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ON REGULARITY OF HYPERGRAPH SEQUENCES

Dedicated to Professor Tadeusz Traczyk

1. Introduction

The motivation of this article is an attempt to show that the general language of the hypergraphs theory equipped with the asymptotic description might be useful in analyzing sequences of data (given for instance by repeating of an experiment etc.) or sequences of information systems. The introduced notions and the obtained results can be applied or interpreted in many domains, eg. theory of numbers, modal logic or information systems. In Section 5 some results for information systems are formulated.

Speaking more precisely, in the paper we generalize two notions connected with the asymptotic behaviour of a hypergraph: the regularity of the hypergraph and the independence of its edges. Then we prove that the corresponding asymptotic regularity is equivalent to average independence of its edges. The obtained characterization may profit in two directions. Assume that for an infinite sequence of hypergraphs H_n $(n \geq 0)$ the average degree of H_n tends to ∞ for $n \to \infty$. Then, if the edges of H_n are "almost independent" then H_n is asymptotically regular. Conversely, if asymptotically regular hypergraph H_n has "almost independent" edges then the averaged degree of H_n tends to infinity provided the condition (11) holds true with some $c, \delta \in \{0, 1\}$.

The idea of asymptotic description is quite typical in application to many fields of mathematics. The possible approach may arise in the following way: assume that $X = \{x_1 \prec x_2 \prec x_3 \prec \ldots\}$ is an infinite ordered set, E is a given set and $\nu : \mathsf{E} \to \mathcal{P}^+(X)$ any function satisfying the condition

$$(1) \qquad \forall x \in X \quad 1 \leq \text{ card } \{e \in \mathsf{E} : \ x \in \nu(e)\} < \infty.$$

Then we can associate with the triple (X, E, ν) the sequence of hypergraphs $H_n = (X_n, \mathsf{E}_n, \nu_n)$ such that the set X_n of vertices of H_n is equal to $\{x \in$

$X : x \prec x_n\}$, the set E_n of edges of H_n is equal to $\{e \in \mathsf{E} : \nu(e) \cap X_n \neq \emptyset\}$. The incidence function ν_n of H_n is given by $\nu_n = \nu|_{\mathsf{E}_n}$. Regarding (X, E, ν) as regular provided H_n is asymptotically regular we remark in Example 2 (see p. 660), that the triple (N_1, P, ν), where N_1 is the set of positive integers greater then unity (with the usual order) and $\nu(p) = \{n \in \mathsf{N}_1,\ n \equiv 0(\mod p)\}$ is regular. This regularity is equivalent to the important fact about the normal order of the well known number theoretical functions $\omega(m)$ and $\Omega(m)$ (see [2]).

2. Basic notions and definitions

X and E are always the finite sets. $\mathcal{P}(X)$ ($\mathcal{P}^+(X)$) is the family of all (nonempty) subsets of X, respectively.
N – is the set of all positive integers, P — the set of all primes.
$|A|$ — denotes the cardinality or absolute value according to whether A is a set or a number.
$\langle a \rangle$ — denotes the integral part of a.

Throughout the paper we use the convention that the statements "the sequence H_n of hypergraphs is regular" and "the hypergraph H_n is asymptotically regular" are equivalent. The statement "almost all $x \in X_n$ are in Y_n" (as $n \to \infty$) means that the following condition holds true:

$$\forall \varepsilon > 0\ \exists N = N(\varepsilon)\ \forall n > N \quad |\{x \in X_n : x \notin Y_n\}| \leq \varepsilon |X_n|.$$

By a hypergraph H we mean the triple (X, E, ν) where $\nu : \mathsf{E} \to \mathcal{P}^+(X)$ and $\nu(\mathsf{E}) = X$. The elements of X are called vertices while the images $\nu(e)$ with $e \in \mathsf{E}$ the edges of the hypergraph H. Any partial hypergraph is given by a restriction of ν. As an example we consider the star of x, $H(x) = H^1(x)$ defined as follows (cf. [1]):

$$H^1(x) = (X^1(x),\ \mathsf{E}^1_x,\ \nu^1_x)$$

where ν^1_x is a restriction of ν to the set E^1_x defined by the equality

$$\mathsf{E}^1_x = \{e \in \mathsf{E} :\ x \in \nu(e)\} \quad \text{while}$$
$$X^1(x) = \bigcup_{e \in \mathsf{E}^1_x} \nu(e) = \bigcup \nu(\mathsf{E}^1_x).$$

Inductively we define for $m \geq 2$ the m-th star of x,

$$H^m(x) = (X^m(x), \mathsf{E}^m_x, \nu^m_x)$$

where

$$\mathsf{E}^m_x = \{e \in \mathsf{E} :\ \nu(e) \cap X^{m-1}(x) \neq \emptyset\}$$
$$X^m_x = \bigcup \nu(\mathsf{E}^m_x), \qquad \nu^m_x = \nu|_{\mathsf{E}^m_x}.$$

The cardinality of E_x^m is called the m-th degree of x in the hypergraph $H = (X, \mathsf{E}, \nu)$ and we write

$$d_H^{(m)}(x) = |\mathsf{E}_x^m|. \tag{2}$$

The first degree of x in H is called simply the degree of x and is denoted by $d_H(x)$. The average degree of the hypergraph $H = (X, \mathsf{E}, \nu)$ is the number

$$d_H = |X|^{-1} \sum_{x \in X} d_H(x) \tag{3}$$

The hypergraph is regular (cf. [1]) iff

$$d_H(x) = d_H \qquad \text{for any} \quad x \in X. \tag{4}$$

Given any hypergraph $H = (X, \mathsf{E}, \nu)$ we define for any $\delta \in [0,1]$ and $m \in \mathbb{N}$ respectively the hypergraphs

$$H(\delta) = (X, \mathsf{E}(\delta), \nu(\delta)) \tag{5}$$

$$H^m = (X, \mathsf{E}, \nu^m) \tag{6}$$

where

$$\mathsf{E}(\delta) = \{e \in \mathsf{E} : \ |\nu(e)| \geq (1-\delta)|X|\}, \quad \nu(\delta) = \nu|_{\mathsf{E}(\delta)};$$

$$\nu^1 = \nu \text{ and inductively } \nu^m(e) = \bigcup_{\nu(e') \cap \nu^{m-1}(e) \neq \emptyset} \nu(e').$$

LEMMA 1. *Under the above notation we have*

$$d_{H^m}(x) = d_H^{(m)}(x) \quad \text{for any} \quad x \in X.$$

Applying Lemma 1 we obtain

LEMMA 2. *For any $m, k \in \mathbb{N}$ we have*

$$\sum_{x \in X} (d_H^{(m)}(x))^k = \sum_{(e_1, e_2, \ldots, e_k) \in \underbrace{\mathsf{E} \times \mathsf{E} \times \ldots \times \mathsf{E}}_{k-\text{tuples}}} |\nu^m(e_1) \cap \nu^m(e_2) \cap \ldots \cap \nu^m(e_k)|$$

3. Independence of hypergraph edges

Let us fix $n \in \mathbb{N}$ and consider the hypergraph $H_n = (X_n, \mathsf{E}_n, \nu_n)$. Having endowed the discrete measure on X_n one regards two distinct edges $E = \nu_n(e)$, $E' = \nu_n(e')$ as independent iff

$$|E \cap E'| - |X_n|^{-1}|E||E'| = 0. \tag{7}$$

In many examples the above equality is valid only approximately (see e.g. Ex. 1, Ex. 2). We shall therefore replace the condition (7) by the average one (over the edges $E \neq E'$) replacing the corresponding equality by the asymptotic equality as $n \to \infty$. Hence we introduce the following

DEFINITION. The sequence of hypergraphs H_n given above has almost independent edges iff

$$\forall\varepsilon > 0 \ \exists N = N(\varepsilon) \ \forall n > N$$

$$(8) \quad \Big| \sum_{\substack{(e,e')\in \mathbf{E}_n\times\mathbf{E}_n \\ E\neq E'}} (|E\cap E'| - |X_n|^{-1}|E||E'|)\Big| < \varepsilon |X_n|^{-1}\Big(\sum_{e\in\mathbf{E}_n} |E|\Big)^2.$$

Remark 1. One should keep in mind that the edges $E = \nu_n(e)$, $E' = \nu_n(e')$ depend on n however this dependence will not be stressed in the sequel.

Assuming the condition

$$(9) \qquad \forall\varepsilon > 0 \ \exists N = N(\varepsilon) \ \forall n > N \ \sum_{e\in\mathbf{E}_n} |E| > \frac{|X_n|}{\varepsilon}$$

one can extend the summation in the left-hand side of inequality (8) to all pairs $(e, e') \in \mathbf{E}_n \times \mathbf{E}_n$ obtaining the equivalent definition

$$(10) \quad \forall\varepsilon > 0 \ \exists N = N(\varepsilon) \ \forall n > N,$$

$$\Big| \sum_{(e,e')\in\mathbf{E}_n\times\mathbf{E}_n} |E\cap E'| - |X_n|^{-1}\Big(\sum_{e\in\mathbf{E}_n}|E|\Big)^2 \Big| < \varepsilon|X_n|^{-1}\Big(\sum_{e\in\mathbf{E}_n}|E|\Big)^2.$$

On the other hand we will prove the following

LEMMA 3. *Assume that for the sequence of hypergraphs* $H_n=(X_n, \mathbf{E}_n, \nu_n)$ *the conditions* (8) *and* (10) *are satisfied. If there exist the constants* $c, \delta \in (0,1)$ *independent of* n *such that*

$$(11) \qquad \sum_{e\in\mathbf{E}_n(\delta)} |E| \le c\delta \sum_{e\in\mathbf{E}_n} |E|$$

then we have that

$$(12) \quad \forall\varepsilon > 0 \ \exists N = N(\varepsilon) \ \forall n > N \quad \text{it holds} \quad \sum_{e\in\mathbf{E}_n} |E| > |X_n|\frac{\delta(1-c)}{\varepsilon}.$$

Proof. Subtracting the left-hand sides of inequalities (8) and (10) we obtain

$$\sum_{e\in\mathbf{E}_n} \Big(|E| - |X_n|^{-1}|E|^2\Big) < 2\varepsilon|X_n|^{-1}\Big(\sum_{e\in\mathbf{E}_n}|E|\Big)^2$$

hence

$$(13) \qquad |X_n|^{-1}\sum_{e\in\mathbf{E}_n}|E|^2 > \sum_{e\in\mathbf{E}_n}|E|\Big(1 - 2\varepsilon|X_n|^{-1}\sum_{e\in\mathbf{E}_n}|E|\Big).$$

The left-hand side above is

$$\leq |X_n|^{-1}\Big(\sum_{e\in \mathbf{E}_n(\delta)} |E||X_n| + (1-\delta)\sum_{e\in \mathbf{E}_n\setminus \mathbf{E}_n(\delta)} |E||X_n|\Big)$$

$$\leq \sum_{e\in \mathbf{E}_n(\delta)} |E| + (1-\delta)\sum_{e\in \mathbf{E}_n} |E|.$$

By inequality (13) we obtain

$$\sum_{e\in \mathbf{E}_n(\delta)} |E| + (1-\delta)\sum_{e\in \mathbf{E}_n} |E| > \sum_{e\in \mathbf{E}_n} |E|\Big(1 - 2\varepsilon|X_n|^{-1}\sum_{e\in \mathbf{E}_n} |E|\Big).$$

hence

$$\sum_{e\in \mathbf{E}_n(\delta)} |E| > \sum_{e\in \mathbf{E}_n} |E|\Big(\delta - 2\varepsilon|X_n|^{-1}\Big)\sum_{e\in \mathbf{E}_n} |E|\Big).$$

In view of the assumption (11) we obtain that

$$c\delta\sum_{e\in \mathbf{E}_n} |E| > \sum_{e\in \mathbf{E}_n} |E|\Big(\delta - 2\varepsilon|X_n|^{-1}\sum_{e\in \mathbf{E}_n} |E|\Big)$$

hence dividing both sides by $\sum_{e\in \mathbf{E}_n} |E|$ we obtain

$$\sum_{e\in \mathbf{E}_n} |E| > |X_n|\frac{\delta(1-c)}{2\varepsilon}$$

which proves (12) (if $\varepsilon \to \varepsilon/2$).

4. Asymptotic regularity of hypergraph

Let $H_n = (X_n, \mathbf{E}_n, \nu_n)$ be any sequence of hypergraphs and $d_n(x)$, d_n, $(d_n^\delta(x), d_n^\delta)$ denote the degree of x, the average degree in H_n, $(H_n(\delta))$ respectively. We shall say that the sequence H_n is regular iff

$$(14)\qquad \forall\varepsilon > 0\ \ \exists N = N(\varepsilon)\ \ \forall n > N\quad \sum_{x\in X_n} (d_n(x) - d_n)^2 < \varepsilon(d_n)^2|X_n|.$$

Remark 2. The above definition coincides with the conventional regularity (4) when H_n is a constant sequence.

As a simple consequence of the above definition we obtain

COROLLARY 1. *If the hypergraph H_n is asymptotically regular then*

$$\forall\,\varepsilon > 0\ \ \exists N = N(\varepsilon)\ \ \forall n > N\ \ \Big|\{x \in X_n :\ |d_n(x) - d_n)| > \varepsilon d_n\}\Big| < \varepsilon|X_n|.$$

In view of Lemma 2 we obtain the following characterization of asymptotic regularity

THEOREM 1. *The hypergraph H_n is asymptotically regular iff*
$\forall\varepsilon > 0\ \exists N = N(\varepsilon)\ \forall n > N$

$$\Big|\sum_{(e,e')\in \mathbf{E}_n\times\mathbf{E}_n} |E\cap E'| - |X_n|^{-1}\Big(\sum_{e\in\mathbf{E}_n}|E|\Big)^2\Big| << \varepsilon|X_n|^{-1}\Big(\sum_{e\in\mathbf{E}_n}|E|\Big)^2.$$

Proof. Squaring out the left-hand side of inequality (14) we obtain

$$0 \le \sum_{x\in X} d_n^2(x) - |X_n|(d_n)^2 < \varepsilon|X_n|(d_n)^2.$$

Now applying Lemma 2 for $m = 1$ and $k = 1,2$ we obtain the desired equivalence.

COROLLARY 2. *If the sequence H_n has almost independent edges and the average degree d_n tends to infinity as n approaches infinity then H_n is asymptotically regular.*

Proof. It is sufficient to apply (9), (10), and observe that the assertion $d_n \to \infty$ is equivalent to the condition (9).

COROLLARY 3. *Assume that there exist the constants $\delta, c \in (0,1)$ independent of n such that $d_n^{(\delta)} \le c\delta\cdot d_n$ as $n \to \infty$. Then if the hypergraph H_n is asymptotically regular and the edges of H_n are almost independent then $d_n \to \infty$ as n approaches infinity.*

Proof. By Lemma 2 and (3) we have that

$$d_n^{(\delta)} = |X_n|^{-1}\sum_{x\in X_n} d_n^{(\delta)}(x) = |X_n|^{-1}\sum_{e\in\mathbf{E}_n(\delta)}|E|$$

and

$$d_n = |X_n|^{-1}\sum_{e\in\mathbf{E}_n}|E|,$$

hence by Lemma 3 we obtain the claim.

5. Applications to information systems

In this section we show the representation of a hypergraph by certain information systems and we discuss consequences of this representation.

By a nondeterministic information system we mean the 4-tuple $S = (X, A, V, F)$ where X, A, V are given finite sets and $F:\ X \times A \to P^+(V)$ (see [4], [5]). Traditionally X is called the universe (or the set of objects), A is called the set of attributes and V is called the set of values of the information function F. We will restrict our consideration only to the information systems satisfying the condition

(15) $$\forall v\in V\ \ \forall x\in X\ \ \exists a\in A:\ v\in F(x,a).$$

Every hypergraph is a particular information system. Namely letting $V = \{0,1\}$ we can identify the hypergraph $H = (X, \mathsf{E}, \nu)$ with the information system $S = (X, \mathsf{E}, V, F)$ such that $F: X \times \mathsf{E} \to \mathcal{P}_1(V)$ satisfies

$$F(x,E) = \begin{cases} \{1\} & \text{if } x \in \nu(E) \\ \{0\} & \text{if } x \notin \nu(E). \end{cases}$$

With any information system satisfying (15) we associate the hypergraph

$$H = (V, X \times A, F)$$

the partial hypergraph

$$H^x = (V, \{x\} \times A, F)$$

and the hypergraph H^v defined as follows

$$H^v = (X, A, X^v), \qquad X^v(a) = \{x \in X : v \in F(x,a)\}.$$

We will prove the following

THEOREM 2. *Assume that the hypergraphs $H_n = (V_n, X_n \times A_n, F_n)$ and $H_n^v = (X_n, A_n, X_n^v)$ are asymptotically regular for any $v \in V_n$ (as $n \to \infty$). Then for almost all $x \in X_n$ the partial hypergraph $H_n^x = (V_n, \{x\} \times A_n, F_n)$ $(x \in X_n)$ is also asymptotically regular (as $n \to \infty$).*

Proof. Let $d_n^v(x)$, (d_n^v) be the degree of x (average degree) in H_n^v, $\tilde{d}_n(v)$, $(\tilde{d}_n)$ the degree of v (average degree) in H_n and $d_n^x(v)$ the degree of v in H_n^x. We set $d_n = |V_n|^{-1} \sum_{v \in V_n} d_n^v$.

Directly from the definition of the hypergraphs H_n^x and H_n^v we have that $d_n^x(v) = d_n^v(x)$. Therefore

$$d_n^v = |X_n|^{-1} \sum_{x \in X_n} d_n^v(x) = |X_n|^{-1} \sum_{x \in X_n} d_n^x(v) = |X_n|^{-1} \tilde{d}_n(v). \tag{16}$$

By summation over $v \in V_n$ we obtain that

$$d_n = |X_n|^{-1} \tilde{d}_n. \tag{17}$$

Now we have

$$\sum_v \sum_x (d_n^v(x) - d_n)^2 \le 2 \sum_v \sum_x (d_n^v(x) - d_n^v)^2 + 2 \sum_x \sum_v (d_n^v - d_n)^2$$

where the summation is taken over $x \in X_n$ and $v \in V_n$ The asymptotic regularity of $H_n^v(H_n)$ provides the estimate for the first (second) sum respectively.

Hence in view of (16), (17) we obtain that

$$\forall \varepsilon > 0 \; \exists N = N(\varepsilon) \; \forall n > N \qquad \text{it holds}$$

$$\sum_v \sum_x (d_n^v(x) - d_n)^2 \le 2 \sum_v \varepsilon |X_n| (d_n^v)^2 + 2 \sum_x \varepsilon |V_n| (d_n)^2.$$

Writing

$$(d_n^v)^2 = (d_n + d_n^v - d_n)^2 = d_n^2 + (d_n^v - d_n)^2 + 2d_n(d_n^v - d_n)$$

we split the sum over v above into the three corresponding sums (the third one being equal to zero), obtaining the contribution not exceeding the value

$$2\varepsilon|X_n|(|V_n|d_n^2 + \varepsilon|V_n|d_n^2).$$

Hence we obtain (for any $\varepsilon > 0$ and all $n > N(\varepsilon)$)

$$\sum_v \sum_x (d_n^v(x) - d_n)^2 \le 2\varepsilon|V_n||X_n|d_n^2(1 + 2\varepsilon). \tag{18}$$

To prove that for almost any $x \in X_n$ the hypergraph H_n^x is asymptotically regular we have to show that the set of exceptions $E_{xc}(X_n)$ is of smaller order of magnitude. Assume on the contrary that there exists some absolute constant $c > 0$ and the subsequence $\{n_k,\ k = 1, 2, \ldots\}$ of natural numbers such that for any $k \in \mathbb{N}$ it holds

$$|E_{xc}(X_{n_k})| > c|X_{n_k}|$$

and for any $x \in E_{xc}(X_{n_k})$ the regularity condition is not valid i.e.

$$\exists \varepsilon' > 0 \ \ \forall K \ \ \exists k > K \ \ \sum_v (d_{n_k}^v(x) - d_{n_k})^2 > \varepsilon'|V_{n_k}|(d_{n_k})^2.$$

Summing the above inequality over $x \in E_{xc}(X_{n_k})$ we obtain that the left-hand side of (18) is at least as large as

$$\sum_{x \in E_{xc}(X_{n_k})} \sum_v (d_{n_k}^v(x) - d_{n_k})^2 > \varepsilon' c|X_{n_k}||V_{n_k}|(d_{n_k})^2$$

which is contrary to the inequality (18) (when $\varepsilon < \min(1, \frac{\varepsilon' c}{6})$ and K is sufficiently large). This completes the proof of Theorem 2.

COROLLARY 4. *Keeping the above notation let us assume that the hypergraphs H_n and $H_n^v (v \in V_n)$ have almost independent edges and for each $v \in V_n$ the average degree d_n^v tends to infinity as n approaches infinity. Then the partial hypergraphs H_n^x are asymptotically regular for almost all $x \in X_n$.*

Proof. By Corollary 2 we have that the hypergraphs H_n^v, $(v \in V_n)$ are asymptotically regular. Moreover since $d_n^v \to \infty$ as n tends to infinity we obtain that the average degree $d_n = |V_n|^{-1} \sum_v d_n^v$ satisfies the same condition. Hence in view of Corollary 2 and (17) the hypergraph H_n is asymptotically regular. Now the thesis comes from Theorem 2.

6. Examples

EXAMPLE 1. Let $Y = \{x_1 \prec x_2 \prec x_3 \prec \dots\}$ be an infinite ordered set

$$\mathcal{P}_3^+(Y) = \{A \in \mathcal{P}^+(Y) : \ |A| \ \text{ is divisible by } 3\}.$$

We consider the triple $(X, \mathsf{E}, \nu) = (\mathcal{P}_3^+(Y), \mathsf{N}, \nu)$ with $\nu(i) = \{A \in \mathcal{P}_3(Y) : x_i \in A\}$.

Then the condition (1) is satisfied. We define the hypergraph $H_n = (X_n, \mathsf{E}_n, \nu_n)$ as follows

$$X_n = \{A \in P_3^+(Y) : \ A \subset \{x_1, x_2, \dots, x_n\}\}$$

$$\mathsf{E}_n = \{i \in \mathsf{N} : \ \nu(i) \cap X_n \neq \emptyset\} = \{1, 2, \dots, n\}$$

$$\nu_n = \nu|_{\mathsf{E}_n}$$

We shall need the following summation formulas which may be proved by induction (see e.g. [3])

$$\sum_{k=0}^{\langle \frac{n}{3} \rangle} \binom{n}{3k} = \frac{1}{3}\Big(2^n + 2\cos\frac{n\pi}{3}\Big)$$

$$\sum_{k=0}^{\langle \frac{n}{3} \rangle} \binom{n}{3k+1} = \frac{1}{3}\Big(2^n + 2\cos\frac{(n-2)\pi}{3}\Big)$$

$$\sum_{k=0}^{\langle \frac{n}{3} \rangle} \binom{n}{3k+2} = \frac{1}{3}\Big(2^n + 2\cos\frac{(n-4)\pi}{3}\Big)$$

hence letting (for simplicity) n be divisible by 6 we obtain

$$|X_n| = \frac{1}{3}(2^n + 2) - 1$$

and denoting $E^{(i)} = \nu_n(i)$ we get

$$\forall\ 1 \le i \le n \quad |E^{(i)}| = \sum_{k=0}^{\langle \frac{n-1}{3} \rangle} \binom{n-1}{3k+2} = \frac{1}{3}(2^{n-1} + 1)$$

$$\forall\ 1 \le i < j \le n \quad |E^{(i)} \cap E^{(j)}| = \sum_{k=0}^{\langle \frac{n-2}{3} \rangle} \binom{n-2}{3k+1} = \frac{1}{3}(2^{n-2} - 1)$$

hence

$$\Big||E^{(i)} \cap E^{(j)}| - |X_n|^{-1}|E^{(i)}||E^{(j)}|\Big| = \Big| - \frac{2^{n+1} + 2^{n-2}}{3(2^n - 1)}\Big| \le \frac{4}{3}.$$

Summing it over $i, j \in \{1, \ldots, n\}$, $i \neq j$, we see that the edges of H_n are almost independent. Moreover since

$$\sum_{i=1}^{n} |E^{(i)}| = \frac{1}{3} n(2^{n-1} + 1) \sim \frac{n}{2} |X_n| \tag{19}$$

we conclude in view of (9), (10) and Theorem 1 that H_n is asymptotically regular (the case $6 \nmid n$ is to be verified in the same manner).

Observing that

$$d_n(A) = |\{i \leq n : \ A \in E^{(i)}\}| = |A|$$

we obtain in view of (19) and Corollary 1 the following conclusion: for any $\varepsilon > 0$ the number of subsets $A \in X_n$ having more than $(1+\varepsilon) \cdot \frac{n}{2}$ elements is less than $\varepsilon |X_n|$ as n tends to infinity.

EXAMPLE 2. We consider the triple

$$(X, \mathsf{E}, \nu) = (\mathsf{N}_1, P, \nu)$$

with $\nu(p) = \{n \in \mathsf{N}_1 : \ p \text{ divides } n\}$. Therefore the hypergraph $H_n = (X_n, \mathsf{E}_n, \nu_n)$ is defined as follows (see section 1)

$$X_n = \{k \in \mathsf{N} : \ 1 < k \leq n\}$$

$$\mathsf{E}_n = \{p \in P : \ p \leq n\}$$

$$\nu_n = \nu|_{\mathsf{E}_n}.$$

Therefore letting $\nu_n(p) = E_p$ we obtain for $p \neq q$

$$|E_p \cap E_q| - |X_n|^{-1} |E_p||E_q| = \langle \frac{n}{pq} \rangle - (n-1)^{-1} \langle \frac{n}{p} \rangle \langle \frac{n}{q} \rangle.$$

If $pq \leq n$ we obtain by an easy calculation that the above difference is ≤ 5 in absolute value. If $pq \geq n$, $(p \neq q)$ then $|E_p \cap E_q| = 0$ hence the above is

$$\leq (n-1)^{-1} \langle \frac{n}{p} \rangle \langle \frac{n}{q} \rangle.$$

Therefore taking the summation over $P \ni p, q \leq n$, $(p \neq q)$ and applying the asymptotic formula (see e.g. [2])

$$\sum_{\substack{p \in P \\ p \leq n}} p^{-1} \sim \log \log n \qquad (\text{as} \quad n \to \infty) \tag{20}$$

we obtain

$$\Big| \sum_{\substack{P \ni p, q \leq n \\ p \neq q}} (|E_p \cap E_q| - |X_n|^{-1} |E_p||E_q|) \Big| \leq \sum\nolimits_1 + \sum\nolimits_2$$

where $\sum_1$ ($\sum_2$) are the subsums restricted by the condition $pq \leq n$ ($n < pq \leq n^2$) respectively. Hence

$$\sum\nolimits_1 \leq 5n \quad \text{and}$$

$$\begin{aligned}\sum\nolimits_2 &\leq \sum_{\substack{p,q\leq n\\ pq\geq n}} (n-1)^{-1}\langle\frac{n}{p}\rangle\langle\frac{n}{q}\rangle \leq 2n \sum_{\substack{p,q\leq n\\ pq\geq n}} (pq)^{-1} \leq \\ &\leq 2n\Big\{\sum_{2\leq p\leq n^{\frac{1}{2}}} \quad \sum_{n^{\frac{1}{2}}\leq q\leq n} (pq)^{-1} + \sum_{n^{1/2}\leq p\leq n}\sum_{1\leq q\leq n} (pq)^{-1}\Big\} \leq \\ &\leq (5\log 2) n \log\log n\end{aligned}$$

by (20) (if n is sufficiently large).

Since

$$\sum_{P\ni p\leq n} |E_p| = \sum_{P\ni p\leq n} \langle\frac{n}{p}\rangle = \sum_{P\ni p\leq n} \Big(\frac{n}{p}+\theta\Big)$$

with some θ : $|\theta| \leq 1$, we obtain the asymptotic equality

$$\sum_{P\ni p\leq n} |E_p| \sim n\log\log n \sim |X_n|\log\log n.$$

Therefore in view of (9), (10) and Theorem 1 we conclude that $H_n = (X_n, \mathsf{E}_n, \nu_n)$ is asymptotically regular. Observing that for $k \leq n$ we have

$$d_n(k) = \sum_{\substack{P\ni p|k\\ p\leq n}} 1 = \sum_{\substack{p\in P\\ p|k}} 1$$

we obtain in view of Corollary 1 the following conclusion: for any $\varepsilon > 0$ the number of positive integers $\leq n$ having more than $(1+\varepsilon)\log\log n$ prime divisors is less than εn as n tends to infinity ([cf.[2]).

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