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A NOTE ON CATEGORIES OF INFORMATION SYSTEMS

Dedicated to Professor Tadeusz Traczyk

1. Introduction

Several notions motivated by the problem of classifying objects according to their values of attributes or features were introduced and examined. We mention for example the logical kit of Semadeni [2], the information system of Pawlak, the context of Wille [3] and the probably most commonly known and applied relational database model of Codd [7]. From some points of view the above notions are equivalent or inter-translatable (see e.g. Wiweger [4], where the relation among logical kits, information systems and contexts is explained); of course there are important differences among them. (In fact small differences in the beginning can give unequal results at the end). In every of the models mentioned we have other classes of questions considered and areas of applications also do not coincide. Category theory has proved to be useful in so many areas that it should also be possible to apply it in the field of information systems, logical kits, contexts etc. In fact there exist results for logical kits and information systems using category theory, see e.g. Semadeni [2], Wiweger [4]; other results of this kind connected to a similar notion, rough sets, can be found in Biegańska [6], Obtulowicz [5] and Banerjee, Chakraborty [9].

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Our aim here is the following: we would like to define some categories related to information systems (or some subcategories of known one's), which can profit in better understanding of structuress associated to these systems. In particular we hope to obtain new insight into indiscernibility of objects, dependence of attributes and problems related to reducing the number of attributes. This note only gives introductory considerations. We suggest that it is worthwhile to develop a theory for the categories introduced in this paper and to find applications for the results obtained. In particular we hope that the tool of category theory can help to analyse situations in which we deal with information systems with incomplete, damaged or lost information. We shall be interested in questions similar to the following: assume we changed some of the values of a matrix (or information system), how does that fact influence the indiscernibility of objects and the dependance of attributes, or determinant of the matrix?

2. Basic definitions

The notion of information system was introduced by Pawlak in [1].

An *information system* is a quadruple $(U, A, (V_a)_{a\in A}, f)$ where U is a set of objects, A stands for a set of attributes, V_a is a set of values for an attribute a, and $f : U \times A \to \bigcup_{a\in A} V_a$ is a function such that $f(x,a) \in V_a$ for any $x \in U$, $a \in A$. The function f is called the information function. Shortly the system will be denoted by (U, A, V, f) where $V = \bigcup_{a\in A} V_a$.

For every set of attributes $B \subseteq A$ an *indiscernibility relation* $\mathrm{Ind}(B) \subseteq U^2$ is defined in the following way: For every $x, y \in U \, \mathrm{Ind}(B)xy$ iff $\forall a \in B f(x,a) = f(y,a)$.

We say that the set of attributes B *depends* on the set of attributes C (denoted by $C \to B$), if $\mathrm{Ind}(C) \subseteq \mathrm{Ind}(B)$.

We also recall that a set of attributes $B \subseteq C$ is a *reduct* of C if $\mathrm{Ind}(B) = \mathrm{Ind}(C)$ and the set B is minimal with respect to inclusion.

A system $(U, A, V, F)_{\perp_0}$ with one distinguished element $\perp_0 \in A$ is called a *pointed system* or, more precisely, the attribute pointed system. We could define simiarly the object pointed system.

Let us observe that a monoid can be seen as a special kind of an information system: $(M, M, M, *)$ such that $+ : M \times M \to M$ and $m * (a * b) = (m * a) * b$.

3. Categories of information systems

We start with the basic definition of our category: We consider as objects information systems and (structure preserving) morphisms between them. In particular we will look at (sub)categories of information systems (U, A, V, f) such that $U = \{x_1, \ldots, x_n\}$, $A = \{a_1, \ldots, a_m\}$ and V are fixed. Now given two information systems $S_1 = (U, A, V_1, f_1)$ and $S_2 = (U, A, V_2, f_2)$, we say that m_σ is a morphism from S_1 to S_2 iff there exists a permutation $\sigma : \{1, \ldots, m\} \stackrel{1-1}{\to} \{1, \ldots, m\}$ such that

$$\forall i \forall j [f_1(x_i, a_j) = f_2(x_i, a_{\sigma(j)})] .$$

Furthermore of course m_σ should map U on U, A on A and V_1 on V_2[1].

Sometimes we shall say that m_σ is a permutation of columns of the information system and we shall write

$$m_\sigma : \{a_1^c, \ldots, a_m^c\} \to \{a_1^c, \ldots, a_m^c\} .$$

Of course σ^{-1} gives a dual morphism.

As a subclass we can also consider morphisms determined by a subset $A_0 \subseteq A$ and permutation morphisms $m_\sigma : A \to A$ such that $m_\sigma | A_0 = \mathrm{Id}_{A_0}$ where Id_{A_0} is the identity on A_0.

In a similar way a morphism determined by a permutation of objects is defined.

We call categories with morphisms of the first kind *natural*; they can also be named the *permuting attributes category*.

We shall say that a morphism m_σ preserves pointed elements in the pointed information systems $(U, A, V_0, f_0)_{\perp_0}$ and $(U, A, V_1, f_1)_{\perp_1}$ if $m_\sigma(\perp_0) = \perp_1$.

We shall call the category of pointed information systems (with pointed morphisms) the *natural pointed category*. To be more precise, we have attribute pointed and object pointed categories.

Let us finally observe that the natural category is just the category of one object set U with morphisms being permutations of U. In symbols: $\mathrm{Cat}((U, A, V, f), \mathrm{perm}(U)) \approx \mathrm{Cat}((U, A', V', f'), \mathrm{perm}(U)) \approx (\mathrm{perm}(U), \circ)$. The meaning of "$\approx$" is intuitively clear.

Now let us assume that the set of objects U is fixed and the sets of attributes are arbitrary. Therefore objects in this category are information systems $(U, A, V, f), (U, A', V', f'), (U, A'', V'', f''), \ldots$ etc. We define morphisms here in the following way

$$(U, A, V, f) \stackrel{m_i}{\to} (U, A', V', f') \quad \text{iff } \mathrm{Ind}(A) = \mathrm{Ind}(A') .$$

This category shall be called the *indiscernibility category*.

Having the same objects we can add some more morphisms: $(U, A, V, f) \stackrel{m_d}{\to} (U, A', V', f')$ iff the set of attributes A depends on the set A' i.e. $\mathrm{Ind}(A) \subseteq \mathrm{Ind}(A')$. The category with these morphisms shall be called *dependency category*.

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[1] In this special case $V_1 = V_2$ would be sufficient.

Let us observe that we can't define a category with morphisms determined by reducts in an analogous way, because in general we shall not have identity morphisms. We may however construct a subcategory, with objects having the property of *independence*, that is for objects (U, A, V, f) such that it is true that for all $B \subseteq A$ $B \neq A \rightarrow \mathrm{Ind}(B) \neq \mathrm{Ind}(A)$. On the other hand it is possible to consider functions $m_r : A \rightarrow A'$ such that $A' \subseteq A$ is a reduct of A and we may call m_r a semi-morphism. Here we not always have identity morphisms and there are no nontrivial compositions. One more word on the notation. Assuming that $U = \{x_1, \ldots, x_n\}$, $A = \{a_1, \ldots, a_m\}$, $V = Z \cup Q \cup R$ we denote by $Z(m)$, $Q(m)$ and $R(m)$ respectively the category with objects of the form (U, A, Z, z), (U, A, Q, q), (U, A, R, r) and with morphisms determined by permutations of attributes (or, more precisely, by permutations of columns named by attributes).

By $M(m)$ we denote the union $Z(m) \cup Q(m) \cup R(m)$. In summary, we have defined the following categories of information systems:

The category **IS** of objects being information systems

$$\mathrm{Obj}\,\mathbf{IS} = \{(U, A, V, f) | U, A, V \in \underset{\mathrm{Fin}}{\underline{\mathrm{Set}}} \,\&\, f : U \times A \rightarrow V\}$$

and morphisms

$$\mathrm{Mor}\,\mathbf{IS} = M_\sigma \cup M_i \cup M_d \cup M_r \text{ and their compositions},$$

where elements of M_σ are morphisms determined by permutations of attributes, elements in M_i are determined by $\mathrm{Ind}(A)$ relations, morphisms in M_d and M_r are defined using respectively dependency relations and reducts. By $\mathbf{IS}_\sigma$, $\mathbf{IS}_i$, $\mathbf{IS}_d$, $\mathbf{IS}_r$ we denote the corresponding subcategories. Note that by definition of a category, we have to add identity morphisms to the reduct morphisms to obtain $\mathbf{IS}_r$.

Now let us describe the structure of Mor **IS**. We need one more definition: a morphism $m_d : (U, A, V, f) \rightarrow (U, A', V', f)$ is *strict* if $\mathrm{Ind}(A) \subset \mathrm{Ind}(A')$.

CONVENTION: Sometimes instead of $f|B$ (i.e. the restriction of f to the set B) we shall write only f.

NOTATION: if $A = \{a_1, \ldots, a_n\}$ then $A_1 = \{a_1\}, \ldots, A_i = \{a_1, \ldots, a_i\}$ and $IS_1 = (U, A_1, V, f), \ldots, IS_i = (U, A_i, V, f)$.

PROPOSITION 3.1. 1. $M_\sigma \subset M_i \subset M_d$
2. $M_r \subset M_i$
3. $M_r \neq M_\sigma$
4. $M_r \cap M_\sigma \neq \emptyset$.

PROPOSITION 3.2. 1. *If for* $i = 1, \ldots, n-1$ $\mathrm{IS}_i \overset{m_d}{\rightarrow} \mathrm{IS}_{i+1}$ *are strict dependent morphisms then* $\mathrm{IS}_i \overset{m_r}{\rightarrow} \mathrm{IS}$.

2. *Let* $\mathrm{IS} = (U, A, V, f)$ *where* $A = \{a_1, \ldots, a_n\}$ *and assume that a morphism* $m_d : \mathrm{IS}_{n-1} \to \mathrm{IS}$ *is strict and there exists* $\{a_{i_1}, \ldots, a_{i_k}\} \neq \{a_1, \ldots, a_{n-1}\}$ *such that* $(U, \{a_{i_1}, \ldots, a_{i_k}, a_n\}, V, f) \overset{m_i}{\to} \mathrm{IS}$ *then there exists* $\{a_{j_1}, \ldots, a_{j_l}\} \subseteq \{a_{i_k}, \ldots, a_{I_k}\}$ *such that* $(U, \{a_{j_1}, \ldots, a_{j_l}, a_n\}, V, f) \overset{m_r}{\to} \mathrm{IS}$.

Proof. Straightforward. ∎

At this point, having defined the basic category we can start developing the theory. It is not difficult to define the product and the comma category. Some simple functors like a forgetful or an inclusion functor are also easy to obtain. At present we consider natural transformations.

4. Natural transformations and information systems

It is often mentioned (e.g. in Lambek and Scott [8]) that the concept of natural transformations is the key concept that necessitated the invention of category.

Many objects of interest to mathematicians may be viewed as functors from small categories to the category of **Sets**. When those functors are seen as objects of a category, the morphisms between two objects are precisely the natural transformations. In order to understand the structure of a mathematical objects like an information system, it is useful to see how it can be described as a functor. Moreover if we also consider only special morphisms between the structures (like the ones proposed in the previous section), we can express that information in the definition of the functor that "describes it", and (possibly) discover some properties of the structure.

Let us consider the following very small and abstract category **PreSIS** of figure 1 and consider a functor from **PreSIS** to $\mathbf{Set}_{\mathrm{FinTot}}$. Such a functor F maps U to some finite set $F(U)$ which we will view as a set of objects. Similarly F maps A to a finite set $F(A)$ of attributes, V to a finite set $F(V)$ of values and I to a finite set $F(I)$ of informations[2]. Furthermore F maps

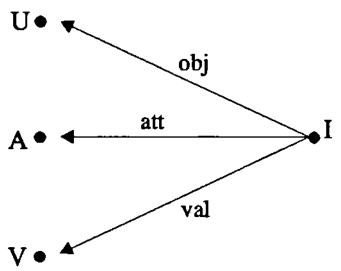

Figure 1: The very small category **PreSIS**

[2] If we view an information system as a matrix determined by information function $f(u, a)$ we can view the set I of information as entries in this matrix.

the arrows obj,val,attr to respectively the total mappings $F(\text{obj}) : F(I) \to F(U)$, $F(val) : F(I) \to F(V)$, $F(attr) : F(I) \to F(A)$. Such a functor F defines a what we call *incomplete information system* (U_F, A_F, V_F, f_F) as follows

$$U_F = F(U), A_F = F(A), V_F = F(V)$$

and $f_F : U_F \times A_F \to V_F$ is such that $f_F(x,a) = v$ iff $\exists i \in F(I)[F(\text{obj})(i) = x \wedge F(\text{val})(i) = v]$.

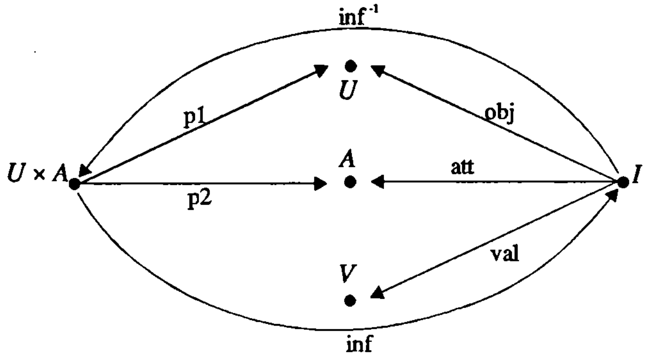

Figure 2: The very small category **SIS**

An incomplete information system is a system for which in general the information function is partial, i.e. for some objects some (possibly all) values of their attributes can be unknown. It is easy too see that for the information systems we defined above the information function is in general partial.

In order to force the information system to be complete (i.e. for all $u \in U_F$, $A \in A_F$ $f_F(u,a)$ is defined) we have to force that $F(I)$ coinsides with $F(U) \times F(A)$ and that the mappings F_{obj} and F_{attr} coincide with respectively the first and second projection on elements of $F(U) \times F(A)$.

Consider the very small category **SIS** of figure 2, where we have added to the category **PreSIS** the product[3] of U and A – consisting of the object $U \times A$ and the projction finctions π_1 and π_2 and the arrow 'inf^{-1} which is uniquely determined by the product – plus the arrow 'inf' which is the inverse of inf^{-1}. It is easy to see that **SIS** is a (very small) category[4]. Let F : **SIS** $\to$ $\mathbf{Set}_{\text{FinTot}}$ be a functor then the information system $S = (U_F, V_F, A_F, f_F)$ determined by F is defined as follows:

- $U_F = F(U)$

[3] For a definition of a *product* of objects in a category consult for example [8].

[4] We did not drow the identity morphisms; the composition of morphisms is the obvious one.

• $V_F = F(V)$
• $A_F = F(A)$
• $f_F = F(\text{val})F(\text{inf})$ (i.e. the composition $F(\text{val}) \circ F(\text{inf})$ of the functions $F(\text{val})$ and $F(\text{inf})$.

PROPOSITION 4.1. *Consider the (product preserving) functors between the categories* **SIS** *and* $\mathbf{Set}_{\mathbf{FinTot}}$. *These functors define precisely the complete information systems.*

Proof. By definition of $\mathbf{Set}_{\mathbf{FinTot}}$ the functions $F(\text{val})$ and $F(\text{inf})$ are total, thus so is f_F. ∎

We conclude that we may construct a category of information systems as being the functor category $\mathbf{Set}_{\mathbf{FinTot}}^{\mathbf{SIS}}$. If we consider the set of all functors as objects then the most natural choise of morphisms will be the set of all natural transformations between the functors from **SIS** to $\mathbf{Set}_{\mathbf{FinTot}}$. We get the following (necessary) condition for a *natural* morphisms $m = (m_U, m_A, m_V)$ between two information systems $S_1(U_1, A_1, V_1, f_1)$ and $S_2 = (U_2, A_2, V_2, f_2)$:

Let $u \in U_1$, $a \in A_1$, $v \in V_1$

$$(*) \qquad m_V(f_1(u,a)) = f_2(m_U(u), m_A(a)).$$

In order to motivate this claim consider natural transformations $\tau = (\tau_I, \tau_U, \tau_A, \tau_V, \tau_{U\times A})$[5] of information systems. The following conditions hold for $\tau : F \to G$ (where $F, G \in \mathbf{Set}_{\mathbf{FinTot}}^{\mathbf{SIS}}$):

1. $G(\text{obj})\tau_I = \tau_U F(\text{obj})$
2. $G(\text{att})\tau_I = \tau_A F(\text{att})$
3. $G(\text{val})\tau_I = \tau_V F(\text{val})$
4. $G(\text{inf}^{-1})\tau_I = \tau_{U\times A} F(\text{inf}^{-1})$
5. $G(\text{inf})\tau_{U\times A} = \tau_I F(\text{inf})$
6. $G(\pi_1)\tau_{U\times A} = \tau_U F(\pi_1)$
7. $G(\pi_2)\tau_{U\times A} = \tau_A F(\pi_2)$.

Equations 1, 2 and 3 give the condition mentioned for morphisms. Equations 4, 5, 6 and 7 are due to the relation between the set of information and the set of all object-attribute pairs. It is easily seen that in this framework the three types of morphisms – for the permuting category, the indiscernability category and the dependence category – defined in section 2, do all satisfy condition $(*)$.

We can characterize the categories of section 2 as subcategories of the functor category we defined above. We will show this for the permuting and the indiscernibility category of information systems.

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[5] For each object of **IS** we have an arrow of a natural transformation.

Let F be a functor in $\mathbf{Set}^{\mathbf{SIS}}_{\mathbf{FinTot}}$ and let $\sigma : F(A) \to F(A)$ be a permutation of the attribute set $F(A)$ (i.e. σ is a one-one function from $F(A)$ to $F(A)$). Let the function $s : F(I) \to F(I)$, induced by σ be as follows:

$$s(p) = q \quad \text{iff} \quad \sigma(F(\text{att})(p)) = F(\text{att})(q) \& F(\text{obj})(p) = F(\text{obj})(q)$$

i.e. s maps an information element p carring information of an object u on attribute a to an information element q that carries information on the same object as p but on the permutated attribute $\sigma(a)$.

A permuting category is completely characterized as being a subcategory of $\mathbf{Set}^{\mathbf{SIS}}_{\mathbf{FinTot}}$ satisfying

$$\forall F, G \exists \sigma [F(\text{att}) s = G(\text{att})$$

and

$$\begin{aligned} F(U) &= G(U) \\ F(A) &= G(A) \\ F(V) &= G(V) \\ F(I) &= G(I) \\ F(U \times A) &= G(U \times A) \\ F(\text{obj}) &= G(\text{obj}) \\ F(\text{val}) &= G(\text{val}) \\ F(\pi_1) &= G(\pi_1) \\ F(\pi_2) &= G(\pi_2) \\ F(\text{inf}) &= G(\text{inf})] \end{aligned}$$

where F, G are functors in $\mathbf{Set}^{\mathbf{SIS}}_{\mathbf{FinTot}}$ and σ a permutation on $G(I)$.

Let F be a functor in $\mathbf{Set}^{\mathbf{SIS}}_{\mathbf{FinTot}}$ and let Ind be a indiscernibility relation on U. Note that Ind is an equivalence relation. Consider the set of equivalence classes U/Ind and choose from each of these equivalence classes one representative $u \in F(U)$. Let $r : F(U) \to F(U)$ be the function that maps each object of U to the representative of its equivalence class. Consider now the function $i : F(I) \to F(I)$ induced by r in the following manner

$$i(p) = q \quad \text{iff} \quad F(\text{att})(p)) = F(\text{att})(q) \& r(F(\text{obj})(p)) = F(\text{obj})(q)$$

i.e. i maps the information p of an object u in some attribute a to the information q of the representative $r(u)$ of the equivalence class of u in the same attribute a.

An indiscernibility category is completely characterized as being a subcategory of $\mathbf{Set}^{\mathbf{SIS}}_{\mathbf{FinTot}}$ satisfying

$$\forall F [F(\text{val}) = F(\text{val}) \circ i]$$

where $F \in \mathbf{Set}^{\mathbf{SIS}}_{\mathbf{FinTot}}$.

Observe that we defined in the two cases above the categories of information systems as functor categories relying on functions from information to information (from $F(I)$ to $F(I)$).

5. Final remarks

1. This paper is only a starting point of futher investigation.

2. We plan to study the representation of information systems categories as a family of functors from other simple structures e.g. from

$$\begin{array}{ccc} U & \rightarrow & A \\ & \searrow & \downarrow \\ & & V \end{array}$$

to $\mathbf{Set_{Fin}}$ (identity morphisms are not shown in the above diagram).

3. It seems to be important to describe the behavior of the category **IS** in terms of (or in relation to) permutation groups $(\mathrm{Perm}(U), \circ)$.

Appendix

A. Definitions of category theory

Definitions in this section are mainly taken from Lambek and Scott [8].

DEFINITION A.1. A. *category* C is a collection of two kinds of entities, called *objects* anf *morphisms*. The morphisms are mappings from one object to another. For each object A in C there exists an *identity morphisms* is_A : $A \to A$. Futhermore morphisms $f : A \to B$ and $G : B \to C'$ may be *composed* to produce a morphism $gf : A \to C$. Composition should satisfy the following two properties: Let $f : A \to B$, $g : B \to C$, $h : C \to D$

- $h(gf) = (hg)(f)$ (*associativity*)
- $fid_A = id_b f = f$ (identity)

Morphisms are also called *arrows*.

DEFINITION A.2. A *functor* $F : \mathcal{A} \to \mathcal{B}$ is a mapping between two categories $\mathcal{A}$ and $\mathcal{B}$ that sends objects of $\mathcal{A}$ to objects of $\mathcal{B}$ and arrows of $\mathcal{A}$ to arrows of $\mathcal{B}$ such that, if there is an arrow $a : A \to A'$ in $\mathcal{A}$ then $F(a) : F(A) \to F(A')$ in $\mathcal{B}$. Moreover a functor preserves identities and composition.

DEFINITION A.3. Given two functors $F, G : \mathcal{A} \to \mathcal{B}$ a *natural transformation* $t : F \to G$ is a family of arrows $t_A : F(A) \to G(A)$ in $\mathcal{B}$, one arrow for each object A of $\mathcal{A}$, such that the following square commutes for arrows $F : A \to B$ in $\mathcal{A}$:

$$\begin{array}{ccc} F(A) & \xrightarrow{t(A)} & G(A) \\ \downarrow F(f) & & \downarrow G(f) \\ F(B) & \xrightarrow{t(B)} & G(B) \end{array}$$

DEFINITION A.4. Given a set I and a family $\{A_i | i \in I\}$ of objects in a category $\mathcal{A}$, their *product* is given by an object P and a family of *projections* $\{p_i : P \to A_i | i \in I\}$ with the following universal property: given any object Q and a family of arrows $\{q_i : Q \to A_i | i \in I\}$, there is a unique arrow $f : Q \to P$ such that $p_i f = q_i$ for all $i \in I$.

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