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# VARIOUS APPLICATIONS OF SYMMETRIC GROUPOIDS

*Dedicated to Professor Tadeusz Traczyk*

## Preliminaries

We investigate the role symmetric groupoids and more general concepts used in different areas of mathematics. In chapter 1 we give a brief outline of [EL2], where we described the use of symmetric groupoids for discrete dynamical systems. In chapter 2 we introduce a new notion of extensions of symmetric groupoids, which allows a general description for homotopy sets of the kind $[S^p \times S^q; S^n]$ (cf. (2.14)), where $S^m$, $m \in \mathbb{N}$, denotes the $m$–sphere. In chapter 3 we apply ($n$–)symmetric groupoids to the theory of topological groups and show, starting from two complete group topologies, how to get a complete infimum group topology (cf. (3.6)). This generalizes parts of the results in [E3].

For $n > 2$, by the following definition we get generalizations of symmetric groupoids. The equivalence stated below is immediate.

0.1. Definition/Proposition. *Let $X$ be a set, $\bullet : X \times X \to X$ a binary operation and $n \in \mathbb{N}$. We call the pair $(X, \bullet)$ an $n$–symmetric groupoid, if the following identities are satisfied for $x, y, z \in X$:*

(i) $x \bullet x = x$ (*idempotency*),

(ii) $\underbrace{x \bullet (x \bullet \ldots \bullet (x}_{n \text{ times}} \bullet y) \ldots) = y$ (*$n$–symmetry*),

(iii) $\underbrace{(x \bullet y) \bullet z = x \bullet (y \bullet (x \bullet \ldots \bullet (x}_{(n-1)\text{times}} \bullet z) \ldots))$ (*$n$–antidistributivity*).

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*The axioms* (i), (ii), (iii) *are equivalent to the system* (i), (ii), (iii′) *with*

(iii′) $\qquad (x \bullet y) \bullet (x \bullet z) = x \bullet (y \bullet z) \qquad$ (left distributivity). ■

For $x_1, \ldots x_k \in X$ we agree to the notation

$$x_1 \bullet x_2 \bullet x_3 \bullet \ldots \bullet x_k := x_1 \bullet (x_2 \bullet (x_3 \bullet \ldots \bullet x_k)).$$

By (iii), any term can be written without brackets.

The following result shows that lots of (quotients of) groups can be equipped with an $n$–symmetric structure. By calculation one can prove

0.2. Proposition. *Let* $n \in \mathbb{N}$, $X$ *be a group,* $G < X$, $\tau : X \to X$ *an endomorphism with* $\tau(G) < G$, *and suppose one of the following conditions to be satisfied:*

(1) $$g\tau(g)^{-1} \in \mathcal{Z}(X) \qquad \forall g \in G,$$

(2) $$\tau(X) < \mathcal{N}(G),$$

*where* $\mathcal{Z}(X)$ *denotes the centre of* $X$ *and* $\mathcal{N}(G)$ *the normalizer of* $G$. *By the assignment*

$$X/G \times X/G \ni (xG, yG) \mapsto x\tau(x)^{-1}\tau(y)G \in X/G$$

*we get on the set* $X/G$ *of left cosets a binary operation* $*$, *which is idempotent and left distributive.*

*In addition, let* $\tau^\ell = \tau^m$ $(\ell, m \in \mathbb{N})$ *with* $\tau^0 := id_X$. *Then* $*$ *satisfies the identity*

$$\underbrace{xG * (xG * \ldots * (xG}_{\ell \text{ times}} * yG) \ldots) = \underbrace{xG * (xG * \ldots * (xG}_{m \text{ times}} * yG) \ldots) \quad \forall x, y \in X.$$

*In particular, if* $\tau$ *is an automorphism of order* $n$, *i.e.* $\tau^n = id_X$, *then* $*$ *is* $n$*–symmetric.* ■

For an $n$–symmetric groupoid $X$ and a set $Y$, which are eventually equipped with basepoints $x_0$ resp. $y_0$, the set of mappings $X^Y$, as well as the set of basepoint preserving mappings $(X, x_0)^{(Y, y_0)}$, become $n$–symmetric groupoids by componentwise definition,

$$(f \bullet g)(y) := f(y) \bullet g(y), \quad y \in Y, \ f, g \in X^Y \text{ or } \in (X, x_0)^{(Y, y_0)}.$$

A further useful example of (2–) symmetric groupoids is given by the concept of symmetric spaces. A symmetric space consists of a topological space $S$ together with a continuous binary operation $\bullet$, which satisfies (0.1), (i)–(iii) for $n = 2$ and one additional topological axiom which is inessential for our purposes.

0.3. DEFINITION. *For $n \in \mathbb{N}$, let $(X, \bullet)$ be an $n$–symmetric groupoid, and $e, x \in X$. By $P(x, \bullet, e)$ (or $P$ in short) we denote the minimal subset of $X$ such that*

$$\text{(i)} \qquad x, e \in P,$$

$$\text{(ii)} \qquad a, b \in P \Rightarrow a \bullet b \in P,$$

*and call $P$ the set of integral powers of $x$ with respect to (w.r.t.) $e$.*

*Now let $n = 2$, i.e. $(X, \bullet)$ is a symmetric groupoid. For $k \in \mathbb{Z}$ we agree to write*

$$x^{\langle k \rangle} := \begin{cases} \underbrace{x \bullet e \bullet \ldots \bullet e \bullet x}_{k \text{ factors}}, & k > 0, k \equiv 1(2), \\ \underbrace{x \bullet e \bullet \ldots \bullet x \bullet e}_{k \text{ factors}}, & k > 0, k \equiv 0(2), \\ \underbrace{e \bullet x \bullet \ldots \bullet e \bullet x}_{|k|+1 \text{ factors}}, & k \le 0, k \equiv 1(2), \\ \underbrace{e \bullet x \bullet \ldots \bullet x \bullet e}_{|k|+1 \text{ factors}}, & k \le 0, k \equiv 0(2), \end{cases}$$

*and call $x^{\langle k \rangle}$ the $k$-th power of $x$ w.r.t. $e$.*

In the situation of (0.3), clearly $P$ forms an $n$–symmetric subgroupoid of $X$. For the powers of a symmetric groupoid w.r.t. any chosen element it can easily be verified

$$x^{\langle k \rangle} \bullet x^{\langle \ell \rangle} = x^{\langle 2k - \ell \rangle}, \quad (x^{\langle k \rangle})^{\langle \ell \rangle} = x^{\langle k\ell \rangle} \qquad (k, \ell \in \mathbb{Z}, x \in X).$$

## 1. Power processes

For a skew field $K$ and a left $K$–module $V$ we introduce the notion of a $k$–th power $x^{[k]}$ for elements $x \in V$ and $k \in \mathbb{N}_0$ by means of symmetric groupoids. In addition, if $K$ is endowed with a valuation and $V$ with a norm, it is in place to investigate domains of boundedness of the iterative $k$–th power process ($k \in \mathbb{N}_0$)

$$x_{\nu+1} := x_\nu^{[k]} + c, \text{ where } x_0 := 0 \in V,\ \nu \in \mathbb{N}_0,$$

i.e. the set of all elements $c \in V$, where the given process remains bounded. We shortly describe how to determine these domains for a certain type of powers on complex vector spaces, where $|\cdot| : \mathbb{C} \to \mathbb{R}_0^+$ is given by $z \mapsto (z\bar{z})^{\frac{1}{2}}$.

In this chapter, a field is not necessarily commutative. For a field $K$ we use the abbreviations $c_K :=$ char $K$, $C_K :=$ centre $K$.

1.1. DEFINITION. *Let $K$ be a field, $J : K \to K$ an antiautomorphism, $V$ a left $K$–module, and $h : V \times V \to K$ a $J$–sesquilinear form (sqlf). We call $\mathcal{N}_h := \{x \in V : h(x, x) = 0\}$ the nullspace of $V$ with respect to the $J$–sqlf $h$*

*and define* $\bullet_h : V \times V \setminus \mathcal{N}_h \to V$ *by*

$$(x,y) \mapsto \frac{1}{h(y,y)}(-h(x,x)y + a(x,y)x),$$

*where* $a(x,y) := h(x,y) + h(y,x)$.

1.2. DEFINITION. *Let* $K$ *be a field,* $c_K \neq 2$, $J : K \to K$ *an antiautomorphism,* $V$ *a left* $K$*-module, and* $h : V \times V \to K$ *a* $J$*-sqlf. We call* $h$ *a symmetrizing* $J$*-sqlf, if for* $x, y \in V$

(i) $$a(x,y) \in C_K,$$

(ii) $$h(x,x)J(h(y,y)) = J(h(x,x))h(y,y),$$

(iii) $$h(x,x)J(a(x,y)) = J(h(x,x))a(x,y).$$

It turns out that for a symmetrizing $J$–sqlf the antiautomorphism $J$ is necessarily involutory ([EL2],(1.5)). Any symmetric bilinear form (blf) on vector spaces is symmetrizing.

1.3. THEOREM. *Let* $K$ *be a field,* $c_K \neq 2$, $J : K \to K$ *an antiautomorphism,* $V$ *a left* $K$*-module, and* $h : V \times V \to K$ *a symmetrizing* $J$*-sqlf. Then* $(V \setminus \mathcal{N}_h, \bullet_h)$ *is a symmetric groupoid* ([EL2], (1.8)). ■

The following example of a symmetric groupoid arises from a special case of (1.3) and plays an important role in chapter 2.

1.4. EXAMPLE. *Let* $\langle \cdot ; \cdot \rangle$ *be a nonsingular symmetric blf on* $\mathbb{R}^{n+1}$, *and*

$$S^n_\alpha := \{x \in \mathbb{R}^{n+1} : \langle x; x \rangle = \alpha\}, \;\; 0 \neq \alpha \in \mathbb{R}.$$

*Define* $x \bullet y := -y + 2\frac{\langle x;y \rangle}{\langle x;x \rangle}x$, $x, y \in S^n_\alpha$. *Then* $(S^n_\alpha, \bullet)$ *is a symmetric groupoid* ([L], p. 66).

According to (0.3), for $k \in \mathbb{Z}$ we can form $k$–th powers of elements of $V \setminus \mathcal{N}_h$ with respect to a certain $e \in V$ with $h(e,e) \neq 0$. It can be proved that this formation of powers can be extended to whole $V$ for non negative $k$ ([EL2],(1.12)), and all powers of $x$ lie in the plane spanned by $x$ and $e$. We denote $k$–th powers of $x$ in $V$ in the sense above by $x^{[k]}$ and call them the $k$–th power of $x$ w.r.t. $h$ and $e$. The term "power" is in place, since our concept of a power in particular situations coincides with the notion of powers in fields or algebras, respectively: In the situation of (1.4), for the canonical inner product and $\alpha := 1$, we get a continuous binary operation on the $n$–sphere $S^n$ with the topology inherited from $\mathbb{R}^{n+1}$, which makes $(S^n, \bullet)$ a symmetric space. For $n = 7$, $S^n$ consists of all Cayley numbers of norm 1. Writing $\cdot$ for the Cayley multiplication, one can prove $x \bullet y = x \cdot y^{-1} \cdot x$ $(x, y \in S^7)$ ([E2], (1.5)). From this one concludes by an easy calculation, that powers w.r.t. $\bullet$ and $e := (1,0,\dots 0) \in S^7$ are the same as powers w.r.t. the Cayley multiplication.

Now we turn to normed complex vector spaces, where $\mathbb{C}$ is equipped with the usual absolute value, and consider iterative processes like that described above, with non negative powers w.r.t. a symmetric blf $h$ and $e$ with $h(e,e) \neq 0$. Let $k \in \mathbb{N}_0$. Using a criterion for boundedness of subsets of a two dimensional subspace of a left $K$–module over a complete valuated field ([EL1], (2.8)) we can show that for any complex vector space and given $h$ and $e$ the behaviour of $(x_\nu)_{\nu\in\mathbb{N}_0}$ with respect to boundedness depends only on the values $\gamma := \frac{h(c,e)}{h(e,e)}$, $\delta := \frac{h(c,c)}{h(e,e)}$, $\gamma, \delta \in \mathbb{C}$, i.e. is independent from both the blf and the chosen $e$ ([EL2],(3.3)). Therefore, we have reduced the problem of determining domains of boundedness to finite dimensions.

By means of a certain $k$–th power process on a 2–dimensional $\mathbb{C}$–vector space we can determine all pairs $(\gamma, \delta) \in \mathbb{C} \times \mathbb{C}$, such that a $k$–th power process with $\gamma = \frac{h(c,e)}{h(e,e)}$, $\delta = \frac{h(c,c)}{h(e,e)}$, remains bounded (cf. [EL2],(3.4)ff.). Roughly speaking, we get something like universal domains of boundedness for $k$–th power processes with respect to symmetric blfs on normed complex vector spaces.

## 2. Homotopy theory

For compactly generated topological spaces $X$ and $Y$, which are equipped with basepoints, we denote the set of homotopy classes $[f]$ of continuous basepoint preserving mappings $f$ from $X$ to $Y$ by $[X;Y]$. For sets of homotopy classes with symmetric spaces as a range it is easy to verify

2.1 Definition/Proposition. *Let $X$ and $S$ be topological spaces with basepoints, $S$ a symmetric space with the binary operation $\bullet$, and $f, g : X \to S$ continuous, basepoint preserving mappings. By*

$$[f] \bullet [g] := [f \bullet g]$$

*we define on $[X;S]$ a binary operation $\bullet$. The pair $([X;S], \bullet)$ is a symmetric groupoid.* ∎

Our aim in this section is a description of the symmetric groupoids $([S^p \times S^q; S^n], \bullet)$, where $\bullet$ is induced by (2.1) and the symmetric structure on $S^n$ presented after (1.4).

2.2. Definition. *Let $(G,+)$ be an abelian group, $R \subseteq G$, and $(R,\bullet)$ a symmetric groupoid. We call the symmetric groupoid $(R,\bullet)$ group related to $(G,+)$, if there is a map $\tau : R \to G$ such that*

$$a \bullet b = a - \tau(a) + \tau(b), \quad a, b \in R.$$

In the sequel, we call $\tau$ the map describing $\bullet$. Without loss of generality, we can assume $0 \in R$, $\tau(0) = 0$, and $\tau^2 = \mathrm{id}_R$ (cf. [E1], after Prop. 6). Examples of group related symmetric groupoids are given by (0.2) with $X$ abelian and $\tau^2 = \mathrm{id}_X$. In particular, $\mathbb{Z}$ is a group related symmetric groupoid with $\tau = -\mathrm{id}_{\mathbb{Z}}$.

2.3. Definition. *Let $(R, \bullet)$ be a symmetric groupoid, $a, b \in R$. We define a relation $\sim_\ell$ on $R$ by*

$$a \sim_\ell b :\iff \exists\, k \in \mathbb{N}\ \exists\, a_1, \ldots a_k \in R : a_k \bullet \ldots \bullet a_1 \bullet a = b.$$

The relation $\sim_\ell$ is an equivalence relation.– The structure of group related symmetric groupoids $R$ was described in ([E1], Theorem 12) as a union of cosets by a certain subgroup $\mathcal{U}$ of G. By calculation one can show that any group related symmetric groupoid is a $SIE$–groupoid, as well as the coincidence of the relation $\sim_\ell$ with the relation $\sim$ on $R$ determined by $\mathcal{U}$ (to be more precise, for $r, s \in R$ we define $r \sim s$ iff $r - s \in \mathcal{U}$), which are congruence relations w.r.t. the symmetric structure by ([R],(3.2)). It is not too hard to prove that ([E1], Theorem 12) is a variant of a special case of ([R],(4.3)); in other words, $R$ forms an $AG$–sum of copies of $\mathcal{U}$.

2.4. Definition. *Let $(R, \bullet)$ be a symmetric groupoid, related to $(G, +)$, and $\tau$ the respective describing map. A subgroup $H$ of $G$ with $H \subseteq R$ is called $\tau$–admissible, if*

(i) $\quad R + H \subseteq R,$

(ii) $\quad \tau|_H = id_H,$

(iii) $\quad \tau(r + h) = \tau(r) + \tau(h), \quad (r \in R, h \in H).$

*We call the set of all $\tau$-admissible subgroups in $R$ the $\tau$–spectrum of $R$ and write $\mathcal{S}_\tau(R)$.*

If $H \in \mathcal{S}_\tau(R)$, by (i) $R$ is a union of cosets of $G$ by $H$. Therefore, $H$ defines an equivalence relation on $R$. For the coset space we write $R/H := \{r + H : r \in R\}$.

2.5. Definition/Proposition. *Let $(R, \bullet)$ be a symmetric groupoid, related to $(G, +)$, and $\tau$ the respective describing map. We put*

$$\mathcal{F}_\tau(R) := \{r + H : r \in R, H \in \mathcal{S}_\tau(R)\},$$

*and call this set the $\tau$-fan of $R$. Assigning*

$$(r_1 + H, r_2 + K) \mapsto r_1 \bullet r_2 + K, \qquad r_1, r_2 \in R;\ H, K \in \mathcal{S}_\tau(R),$$

*a binary operation $\bullet_\tau : \mathcal{F}_\tau(R) \times \mathcal{F}_\tau(R) \to \mathcal{F}_\tau(R)$ is defined. The pair $(\mathcal{F}_\tau(R), \bullet_\tau)$ is a symmetric groupoid.*

*Proof.* $\bullet_\tau$ is well defined, since for $r_1, r_2 \in R$ and $H, K \in \mathcal{S}_\tau(R)$, $h \in H$, $k \in K$ we calculate

$$\begin{aligned}(r_1 + h) \bullet (r_2 + k) + K &= r_1 + h - \tau(r_1 + h) + \tau(r_2 + k) + K \\ &\underset{(2.4),(ii),(iii)}{=} r_1 + h - \tau(r_1) - h + \tau(r_2) + k + K \\ &= r_1 - \tau(r_1) + \tau(r_2) + K \\ &= r_1 \bullet r_2 + K. \ \blacksquare\end{aligned}$$

2.6. Definition. *Let $(R, \bullet)$, $(S, \bullet')$ be group related symmmetric groupoids, and $\tau'$ the map describing $\bullet'$; furthermore, let $\lambda : R/\sim_\ell \to \mathcal{S}_{\tau'}(S)$ and $h^* : R \times R \to \mathcal{F}_{\tau'}(S)$ with $h^*(r_1, r_2) \in S/\lambda(q(r_2))$ be mappings, where $q : R \to R/\sim_\ell$ denotes the canonical projection. We call $(R, S, \lambda, h^*)$ a fan extension of $R$ by $S$, if*

$$\bigcup_{\bar{r} \in R/\sim_\ell} \bar{r} \times (S/\lambda(\bar{r})) \subseteq R \times \mathcal{F}_{\tau'}(S)$$

*is a symmetric groupoid by means of*

$$(r_1, s_1) \bullet'' (r_2, s_2) := (r_1 \bullet r_2, s_1 \bullet_{\tau'} s_2 + h^*(r_1, r_2)).$$

Now we come to the announced description of $[S^p \times S^q; S^n]$. To this end, we first gather results of [E2] concerning the homotopy classification of product mappings on spheres. For $m, n \in \mathbb{N}$, homotopy groups $\pi_m(S^n)$ are always abelian groups with $[S^m; S^n]$ as underlying set. In ([E2],(2.5)) we show

2.7. Theorem. *$([S^m; S^n], \bullet)$ is a symmetric groupoid, group related to $\pi_m(S^n)$, with describing map $\tau := \tau_{m,n}$,*

$$\tau_{m,n} : [S^m; S^n] \to \pi_m(S^n), \quad \alpha \mapsto ((-1)^n \iota_n) \circ \alpha,$$

*where $\iota_n \in \pi_n(S^n)$ denotes the homotopy class of the identity map. If we take $\pi_m(S^n)$ (instead of $[S^m; S^n]$) as domain of $\tau$, then $\tau$ proves to be an involutory group automorphism.* $\blacksquare$

A first approximation to the homotopy classification of product mappings on spheres (cf. (2.13)) is given by the type of a mapping.

2.8. Proposition/Definition. *Let $u \in \varphi \in [S^p \times S^q; S^n]$, denote by $\varphi_1$ resp. $\varphi_2$ the homotopy class of the compositions*

$$S^p \xrightarrow{\iota_1} S^p \times S^q \xrightarrow{u} S^n \quad \textit{resp.} \quad S^q \xrightarrow{\iota_2} S^p \times S^q \xrightarrow{u} S^n,$$

*where we write $\iota_1$ resp. $\iota_2$ for the canonical injection of $S^p$ resp. $S^q$ into $S^p \times S^q$. The mapping*

$$\mathfrak{t} : [S^p \times S^q; S^n] \to [S^p; S^n] \times [S^q; S^n],$$

*defined by the assignment* $\varphi \mapsto (\varphi_1, \varphi_2)$, *is a homomorphism of symmetric groupoids, where the algebraic structure of the range of* $\mathfrak{t}$ *is given by the direct product of the group related symmetric groupoids* $([S^p; S^n], \bullet)$ *and* $([S^q; S^n], \bullet)$. *We call* $\mathfrak{t}(\varphi)$ *resp.* $\mathfrak{t}([u])$ *the type of the homotopy class* $\varphi$ *resp. of the product map* $u$; *furthermore we agree to write*

$$\mathcal{T} := \mathfrak{t}([S^p \times S^q; S^n]).$$

2.9. Remark. *Since* $\mathfrak{t}$ *is a homomorphism of symmetric groupoids, by definition of the relation* $\sim_\ell$ *follows for* $\psi, \psi' \in [S^p \times S^q; S^n]$

$$\psi \sim_\ell \psi' \Rightarrow \mathfrak{t}(\psi) \sim_\ell \mathfrak{t}(\psi').$$

In the following we shall use without further explanation two notions from algebraic topology, namely the seperation element of two product mappings of the same type, the properties of which one can find in [J], and the Whitehead product of two elements of homotopy groups of spheres [Wh].

2.10. Lemma/Definition. *Let* $p, q, n \in \mathbb{N}$, *and* $\omega \in \mathcal{T}$ *with* $\omega = (\omega_1, \omega_2)$, *write* $\bar{\omega}$ *for the equivalence class of* $\omega$ *modulo* $\sim_\ell$ *in* $\mathcal{T}$, *and denote by* $\Delta_\omega < \pi_{p+q}(S^n)$ *the subgroup generated by*

$$-[\omega_1, \xi] + (-1)^{q+1}[\eta, \omega_2], \qquad \xi \in \pi_{q+1}(S^n), \eta \in \pi_{p+1}(S^n),$$

*where* $[\cdot, \cdot]$ *stands for the Whitehead product. For* $T \in \mathcal{T}/\sim_\ell$ *and* $\chi, \omega \in T$ *holds* $\Delta_\chi = \Delta_\omega$; *thus* $\Delta_T := \Delta_{\bar{\omega}} := \Delta_\omega$ *is well defined* ([E2], (3.5)). ∎

2.11. Theorem. *For* $\varphi \in [S^p \times S^q; S^n]$ *holds*

$$u, u' \in \varphi \iff d(u, u') \in \Delta_{\overline{\mathfrak{t}(\varphi)}}.$$

*If* $n$ *is odd and* $u, u' \in \varphi$, *we have the identity*

$$(-\iota_n) \circ d(u, u') = d(u, u').$$

*This implies together with* (2.7) *that* $\tau | \Delta_{\overline{\mathfrak{t}(\varphi)}} = id | \Delta_{\overline{\mathfrak{t}(\varphi)}}$ ([BB]; [E2], (3.2); [E2], (3.8)(ii)). ∎

2.12. Proposition/Definition. *Let* $\mathfrak{s} : \mathcal{T} \to [S^p \times S^q; S^n]$ *be a right inverse of* $\mathfrak{t}$, *i.e.* $\mathfrak{t} \circ \mathfrak{s} = id_{\mathcal{T}}$. *We call* $s : [S^p \times S^q; S^n] \to [S^p \times S^q; S^n]$, $s := \mathfrak{s} \circ \mathfrak{t}$, *a type representing map.*

*A type representing map satisfies* $(\varphi, \varphi' \in [S^p \times S^q; S^n])$

(1) $\mathfrak{t}(s(\varphi)) = \mathfrak{t}(\varphi)$,
(2) $\mathfrak{t}(\varphi) = \mathfrak{t}(\varphi') \Rightarrow s(\varphi) = s(\varphi')$.

Every right inverse of $\mathfrak{t}$ determines a type representing map. For the following we fix such a map $s$, as well as representatives $u_\varphi$ for any $\varphi \in$

$[S^p \times S^q; S^n]$ in a way that we can form the seperation element of any two representatives of the same type.

Unfortunately, the seperation element of two representatives $u, u'$ of homotopy classes of the same type is not good enough for the homotopy classification of product mappings on spheres, since $d(u, u') = 0$ implies the homotopy of $u$ and $u'$, but not vice versa (cf. (2.11)). If we put $d(\psi, \psi') := d(u_\psi, u_{\psi'})$ for $\psi, \psi' \in [S^p \times S^q; S^n]$ of the same type, one can show (cf. [E2]) that at least the class $d(\psi, \psi') + \Delta_{\overline{\mathfrak{t}(\psi)}}$ does not depend from our choice of representatives.

2.13. THEOREM. *With the conventions and notations from above,*

$$[S^p \times S^q; S^n] = \dot{\bigcup_{T \in \mathcal{T}/\sim_t}} T \times \pi_{p+q}(S^n)/\Delta_T.$$

*Any $\psi \in [S^p \times S^q; S^n]$ can be represented by a triple $(\psi_1, \psi_2, \psi_3)$ where*

$$(\psi_1, \psi_2) := \mathfrak{t}(\psi),$$
$$\psi_3 := \psi_3' + \Delta_{\overline{\mathfrak{t}(\psi)}} \quad \text{and } \psi_3' := d(\psi, s(\psi)).$$

*With another $\varphi \in [S^p \times S^q; S^n]$, we get the product formula*

$$\varphi \bullet \psi = \varphi_1 \bullet_1 \psi_1, \varphi_2 \bullet_2 \psi_2, \varphi_3' \bullet_3 \psi_3' + d(s(\varphi) \bullet s(\psi), s(\varphi \bullet \psi)) + \Delta_{\overline{\mathfrak{t}(\psi)}},$$

*where according to (2.2) and (2.7), $\bullet_1, \bullet_2, \bullet_3$ are determined by $\tau_1 := \tau_{p,n}$, $\tau_2 := \tau_{q,n}$, $\tau_3 := \tau_{p+q,n}$* ([E2], (3.10)). ∎

If we put

$$\lambda(\overline{\mathfrak{t}(\psi)}) := \Delta_{\overline{\mathfrak{t}(\psi)}}, \quad h^*(\mathfrak{t}(\varphi), \mathfrak{t}(\psi)) := d(s(\varphi) \bullet s(\psi), s(\varphi \bullet \psi)) + \Delta_{\overline{\mathfrak{t}(\psi)}},$$

then (2.13) becomes in view of (2.6)

2.14. COROLLARY. *$([S^p \times S^q; S^n], \bullet)$ is isomorphic to a fan extension $(R, S, \lambda, h^*)$, where $R$ is group related to $\pi_p(S^n) \times \pi_q(S^n)$, $S = [S^{p+q}; S^n]$ is group related to $\pi_{p+q}(S^n)$. For $n \equiv 0(2)$ always holds $2h^*(\cdot, \cdot) = 0$.* ∎

The last assertion of (2.14) has been shown in ([E2], (3.12)).

## 3. Completeness of group topologies

For a group $X$ and complete group topologies $\mathfrak{T}_1$ and $\mathfrak{T}_2$, their group topological infimum $\mathfrak{T}_1 \wedge \mathfrak{T}_2$ is not necessarily complete [R]. Under certain circumstances we get sufficient conditions for the transfer of completeness to the infimum topology, e.g. if it is possible to induce on $X$ the structure of an $n$–symmmetric groupoid. This leads to an $n$–symmetric binary operation (by component–wise definition) on the set of self–mappings on $X$, which in addition carries the monoidal structure given by composition of maps. On this set we investigate a certain relation which corresponds to the notion of being relatively prime in principal ideal domains. By means of ([E3], Lemma 1) and suitably chosen pairs of relatively prime power mappings we get the desired sufficient conditions.

We describe how to proceed for the case $n = 2$. This procedure can be applied to the general case of an $n$–symmetric structure in an analogous but more complicated way and will be discussed elsewhere.

3.1 DEFINITION. *Let $X$ be a set and $*, \otimes : X \times X \to X$ binary operations.*

(a) *The triple $(X, *, \otimes)$ is called a circlet, if*

(i) $$(X), \otimes) \text{ is a monoid,}$$

(ii) $$x \otimes (y * z) = (x \otimes y) * (x \otimes z) \quad \forall\, x, y, z \in X.$$

*We denote the neutral element w.r.t. the associative multiplication $\otimes$ by $\mathfrak{e}$.*

(b) *Let $n \in \mathbb{N}$, and $(X, *, \otimes)$ be a circlet. If in addition, $(X, *)$ is an $n$–symmetric groupoid, we call $(X, *, \otimes)$ an $n$–circle.*

For an $n$–symmetric groupoid $(Y, \bullet)$ and $f, g \in X := Y^Y$ (the set of self–mappings on $Y$), we let $f \otimes g := g \circ f$. Then $(X, \bullet, \otimes)$ is an $n$–circle.

3.2. DEFINITION. *Let $n \in \mathbb{N}$. For a circlet $(X, *, \otimes)$, two elements $a, b \in X$ are called relatively prime, if there exist $x, y \in X$ s.th.*

$$(a \otimes x) * (b \otimes y) = \mathfrak{e}.$$

*For relatively prime elements $a, b \in X$ we write in short $a \dagger b$.*

Our next aim is to find pairs of relatively prime elements in a subset of self–mappings of an $n$–symmetric groupoid (arising from a group as described in (0.2)) which is defined as follows.

3.3. DEFINITION. *For $n \in \mathbb{N}$, let $(X, \bullet)$ be an $n$–symmetric groupoid, $e \in X$. By $\mathcal{P}(X, e)$ (or $\mathcal{P}$ in short) we denote the minimal subset of self–mappings on $X$, s.th.*

(i) $$id_X, \ const(e) \in \mathcal{P},$$

(ii) $$f, g \in \mathcal{P} \Rightarrow f \bullet g, f \otimes g \in \mathcal{P}.$$

*We call $\mathcal{P}$ the set of power mappings on $X$ w.r.t. $e$.*

It turns out that $\mathcal{P}(X, e) = P(\mathrm{id}_X, \bullet, \mathrm{const}(e))$ and $f(e) = e$ for all $f \in \mathcal{P}$.– From now on, we restrict ourselves to the consideration of (2–) symmetric groupoids. By $(\mathbb{Z}, \bullet, \cdot)$ we understand the 2–circle equipped with $\bullet$ as defined after (2.2) and the canonical ring multiplication in $\mathbb{Z}$.

3.4. PROPOSITION. *Let $X$ be a group with an involutory automorphism $\tau : X \to X$ (i.e. $\tau^2 = id_X$), and denote by $e$ the neutral element of $X$ w.r.t. the group multiplication. Then by (0.2), $(X, \bullet)$ is a symmetric groupoid, if we* *put* $x \bullet y := x\tau(x)^{-1}\tau(y)$ $(x, y \in X)$. *With the notation from above and the remark after* (0.3), $(\mathcal{P}, \bullet, \otimes)$ *is an epimorphic image of the* 2*–circle* $(\mathbb{Z}, \bullet, \cdot)$, *given by the assignment* $\mathbb{Z} \ni m \mapsto \varphi_{\langle m\rangle} \in \mathcal{P}$, *where* $\varphi_{\langle m\rangle}(x) := x^{\langle m\rangle}$, *formed w.r.t. e. Furthermore, for* $p, q \in \mathbb{Z}$ *we have*

$$p \dagger q \text{ (in } (\mathbb{Z}, +, \cdot)) \iff p \dagger q \text{ (in } (\mathbb{Z}, \bullet, \cdot)) \Rightarrow \varphi_{\langle p\rangle} \dagger \varphi_{\langle q\rangle} \text{ (in } (\mathcal{P}, \bullet, \otimes)) .$$

*Proof.* We first show that the two notions of being relatively prime are equivalent.

"$\Rightarrow$" Let $\alpha, \beta \in \mathbb{Z}$ and $p\alpha + q\beta = 1$. If $\alpha$ or $\beta$ is even, the implication is trivial. In case $\alpha \equiv 1(2) \equiv \beta$, exactly one of $p, q$, say $q$, is odd. Therefore, $\alpha + q$ is even, and by

$$p(\alpha + q) + q(\beta - p) = 1$$

we get

$$\left(p\,\frac{\alpha + q}{2}\right) \bullet \Big(q(p - \beta)\Big) = 1,$$

which shows that $p \dagger q$ in $(\mathbb{Z}, \bullet, \cdot)$.– "$\Leftarrow$" is trivial.

In order to show the second assertion, let $p, q, \alpha, \beta \in \mathbb{Z}$ and $(p\alpha) \bullet (q\beta) = 1$. By the remark after (0.3) we have

$$x = x^{(p\alpha)\bullet(q\beta)} = (x^p)^\alpha \bullet (x^q)^\beta \quad \forall\, x \in X,$$

which yields

$$\mathrm{id}_X = (\varphi_{\langle p\rangle} \otimes \varphi_{\langle \alpha\rangle}) \bullet (\varphi_{\langle q\rangle} \otimes \varphi_{\langle \beta\rangle}). \ \blacksquare$$

In order to make this chapter self–contained, we formulate ([E3], Lemma 1) according to our purposes.

3.5. LEMMA. *Let* $X$ *be an abelian group,* $\mathfrak{T}_1$ *and* $\mathfrak{T}_2$ *group topologies on* $X$, *and* $f : (X \times X, \mathfrak{T}_1 \times \mathfrak{T}_2) \to (X, \mathfrak{T}_1 \wedge \mathfrak{T}_2)$ *a continuous surjection. If there is a right inverse* $\imath : X \to X \times X$, *which is uniformly continuous w.r.t. the uniformities belonging to* $\mathfrak{T}_1 \wedge \mathfrak{T}_2$ *and* $\mathfrak{T}_1 \times \mathfrak{T}_2$, *the completeness of* $(X, \mathfrak{T}_1)$ *and* $(X, \mathfrak{T}_2)$ *implies the completeness of* $(X, \mathfrak{T}_1 \wedge \mathfrak{T}_2)$. $\blacksquare$

We note that any group topology on an abelian group uniquely determines a uniformity ([RD],(2.1)). For the uniformity belonging to the infimum of two group topologies, cf. ([RD],(2.3)). Now (3.4) yields in combination with (3.5).

3.6. THEOREM. *Let* $X$ *be an abelian group,* $\mathfrak{T}_1$, $\mathfrak{T}_2$ *group topologies on* $X$, *and* $\tau : X \to X$ *a* $\mathfrak{T}_i$*–continuous involutory automorphism* $(i = 1, 2)$. *For* $n \in \mathbb{Z}$, *by* $\mathfrak{T}^{(n)}$ *resp.* $\mathfrak{T}_{(n)}$ *we denote the initial resp. final group topology on* $X$ *w.r.t.* $\mathfrak{T}_1$ *and* $\varphi_{\langle n\rangle}$. *If* $\mathfrak{T}_1$, $\mathfrak{T}_2$ *are complete topologies, and there are* $p, q \in \mathbb{Z}$ *with* $p \dagger q$, *such that* $\mathfrak{T}^{(p)} \subseteq \mathfrak{T}_2 \subseteq \mathfrak{T}_{(q)}$, *then* $X$ *is* $\mathfrak{T}_1 \wedge \mathfrak{T}_2$*–complete.* $\blacksquare$

Proof. The binary operation $\bullet : X \times X \to X$ induced by $\tau$ makes $(X, \bullet)$ a symmetric groupoid and is $(\mathfrak{T}_1 \times \mathfrak{T}_2, \mathfrak{T}_1 \wedge \mathfrak{T}_2)$-continuous as a composition of continuous mappings, given by

$$(X,\mathfrak{T}_1) \times (X,\mathfrak{T}_2) \to (X,\mathfrak{T}_1) \times (X,\mathfrak{T}_2), \quad (x,y) \mapsto (x\tau(x)^{-1}, \tau(y)^{-1}),$$
$$(X,\mathfrak{T}_1) \times (X,\mathfrak{T}_2) \to (X,\mathfrak{T}_1 \wedge \mathfrak{T}_2), \quad (x,y) \mapsto xy^{-1},$$

the latter being continuous by ([RD],(5.24)(b)). For $p, q \in \mathbb{Z}$ as above define $\imath : X \to X \times X$, assigning $x \mapsto (\varphi_{\langle p\rangle}(x), \varphi_{\langle q\rangle}(x))$. Since $p \dagger q$, (3.4) implies $\varphi_{\langle p\rangle} \dagger \varphi_{\langle q\rangle}$, hence $\imath$ is a right inverse for $\bullet$. Since $\mathfrak{T}^{(p)} \subseteq \mathfrak{T}_2$, and $\varphi_{\langle p\rangle} : (X, \mathfrak{T}^{(p)}) \to (X, \mathfrak{T}_1)$ is continuous, $\varphi_{\langle p\rangle} : (X, \mathfrak{T}_2) \to (X, \mathfrak{T}_1)$ is continuous. A similar argument shows that $\varphi_{\langle q\rangle} : (X, \mathfrak{T}_1) \to (X, \mathfrak{T}_2)$ is continuous as well. In addition, both mappings are homomorphisms and continuous also w.r.t. $\mathfrak{T}_i$ $(i = 1, 2)$. Since continuous homomorphisms of abelian groups are uniformly continuous, we conclude that $\imath$ is uniformly continuous w.r.t. the uniformities determined by $\mathfrak{T}_1 \wedge \mathfrak{T}_2$ and $\mathfrak{T}_1 \times \mathfrak{T}_2$. Applying (3.5) completes the proof. ■

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