \ominus a) \ominus (b \ominus a) = c \ominus b$;
(vii) $b \leq c,\ a \leq c \ominus b \Rightarrow b \leq c \ominus a$ *and* $(c \ominus a) \ominus b = (c \ominus b) \ominus a$;
(viii) $a \leq b \leq c \Rightarrow a \leq c \ominus (b \ominus a)$ *and* $(c \ominus (b \ominus a)) \ominus a = c \ominus b$.

PROPOSITION 2.3 [10]. *A partially ordered set J with the least an greatest elements 0 and 1, respectively, and with a partial binary operation $\ominus : J \times J \to J$ such that $b \ominus a$ is defined iff $a \leq b$ and the following two axioms hold for any $a, b, c \in J$:*

(i) $a \ominus 0 = a$;
(ii) $a \leq b \leq c \Rightarrow c \ominus b \leq c \ominus a$ *and* $(c \ominus a) \ominus (c \ominus b) = b \ominus a$;

is a D-poset.

Now let us recall the definition of an orthoalgebra (see [7]).

DEFINITION 2.4. Let K be a set containing two distinct elements $0, 1$ and let K be endowed with a partial binary operation $\oplus$ which satisfies the following four axioms:

(OA i) if $a \oplus b$ is defined, then $b \oplus a$ is defined and $a \oplus b = b \oplus a$;
(OA ii) if $b \oplus c$ is defined and $a \oplus (b \oplus c)$ is defined, then $a \oplus b$ is defined, $(a \oplus b) \oplus c$ is defined and $a \oplus (b \oplus c) = (a \oplus b) \oplus c$;
(OA iii) for every $a \in K$ there exists a unique $b \in K$ such that $a \oplus b$ is defined and $a \oplus b = 1$;
(OA iv) if $a \oplus a$ is defined, then $a = 0$.

Then $(K, 0, 1, \oplus)$ is called an orthoalgebra.

Finally, let us recall the definition of an orthomodular poset (see, e.g., [1, 11]).

DEFINITION 2.5. Let L be a partially ordered set with a partial order $\leq$, the greatest and least elements 1 and 0, respectively, and a unary operation (orthocomplementation) $' : L \to L$ such that, for any $a, b \in L$, the following axioms are satisfied:

(OM i) $a'' = a$;
(OM ii) $a \vee a' = 1$;
(OM iii) $a \leq b \Rightarrow b' \leq a'$;
(OM iv) $a \leq b' \Rightarrow a \vee b$ exists in L;
(OM iv) $a \leq b \Rightarrow a \vee (a \vee b')' = b$ (orthomodular law).

Then L is called an orthomodular poset.

Let us show the interrelations between D-posets, orthoalgebras and orthomodular posets.

Let a, b be two elements in an orthoalgebra K. We say that (i) a is *orthogonal to* b and write $a \perp b$ iff $a \oplus b$ is defined; (ii) a is *less or equal* b and write $a \leq b$ iff there exists an element $c \in K$ such that $a \oplus c = b$, (iii) b is *the orthocomplement of* a iff b is the unique element of K such that $a \oplus b = 1$ and is written as a'. If $a \leq b$, for the element c such that $a \oplus c = b$, we write $c := b \ominus a$. The element c is well-defined, because if c_1 is such that $a \oplus c_1 = b$, then $1 = (a \oplus c) \oplus b' = (a \oplus c_1) \oplus b' \Rightarrow c \oplus (a \oplus b') = c_1 \oplus (a \oplus b')$, and (OA iii) implies that $c = c_1 = (a \oplus b')'$. When a difference $\ominus$ is defined by $b \ominus a = (a \oplus b')'$, K becomes a D-poset. Indeed, (DP i) and (DP ii) are trivially satisfied and (DP iii) can be derived as follows:

$$a \leq b \leq c \Rightarrow b = a \oplus (a \oplus b')',$$
$$c = b \oplus (b \oplus c')' = (a \oplus (a \oplus b')') \oplus (b \oplus c')',$$

whence

$$c \ominus a = (a \oplus b')' \oplus (b \oplus c')' = (b \ominus a) \oplus (c \ominus b),$$

which yields $c \ominus b \leq c \ominus a$, and $(c \ominus a) \ominus (c \ominus b) = b \ominus a$.

In [10], it has been proved that a D-poset becomes an orthoalgebra iff the following additional condition is satisfied:

$$a \leq 1 \ominus a \Rightarrow a = 0. \tag{2.1}$$

If L is an orthomodular poset and we define $a \oplus b = a \vee b$ iff $a \perp b$ in L, then L with $0, 1, \oplus$ is an orthoalgebra (see, e.g., [7]). On the other hand, an orthoalgebra K is an OMP iff $a \perp b \Rightarrow a \vee b$ exists in K ([7, 8]).

The following proposition shows how the axioms of an orthoalgebra can be weakened to define a D-poset.

PROPOSITION 2.6. *Let J be a set with two particular elements* 0 *and* 1, *endowed with a partial binary operation $\oplus$ such that the axioms* (OA i), (OA ii), (OA iii) *and, in addition, the following axiom is satisfied:*

(iv) $1 \oplus a$ *is defined* $\Rightarrow a = 0$.

Then J with the partial order defined by $a \leq b$ iff $a \oplus b'$ exists, where b' is the unique element such that $b \oplus b' = 1$, and with the difference operation $\ominus$ defined by $b \ominus a := (a \oplus b')'$ iff $a \leq b$, becomes a D-poset.

Proof. We note that $a \leq b$ iff $\exists c$ such that $a \oplus c = b$. Indeed, similarly as in an orthoalgebra (see remarks after Definition 2.5), we show that c is uniquely defined and $c = (a \oplus b')'$. If $a \oplus b'$ exists, then by (OA iii), $b = a \oplus (a \oplus b')'$. We have to show that $\leq$ defines a partial order in J. By (OA iii), $a'' = a$, and $a \leq a$ holds.

Now $a \oplus b = a$ holds iff $b = 0$. Indeed, for every a, $1 = (a' \oplus a) \oplus ((a \oplus a')')$ implies by (OA iii) and (iv) that $a = a \oplus (a \oplus a')') = a \oplus 0$. Let $a \oplus b = a$. Then $(a \oplus b) \oplus a' = (a \oplus a') \oplus b = 1 \oplus b$, and by (iv), $b = 0$.

Let $a \leq b$ and $b \leq a$. Then there are c, d such that $b = a \oplus c$, $a = b \oplus d$. Hence $a = (a \oplus c) \oplus d = a \oplus (c \oplus d)$, which implies $c \oplus d = 0$, whence $c' = c' \oplus (c \oplus d) = 1 \oplus d$, and by (iv), $d = 0$, which yields that also $c = 0$. Consequently, $a = b$.

If $a \leq b$, $b \leq c$, then $b = a \oplus d$, $c = b \oplus e$. Hence $c = (a \oplus d) \oplus e = a \oplus (d \oplus e)$, which means that $a \leq c$.

We have proved that $\leq$ defines a partial order in J and $0 \leq a \leq 1$ for any $a \in J$. Now it suffices to prove (i) and (ii) of Proposition 2.3. 2.3 (i) follows by $a \oplus 0 = a$. 2.3 (ii):

$$\begin{aligned} a \leq b \leq c &\Rightarrow c = b \oplus (c \ominus b) = (a \oplus (b \ominus a) \oplus (c \ominus b) \\ &\Rightarrow c \ominus a = (b \ominus a) \oplus (c \ominus b) \\ &\Rightarrow c \ominus b \leq c \ominus a \end{aligned}$$

and

$$(c \ominus a) \ominus (c \ominus b) = b \ominus a. \ \blacksquare$$

We note that (OA i), (OA ii), (OA iii), (iv) are satisfied in every D-poset (see next section). Similarly as in orthoalgebras, we say that two elements a, b in a D-poset J are *orthogonal* ($a \perp b$) if $a \oplus b$ is defined. Clearly, $a \perp b$ iff $a \leq b'$.

Remark. 2.7. Let J be a D-poset. Put

$$N = \{x \in J : x \perp x \text{ or } x' \perp x'\} \setminus \{0, 1\}.$$

Clearly, $x \in N \Rightarrow x' \in N$, $0, 1 \notin N$. Moreover, $x \perp x$, $y \leq x \Rightarrow y \perp y$ and $x' \perp x'$, $x \leq y \Rightarrow y' \perp y'$.

Let $A := J \setminus N$ be equipped with the partial operation $\oplus_A$ defined as follows: $x \oplus_A y$ is defined iff $x \oplus y$ is defined and $x \oplus y \notin N$, and then $x \oplus_A y = x \oplus y$.

Then $(A, \oplus_A, 0, 1)$ is an orthoalgebra. Indeed, we have to verify the axioms (OA i)–(OA iv). (OA i), (OA iii) and (OA iv) are clear.

(OA ii): if $x \oplus_A y$ and $(x \oplus_A y) \oplus_A z$ are defined, then $x \oplus y \notin N$ and $(x \oplus y) \oplus z \notin N$. Therefore $x \oplus (y \oplus z) \notin N$. If $y \oplus z \in N$, then $(y \oplus z)' \perp (y \oplus z)'$ (since $y \oplus z \perp y \oplus z$ and $y \leq y \oplus z \Rightarrow y \perp y$, contradicting $y \in A$). But $y \oplus z \leq x \oplus (y \oplus z) \Rightarrow x \oplus (y \oplus z) \in N$, a contradiction. Therefore $y \oplus_A z$ is defined, and $x \oplus_A (y \oplus_A z) = x \oplus (y \oplus z) = (x \oplus_A y) \oplus_A z$.

This shows that to every D-poset J there exists an associated orthoalgebra A_J. For example, if J is the interval $[0,1]$ of real numbers with the usual ordering and $\ominus$ defined as the difference of numbers, then $A_J = \{0,1\}$. On the other hand, if J is the D-poset of all effects on a Hilbert space H, then the orthoalgebra A_J contains all projections on H.

3. An axiom system for orthomodular, D-orthomodular and A-orthomodular partial algebras

The aim of this section is to give a unified frame to the different axiom systems from the preceding section.

In what follows, we use the notations from [4], with the exception that suprema and infima will be denoted by $\vee$ and $\wedge$, respectively.

Before formulating the axiom systems we recall some basic definitions and terminology from the theory of partial algebras ([3a, b]). Let $\underline{T}(X, \Sigma)$ be any term algebra of type Σ on some set X, let $t, t_1, t_2 \in \underline{T}(X, \Sigma)$ be any terms, and let $\underline{A}$ be any partial algebra of type Σ. Thus, in $\underline{A}$ there are defined some operations, among them there may be proper partial operations (the domain of which is not all of A^n – n being the arity of the operation – but only some proper subset of A^n). We recall that an *existence equation* $t_1 \stackrel{e}{=} t_2$ holds in the partial algebra $\underline{A}$, iff for every valuation $v : X \to A$ the induced – i.e., as usual, recursively defined, but now along the partial structure of $\underline{A}$ – interpretations $\tilde{v}(t_1)$ and $\tilde{v}(t_2)$ of t_1 and t_2 exist and are equal. As set X of variables from which the valuation starts one usually chooses the set of variables occurring freely in the formula, if not stated otherwise (e.g., in connection with axiom (A0) below, only valuations starting from the empty set of variables are considered, see [4], footnote on p. 2).

In particular, the *term existence statement* $t \stackrel{e}{=} t$ holds in $\underline{A}$ iff for every valuation $v : X \to A$ the interpretation $\tilde{v}(t)$ exists (i.e. t induces in $\underline{A}$ a *total* term operation). We shall abbreviate the term existence statement $t \stackrel{e}{=} t$ by $\exists t$, i.e.,

$$\exists t \Longleftrightarrow t \stackrel{e}{=} t.$$

DEFINITION 3.1. ([4]) By an orthomodular partial algebra we understand a partial algebra $\underline{A} := (A; \oplus; '; 0)$ of type $(2,1,0)$ such that the following list of axioms is satisfied in $\underline{A}$ for any $x, y, z \in X$ for any given countably infinite set X of variables:

(A0) $\exists 0$.
(A1) $x'' \overset{e}{=} x$.
(A2) $x \oplus x' \overset{e}{=} 0'$.
(A3) $x \oplus 0 \overset{e}{=} x$.
(A4) $\exists x \oplus y \Rightarrow x \oplus y \overset{e}{=} y \oplus x$.
(A5) $\exists((x \oplus y) \oplus z) \Rightarrow (x \oplus y) \oplus z \overset{e}{=} x \oplus (y \oplus z)$.
(A6) $\exists(x \oplus y)$ and $\exists(y' \oplus z) \Rightarrow \exists(x \oplus z)$.
(A7) $\exists(x \oplus y')$ and $\exists(x' \oplus y) \Rightarrow x \overset{e}{=} y$.
(A8) $\exists(x \oplus y)$ and $\exists(y \oplus z)$ and $\exists(x \oplus z) \Rightarrow \exists(x \oplus (y \oplus z))$.
(A9) $\exists(x \oplus y') \Rightarrow x \oplus (x \oplus y')' \overset{e}{=} y$.

Now we introduce the definitions of a D-orthomodular partial algebra and an A-orthomodular partial algebra.

DEFINITION 3.2. By a *D*-orthomodular partial algebra we understand a partial algebra $\underline{A} := (A; \oplus; '; 0)$ of type $(2,1,0)$ such that the list of axioms of Definition 3.1 is satisfied in $\underline{A}$ with the exception of the axiom (A8).

DEFINITION 3.3 By an A-orthomodular partial algebra we understand a partial algebra $\underline{A} := (A, \oplus, '.0)$ of type $(2,1,0)$ such that the list of axioms of Definition 3.1 is satisfied in $\underline{A}$ with the exception of the axiom (A8), and, in addition, the following axiom is satisfied:

(A10) $\exists x \oplus x \Rightarrow x \overset{e}{=} 0$.

In what follows, instead of "partial algebras" we will speak simply about "algebras".

The axioms (A0) through (A10) are existentially conditioned existence equations, and they define ECE-varieties (see [4]); the ECE variety of all orthomodular algebras has been denoted by OMA in [4]; the ECE variety of A-orthomodular algebras will be denoted by AOMA; and the ECE variety of all D-orthomodular algebras will be denoted by DOMA. We have OMA $\subset$ AOMA $\subset$ DOMA. Indeed, AOMA $\subset$ DOMA is clear. To prove OMA $\subset$ AOMA, we have to show that the axioms (A0) through (A9) imply (A10). But it is shown in [4, Lemma 2.3].

The axiom in question is (A8), therefore all results derived from the axioms (A0)–(A7) remain valid for DOMAs and AOMAs. In particular (see [4]),

(i) the unary operation $'$ is a total bijection (see (A1)); a' is called the orthocomplement of $a \in A$;
(ii) for each $a \in A$, $a \oplus a'$ always exists with the constant value $0' = 1$ (see (A2));

(iii) the constant 0 always exists (see (A0)), and, for each element $a \in A$, $a \oplus 0$ always exists and yields a as value (see (A3));
(iv) the operation $\oplus$ is commutative (see (A4)) and associative (see (A5)), whenever it exists.
(v) If one defines a relation "$\leq$" on an arbitrary D-orthomodular algebra $\underline{A}$ by

$$a \leq b \Longleftrightarrow \exists(a \oplus b'),$$

one easily realizes that from (A2) there follows reflexivity, that (A6) implies transitivity and (A7) means asymmetry of the relation "$\leq$", i.e., in any D-orthomodular algebra $\underline{A}$, the relation "$\leq$" defined above is always a partial order relation on A; moreover, the axiom (A3) – together with (A1) – implies that $1(= 0')$ is the greatest element and 0 is the least element.

Remark 3.4. The axioms (A0)–(A9) are not independent. Indeed, (A3) follows from (A9), (A2) and (A1): $\exists x \oplus x' \Rightarrow x \oplus (x \oplus x')' \stackrel{e}{=} x \Rightarrow x \oplus 0 \stackrel{e}{=} x$, and (A6) follows from (A9), (A1), (A4) and (A5). Indeed, we get $y' \oplus z = (x \oplus (x \oplus y)') \oplus z = (x \oplus z) \oplus (x \oplus y)'$. Hence (A3) and (A6) are implied by the other axioms not involving (A8).

In [4], the following statement has been proved.

THEOREM 3.5 [4]. *Let $(A; \oplus;'; 0)$ be an orthomodular algebra. If we define*

$$a \leq b \Longleftrightarrow \exists a \oplus b', \tag{3.1}$$

then $(A; \leq;'; 0; 1)$ is an orthomodular poset.

Conversely, if $(A; \leq;'; 0; 1)$ is an orthomodular poset, and we define the partial operation $\oplus$ by

$$a \oplus b = c \text{ whenever } a \leq b' \text{ and } a \vee b = c, \tag{3.2}$$

then $(A; \oplus;'; 0)$ is an orthomodular algebra.

Moreover, going back and forth with these constructions starting from either of the two kinds of structures always yields back the original one.

We will prove similar statements for D-orthomodular algebras and D-posets and for A-orthomodular algebras and orthoalgebras.

THEOREM 3.6. *Let $\underline{A} = (A; \oplus;'; 0)$ be a D-orthomodular algebra. If we define*

$$a \leq b \Longleftrightarrow \exists a \oplus b' \tag{3.1}$$

and the partial binary operation $\ominus$ by

$$\exists b \ominus a \text{ iff } a \leq b \text{ and } b \ominus a := (a \oplus b')' \tag{3.3}$$

then $(A, \leq, \ominus, 1)$, where $1 = 0'$, is a D-poset.

Conversely, if $(A; \leq; \ominus; 1)$ *is a D-poset, and we define the unary operation* $'$ *by*

$$(3.4) \qquad a' := 1 \ominus a$$

and the binary operation $\oplus$ *by*

$$(3.5) \qquad a \oplus b \text{ exists iff } a \leq b' \text{ and } a \oplus b := (b' \ominus a)'$$

then $(A, \oplus;'; 0)$, *where* $0 = 1' = 1 \ominus 1$, *is a D-orthomodular algebra.*

Moreover, going back and forth with these constructions starting from either of the two kinds of structures always yields back the original one.

Proof. If $(A; \oplus;'; 0)$ is a D-orthomodular algebra, we have already proved that the relation "$\leq$" defined by (3.1) is a partial order relation on A, 0 is the least and $1(= 0')$ is the greatest element of it. Therefore, it suffices to prove (i) and (ii) of Proposition 2.3. By (A3) $x' \oplus 0 \stackrel{e}{=} x'$, so that by (3.3), $x \ominus 0 = x$, which proves 2.3(i). To prove 2.3(ii), observe first that $a \oplus b \stackrel{e}{=} c$ implies $a \stackrel{e}{=} c \ominus b$. Indeed, by (A9), (A1) and (A4),

$$a' = b \oplus (b \oplus a)' = b \oplus c' = (c \ominus b)'.$$

Moreover, $a \leq a \oplus b$ holds whenever $a \oplus b$ exists, since by (A9), $\exists a \oplus b \Rightarrow \exists a \oplus (a \oplus b)'$.

Let $a \leq b \leq c$. By (3.1), (3.3) and (A9),

$$b = a \oplus (b \ominus a), \ c = b \oplus (c \ominus b) = (a \oplus (b \ominus a)) \oplus (c \ominus b),$$

which implies that

$$c \ominus a = (b \ominus a) \oplus (c \ominus b),$$

hence

$$c \ominus b \leq c \ominus a, \text{ and } (c \ominus a) \ominus (c \ominus b) = b \ominus a.$$

Conversely, let $(A; \leq; \ominus; 1)$ be a D-poset. Then (A0) holds by Proposition 2.2(i), (A1) holds by (DP ii).

(A2): $x \leq x \Rightarrow x \oplus x'$ exists, and by (3.5), (iii) of Proposition 2.2 and (A1), $x \oplus x' = (x \ominus x)' = 0' = 1$.

(A4): $\exists x \oplus y \Rightarrow x \leq y'$ and $x \oplus y = (y' \ominus x)'$ by (3.5). By (3.4), $y' \ominus x = (1 \ominus y) \ominus x = (1 \ominus x) \ominus y$ by Proposition 2.2 (vii). This implies, by (3.4) and (3.5), that $y \oplus x \stackrel{e}{=} x \oplus y$.

(A5): $\exists x \oplus y$ and $\exists (x \oplus y) \oplus z \Rightarrow x \leq y'$, and $x \oplus y \leq z'$ by (3.5). Hence

$$x \leq y', \ z \leq (x \oplus y)' = y' \ominus x \Rightarrow x \leq y' \ominus z$$

and

$$(y' \ominus z) \ominus x = (y' \ominus x) \ominus z$$

by Proposition 2.2(vii). Applying (3.5) we obtain $(y \oplus z)' \ominus x = (y \oplus x)' \ominus z$, and applying (3.5) again gives, together with (A1), that $(y \oplus z) \oplus x = (y \oplus x) \oplus z$, Now (A4) yields the desired result.

(A7) holds by (3.5) and reflexivity of the partial order in A.

(A9): $\exists(x \oplus y')$ implies $x \leq y$ by (3.5). Applying (DP iii) to $0 \leq x \leq y$ gives $(y \ominus 0) \ominus (y \ominus x) = x \ominus 0$, that is, $y \ominus (y \ominus x) = x$. By (3.5), it follows that $y' \oplus (y' \oplus x)' = x'$. Since $x \leq y \iff y' \leq x'$ by (3.4) and (DP iii) applied to $x \leq y \leq 1$, (A9) is satisfied.

(A3) and (A6): see Remark 3.4.

The last statement easily follows from what has been done so far. ∎

THEOREM 3.7. *Let $(A; \oplus; '; 0)$ be an A-orthomodular algebra. If we define*

$$1 := 0' \tag{3.6}$$

then $(A; \oplus; 0; 1)$ is an orthoalgebra.

Conversely, if $(A; \oplus; 0; 1)$ is an orthoalgebra, and we define, for every $a \in A$, a' as the unique element b in A such that $a \oplus b = 1$, then $(A; \oplus; '; 0)$ is an A-orthomodular algebra.

Moreover, going back and forth with these constructions starting from either of the two kinds of structures always yields back the original one.

Proof. If $(A; \oplus; '; 0)$ is an A-orthomodular algebra, then the axioms of Definition 2.4 are satisfied. Indeed, (OA i) follows by (A4); (OA ii) follows by (A5); (OA iii) follows by (A2) and the following implication: $x \oplus y = 1$ implies by (A9), (A1) and (A3) that $x' = y \oplus 1' = y \oplus 0 = y$, which proves uniqueness of x'. (OA iv) follows by (A10).

Conversely, if $(A; \oplus; 0; 1)$ is an orthoalgebra, then by [10], A is a D-poset with the operation $\ominus$ defined as follows:

$$a \ominus b \text{ exists iff } a \leq b \text{ and } a \ominus b = (b' \oplus a)'$$

satisfying the additional axiom (2.1). This implies that the axioms (A0) through (A7) and (A9) are satisfied (see proof of Theorem 3.6), and (A10) is satisfied owing to (2.1). The rest of the proof is straightforward. ∎

In [4], the following additional axiom for OM algebras has been introduced:

$$\begin{aligned} &\exists(x_1 \oplus y_1) \text{ and } \exists(x_1 \oplus y_2) \text{ and } \exists(x_2 \oplus y_1) \text{ and } \exists(x_2 \oplus y_2) \\ \text{(R)} \quad &\Rightarrow (\exists z)(\exists(x_1 \oplus z) \text{ and } \exists(x_2 \oplus z) \text{ and } \exists(y_1 \oplus z') \text{ and } \exists(y_2 \oplus z')). \end{aligned}$$

An orthomodular algebra satisfying (R) is called *rich*.

PROPOSITION 3.8. *If an AOM algebra $\underline{A}$ satisfies* (R), *then $\underline{A}$ is an OM algebra.*

Proof. By Theorem 3.7, $\underline{A}$ can be considered as an orthoalgebra. Assume that $a \oplus b$ exists, and let c be any upper bound of a, b. By (A9) and (3.1), $a \oplus b$ is an upper bound of a, b. Consider the elements $a, b, (a \oplus b)', c'$. Then, owing to (R),

$$\exists(a \oplus (a \oplus b)') \text{ and } \exists(a \oplus c') \text{ and } \exists(b \oplus (a \oplus b)') \text{ and } \exists(b \oplus c')$$
$$\Rightarrow (\exists z)(\exists(a \oplus z) \text{ and } \exists(b \oplus z) \text{ and } \exists((a \oplus b)' \oplus z') \text{ and } \exists(c' \oplus z')).$$

By the definition of partial order, we have

$$a \leq z',\ b \leq z',\ (a \oplus b)' \leq z,\ c' \leq z.$$

From $z' \leq a \oplus b$, and the fact that $a \oplus b$ is a minimal upper bound (see, e.g., [7]), we get $z' = a \oplus b$, and hence $a \oplus b \leq c$. This proves that $a \vee b$ exists, hence $\underline{A}$ is an orthomodular algebra. ∎

EXAMPLE 3.9. Let us consider the interval of real numbers $[0,1]$. With the usual order and the operation $b \ominus a = b - a$ for $a \leq b$, $[0,1]$ is a D-poset satisfying (R), which is not an orthoalgebra.

By definition, DOMA and AOMA are ECE-varieties. Hence, for each set X, there exists a free algebra $\underline{F}(X, DOMA)$ and $\underline{F}(X, AOMA)$. Proof of the following theorem is the same as the proof of Theorem 4.1 in [4].

THEOREM 3.10. *Let X be any set of variables, then the DOMA free DOMA algebra and the AOMA free AOMA algebra exist and coincide with the OMA free OMA algebra:*

$$F(X; OMA) = X \dot{\cup} X^* \dot{\cup} \{0,1\}$$
$$dom\oplus := \{(0,0),(0,1),(1,0)\} \cup$$
$$\cup \bigcup_{x \in X} \{(x,0),(0,x),(x^*,0),(0,x^*),(x,x^*),(x^*,x)\},$$

and

$$0 \oplus 0 := 0,\ 0 \oplus 1 := 1,\ 1 \oplus 0 := 1,$$
$$0 \oplus y := y \oplus 0 := y \text{ for all } y \in X \dot{\cup} X^*;$$
$$x \oplus x^* := x^* \oplus x := 1 \text{ for all } x \in X;$$
$$0' := 1; 1' := 0; x' := x^*; (x^*)' := x \text{ for all } x \in X.$$

0 is the least (and 1 is the greatest) element.

Remark 3.11. Let $(L, \leq, ', 0, 1)$ be a partially ordered orthocomplemented set in which $x \vee y$ exists provided that $x \leq y'$. It is not difficult to prove that it can be characterized as an algebra $\underline{A} := (A; \oplus; '; 0)$ of type (2,1,0) such that axioms (A0)–(A8) are satisfied (note that in this case (A3) and (A6) cannot be omitted — see Remark 3.4).

4. Probability measures on orthomodular algebras

Let $\underline{A} := (A; \oplus;'; 0)$ be a D-orthomodular, A-orthomodular, or orthomodular partial algebra.

DEFINITION 4.1. A mapping

$$m : A \to [0,1]$$

is said to be a probability measure on $\underline{A}$ if

$$m(a \oplus b) = m(a) + m(b),$$

whenever $a \oplus b$ exists, and

$$m(a') = 1 - m(a);\ m(0) = 0.$$

DEFINITION 4.2. (i) A set M of probability measures on $\underline{A}$ is said to be full if, for any $a, b \in A$,

$$(\forall m \in M : m(a) + m(b) \leq 1) \Rightarrow a \oplus b \text{ exists.}$$

(ii) A set M of probability measures on $\underline{A}$ is unital if for any $a \in A$,

$$a \neq 0 \Rightarrow (\exists m \in M : m(a) = 1).$$

(iii) A set M of probability measures on $\underline{A}$ is strong (or rich) if, for any $a, b \in A$,

$$a \oplus b' \text{ does not exist } \Rightarrow (\exists m \in M : m(a) = 1, m(b) \neq 1).$$

We note that a strong set of probability measures is both full and unital. Indeed, let $m(a) + m(b) \leq 1$ $\forall m \in M$ and let $a \oplus b$ do not exist. Then there is $m \in M$ with $m(a) = 1,\ m(b') \neq 1$, hence $m(a) = 1,\ m(b) \neq 0$, a contradiction with $m(a)+m(b) \leq 1$. Taking $b = 0$ in (iii) yields unitality. The following definition generalizes the definition of a numerical OMA (see [4]).

DEFINITION 4.3. Let S be a nonempty set, and let $L \subseteq [0,1]^S$ be a set of functions from S into $[0,1]$. L is said to be a numerical Y-orthomodular algebra (where Y stands for D or A) if it is an Y-orthomodular algebra with respect to the partial operation $\oplus$ defined by

$$f \oplus g := f + g \text{ iff } f + g \leq 1,$$

and the unary operation $'$ given by

$$f' := 1 - f,$$

with the constant $0 = 0_L$ being the function taking the value 0 for all $x \in S$.

THEOREM 4.4. *Let $L \subseteq [0,1]^S$, $S \neq \emptyset$ have the following properties:*

$\underline{1}$ (i) $1 \in L$;

(ii) $f, g \in L,\ f \leq g \Rightarrow g - f \in L$.

Then $(L; \oplus; 1; 0)$ becomes a numerical DOMA.

2 (i) $1 \in L$;
(ii) $f, g \in L,\ f \leq g \Rightarrow g - f \in L$;
(iii) $(\forall f \in L)(f \neq 0 \Rightarrow (\exists \alpha \in S : f(\alpha) > \frac{1}{2}))$.

Then $(L; \oplus; '; 0)$ *becomes a numerical AOMA.*

3 (i) $1 \in L$;
(ii) $f, g \in L,\ f \leq g \Rightarrow g - f \in L$;
(iii) *if* $f_1, f_2, f_3 \in L$ *and if* $f_i + f_j \leq 1$ *for* $i \neq j$, *then* $f_1 + f_2 + f_3 \in L$.

Then $(L; \oplus; '; 0)$ *becomes a numerical OMA.*

4 (i) $1 \in L$;
(ii) $(\forall f \in L)(f \neq 0 \Rightarrow (\exists \alpha \in S) f(\alpha) > \frac{1}{2}))$;
(iii) $f, g \in L,\ f \leq g \Rightarrow g - f \in L$;
(iv) *if* $f_1, f_2, g_1, g_2 \in L$ *and* $\max\{f_1, f_2\} \leq \min\{g_1, g_2\}$ *then there exists* $h \in L$ *such that* $\max\{f_1, f_2\} \leq h \leq \min\{g_1, g_2\}$.

Then $(L, \oplus, ', 0)$ *becomes a rich numerical OMA.*

Proof. 1 see [9]; 2 see [9] and [10] (observe that $f \oplus f$ exists, resp. $f \leq 1 \ominus f$, is equivalent to $f \leq \frac{1}{2}$); 3 and 4 see [4] (we note that 3^o in [4] Theorem 5 is redundant, it follows from 4^o and 1^0). ∎

Let $(A; \oplus; '; 0)$ be an orthomodular algebra and let M be a full set of probability measures on A. For every $a \in A$, let $\overline{a} : M \to [0, 1]$ be a function defined by $\overline{a}(m) := m(a)$ for all $m \in M$. The set of all functions $(\{\overline{a} : a \in A\}; \oplus; '; 0)$ with $\oplus$ and $'$ defined as in Definition 4.3, will be called a *numerical realization* of A. Next theorem generalizes the results obtained in [4].

THEOREM 4.5. *The following statements hold true:*

1 *Every DOMA with a full set of probability measures is isomorphic to a numerical DOMA.*

2 *Every DOMA with a full and unital set of probability measures is isomorphic to a numerical AOMA.*

3 *Every OMA with a full set of probability measures is isomorphic to a numerical OMA.*

4 *Every rich OMA with a full and unital set of probability measures is isomorphic to a numerical rich OMA.*

5 *Every DOMA with a strong set of probability measures is isomorphic to a numerical OMA.*

Proof. 1. Consider the numerical realizations $\{\overline{a} : a \in A\}$. It is straightforward to check that the conditions 1 (i) and (ii) of Theorem 4.4 are satisfied.

2. Observe that unitality guaranties 2 (iii) of Theorem 4.4.

3 and 4 have been proved in [4].

5. Let M be the strong set of probability measures on an DOMA $\underline{A}$. Let $\overline{a}_1, \overline{a}_2, \overline{a}_3$ be such that $\overline{a}_i + \overline{a}_j \leq 1$ for $i \neq j$. Then $\overline{a}_3 \leq 1 - \overline{a}_1$, $\overline{a}_3 \leq 1 - \overline{a}_2$ gives $\overline{a}_3(m) = 1 \Rightarrow (\overline{a}_1 + \overline{a}_2)(m) = 0$. Since M is strong, this yields $\overline{a}_3 \leq 1 - (\overline{a}_1 + \overline{a}_2)$, which implies 3 (iii) of Theorem 4.4. ■

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MATHEMATICAL INSTITUTE
SLOVAK ACADEMY OF SCIENCES
Štefánikova 49,
814 73 BRATISLAVA, SLOVAKIA

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