DEMONSTRATIO MATHEMATICA
Vol. XXVII No 3–4 1994

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ON IDEALS OF BCK-ALGEBRAS

Dedicated to Professor Tadeusz Traczyk

1. Introduction

BCK-algebras were introduced as an algebraic formulation of a propositional calculus by K. Iseki and E. Y. Imai in 1966 [7]. A lot of literature dealing with algebraic theory using first order properties (see e.g. [4, 5, 10, 13]) and ideal theory of BCK-algebras (see e.g. [1, 2, 6, 9, 12]) is available. The theory of prime ideals has been of great interest in this context. The main purpose of this paper is to study some further properties of ideals (in particular prime ideals) of BCK-algebras. More precisely, let X be a commutative BCK-algebra, A be an ideal of X and x be an element of X. Put $x^{-1}A = \{y \in X : x \wedge y \in A\}$. We prove that $x^{-1}A$ is an ideal which contains A and A is prime if and only if $x^{-1}A = A$ for all $x \in X - A$. We use this characterization to show that every maximal ideal in a commutative BCK-algebra is prime. This generalizes a result of Iseki [8] for bounded implicative BCK-algebras. Thaheem [12] proved the converse of Iseki's result [8] and showed that maximal and prime ideals coincide over bounded implicative BCK-algebras. In this paper, we prove (Proposition 3.6) that Thaheem's result holds for even a larger class of bounded "involutory" BCK-algebras (cf. section 2). We also partially resolve a problem proposed in [3] to find a class of ideals that are involutory (Corollary 3.9). These results are contained in section 3 of this paper. In section 2, we include some preliminaries and establish our notations and terminology that we require for our results.

2. Preliminaries

A BCK-algebra is a system $\langle X * 0, \leq \rangle$ satisfying (i) $(x*y)*(x*z) \leq (z*y)$, (ii) $x*(x*y) \leq y$, (iii) $x \leq x$, (iv) $0 \leq x$, (v) $x \leq y, y \leq x$ imply $x = y$, where $x \leq y$ if and only if $x*y = 0$, $x, y, z \in X$. If X contains an element 1 such that $x \leq 1$, for all $x \in X$ then X is said to be bounded. X is called commutative if $x \wedge y = y \wedge x$ for all $x, y \in X$ where $x \wedge y = y * (y * x)$. A bounded commutative BCK-algebra X is a distributive lattice with respect to $\wedge$ and $\vee$, where $x \vee y = N(Nx \wedge Ny)$ for all $x, y \in X$, and $Nx = 1 * x$ (see for instance [4], [10], [13]). X is called implicative if $x * (y * x) = x$ for all $x, y \in X$. It is well-known that every implicative BCK-algebra is commutative but the converse is not true in general [10]. In any commutative BCK-algebra X the inequality $(x \wedge y) * (x \wedge z) \leq x \wedge (y * z)$ holds. Indeed, $(x \wedge y) * (x \wedge z) = (x * (x * y)) * (x * (x * z)) \leq (x * z) * (x * y) \leq (y * z)$. Also $(x \wedge y) * (x \wedge z) \leq (x \wedge y) \leq x$. It follows that $(x \wedge y) * (x \wedge z) \leq x \wedge (y * z)$. This inequality will be repeatedly used. A nonempty subset A of a BCK-algebra X is called an ideal if $0 \in A$ and $x, y * x \in A$ imply $y \in A$. It follows that if A is an ideal, $x \in A$ and $y \leq x$ then $y \in A$. A proper ideal A of a commutative BCK-algebra X is said to be prime if $x \wedge y \in A$ implies $x \in A$ or $y \in A$. Equivalently (see for instance [4]) A is prime if and only if $I \cap J \subseteq A$ implies $I \subseteq A$ or $J \subseteq A$ for any ideals I and J of X. Maximal ideals of BCK-algebras have the usual meaning. Let X be a commutative BCK-algebra, K be a subset then following [3] we define $K^* = \{x \in X : x \wedge k = 0 \text{ for all } k \in K\}$, called the annihilator of K. K^* is an ideal of X. If $K = \{k\}$ (singleton) then we write $\{k\}^* = (k)^*$. In general, for any ideal A, $A \cap A^* = \{0\}$ and $A \subseteq A^{**}$, ($A^{**} = (A^*)^*$, the double annihilator of A). If A and B are subsets of X such that $A \subseteq B$ then $B^* \subseteq A^*$. If $A = A^{**}$ then A is called an involutory ideal. A commutative BCK-algebra all of whose ideals are involutory is called an involutory BCK-algebra. For instance, any finite commutative BCK-algebra or any implicative BCK-algebra is an involutory BCK-algebra (see [3]). For more information on annihilators and involutory ideals, we refer to [3]. For some further properties of BCK-algebras and undefined terminology and notions used here, we refer to [9, 10, 13].

3. The ideals of the type $x^{-1}A$

Throughout this section, X will denote a commutative BCK-algebra unless explicitly mentioned otherwise. We begin with the following

Definition 3.1 . Let A be an ideal of X and $x \in X$. We define $x^{-1}A = \{y \in X : x \wedge y \in A\}$. Clearly $x^{-1}A$ is nonempty because $0 \in x^{-1}A$.

First, we provide an example of ideals of the type $x^{-1}A$ which also elaborates certain general results on these ideals contained in this section. This example is a special case of the more general example [9, Example 3, p. 353] of an infinite commutative BCK-algebra. We choose the finite case for simplicity.

EXAMPLE. Let $X = \{0, a, b, c, d\}$ be a set with least element 0 such that every pair of nonzero elements is incomparable. Define the binary operation $*$ on X as in Table 1.

$*$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	a	a	a
b	b	b	0	b	b
c	c	c	c	0	c
d	d	d	d	d	0

Table 1

$\wedge$	0	a	b	c	d
0	0	0	0	0	0
a	0	a	0	0	0
b	0	0	b	0	0
c	0	0	0	c	0
d	0	0	0	0	d

Table 2

Then $(X, *, 0)$ is a commutative BCK–algebra (cf. Table 2). Any set containing 0 is an ideal [9, p. 358].

(i) Consider an ideal $A = \{0, a\}$. For $b \notin A$, we observe that (cf. Table 2)

$$b^{-1}A = \{0, a, c, d\} \text{ is an ideal, and } b^{-1}A \neq X \text{ and } A \subset b^{-1}A.$$

This provides a non–trivial example of ideal of the type $x^{-1}A$.

(ii) Also, $0^{-1}A = a^{-1}A = X$. That is, for $x \in A, x^{-1}A \neq A$. Thus, we conclude from (i) and (ii) that equalities $x^{-1}A = X, x^{-1}A = A$ are not always true; however $A \subseteq x^{-1}A$ for all $x \in X$ (see Proposition 3.2 for a more general result).

(iii) If we choose $A = \{0, a, b, c\}$, then A is a prime ideal of X. For $d \in X - A, d^{-1}A = A$. This explains the more general result which states that an ideal A of a commutative BCK-algebra X is prime if and only if $x^{-1}A = A$ for all $x \in X - A$ (see Proposition 3.4).

PROPOSITION 3.2. *$x^{-1}A$ is an ideal which contains A.*

Proof. It is obvious that $0 \in x^{-1}A$. Now assume that $z, y * z \in x^{-1}A$. Then $x \wedge z, x \wedge (y * z) \in A$. Since $(x \wedge y) * (x \wedge z) \leq x \wedge (y * z)$ (cf. section 2), $x \wedge (y * z) \in A$ and A is an ideal, therefore $(x \wedge y) * (x \wedge z) \in A$. Again using the fact that A is an ideal and $(x \wedge z) \in A$, we get that $x \wedge y \in A$. This means $y \in x^{-1}A$ which proves that $x^{-1}A$ is an ideal. To prove that $A \subseteq x^{-1}A$, let $y \in A$. Then $x \wedge y \leq y$ implies that $x \wedge y \in A$ and hence $y \in x^{-1}A$. This completes the proof.

We include some properties of these ideals in the following proposition. The proof is simple and, therefore, we omit it.

PROPOSITION 3.3. *The following statements hold:*

(1) *$x^{-1}A = X$ if and only if $x \in A$.*
(2) *If $x \leq y$ then $y^{-1}A \subseteq x^{-1}A$.*
(3) *If A and B are ideals of X such that $A \subseteq B$ then $x^{-1}A \subseteq x^{-1}B$ for all $x \in X$.*
(4) *$(x)^* \subseteq x^{-1}A$ for all $x \in X$.*

(5) *For any ideals* A, B *of* X *and any* $x \in X$, $x^{-1}(A \cap B) = x^{-1}A \cap x^{-1}B$.
(6) *Let* A *be an ideal and* P *be a prime ideal such that* $A \subseteq P$. *Then* $x^{-1}A \subseteq P$ *for all* $x \in X - P$.
(7) $(x \wedge y)^{-1}A = x^{-1}(y^{-1}A)$ *for all* $x \in X$.
(8) *If* X *is a bounded commutative* BCK*-algebra, then* $(x \vee y)^{-1}A = x^{-1}A \cap y^{-1}A$.

Notice that if $x = y$ in (7) then $x^{-1}(x^{-1}A) = x^{-1}A$. This gives a special characteristic of the ideal $x^{-1}A$. If $x \vee y = 1$ then by (8), $A = x^{-1}A \cap y^{-1}A$ which gives a decomposition of A in terms of the ideals of the type $x^{-1}A$.

The following proposition gives a characterization of prime ideals.

PROPOSITION 3.4. *An ideal* A *of* X *is prime if and only if* $x^{-1}A = A$, *for all* $x \in X - A$.

Proof. Suppose that A is a prime ideal of X and $x \in X - A$. The inclusion $A \subseteq x^{-1}A$ follows easily. To prove the reverse inclusion, let $y \in x^{-1}A$. This implies that $x \wedge y \in A$ and A being a prime ideal implies that $y \in A$ (because $x \notin A$ by assumption). This proves that $x^{-1}A = A$. Conversely, assume that $x^{-1}A = A$ for all $x \in X - A$. Let $y \wedge z \in A$ and $z \notin A$. By hypothesis $z^{-1}A = A$ and consequently $y \in z^{-1}A = A$. This proves that A is a prime ideal.

Iseki [8] proved that every maximal ideal in a bounded implicative BCK-algebra is prime. Palasinski [11] generalized this result for commutative BCK-algebras using several technical identities and a separation-type result for ideals ([11], Corollary 3]). We provide a simple proof of this result as an immediate application of the above proposition.

PROPOSITION 3.5. *Every maximal ideal in a commutative* BCK*-algebra is prime.*

Proof. Let A be a maximal ideal in a commutative BCK-algebra X. To show that A is prime, it is sufficient to prove that $x^{-1}A = A$ for all $x \in X - A$ (by Proposition 3.4). As proved earlier $A \subseteq x^{-1}A$. If $A \neq x^{-1}A$ then the maximality of A implies that $x^{-1}A = X$. This happens only when $x \in A$ (by Proposition 3.3) which is a contradiction because $x \notin A$. This shows that $x^{-1}A = A$ and consequently A is a prime ideal.

Thaheem [12] established the converse of Iseki's result [8] and proved that maximal and prime ideals coincide over bounded implicative BCK-algebras. In the following we show that Thaheem's result holds even for a larger class of bounded involutory BCK-algebras.

PROPOSITION 3.6. *An ideal of a bounded involutory* BCK*-algebra is maximal if and only if it is prime.*

Proof. Let P be an ideal of a bounded involutory BCK-algebra X. Suppose that P is maximal. Then P is prime by the above proposition. Conversely, assume that P is prime. Let M be a proper maximal ideal that contains P (see e.g. [10, Proposition 3]). We now show that $M = P$. Assume that $M \not\subseteq P$. Now $M \cap M^* = \{0\} \subseteq P$. P being a prime ideal implies that $M \subseteq P$ or $M^* \subseteq P$. As $M \not\subseteq P$, therefore $M^* \subseteq P$. Since $P \subseteq M$, therefore $M^* \subseteq P^*$. We get that $M^* \subseteq P \cap P^* = \{0\}$. That is $M^* = \{0\}$ and hence $M^{**} = X$. As X is involutory we have $M^{**} = M = X$, a contradiction. Therefore, $M \subseteq P$ and consequently $M = P$. This proves the result.

Recall that an element a in a BCK-algebra X is said to be an atom if $x \leq a$ for some $x \in X$ implies $x = 0$ or $x = a$ (see [13]).

PROPOSITION 3.7. *If x is an atom in X then $(x)^* = x^{-1}A$ for every ideal A which does not contain x, and $(x)^*$ is a prime and maximal ideal.*

Proof. $(x)^* \subseteq x^{-1}A$ by Proposition 3.3(4). If $y \in x^{-1}A$ then $y \wedge x \in A$. Since x is an atom and does not belong to A, $y \wedge x = 0$. Hence $y \in (x)^*$. Then $(x)^* = x^{-1}A$.

If the ideal $(x)^*$ were not maximal, then there would exist a proper ideal A and $y \in A$ such that $(x)^* \subseteq A$ and $y \notin (x)^*$. Then $y \wedge x \neq 0$. Since x is an atom, $y \wedge x = x \in A$, a contradiction. So $(x)^*$ is maximal. By Proposition 3.6 it is prime as well.

PROPOSITION 3.8. *Let A be an ideal in X. Then*

$$A^{**} = \bigcap_{b \in A^*} b^{-1}A.$$

Proof. Let x be any element in A^{**}. Then $x \wedge b = 0$ for all $b \in A^*$. This means that $x \in b^{-1}A$ for all $b \in A^*$ and consequently $x \in \bigcap_{b \in A^*} b^{-1}A$. That is, $A^{**} \subseteq \bigcap_{b \in A^*} b^{-1}A$. Conversely, let $x \in \bigcap_{b \in A^*} b^{-1}A$. Then $x \in b^{-1}A$ for all $b \in A^*$. This implies that $x \wedge b \in A, b \in A^*$ and hence $x \wedge b = (x \wedge b) \wedge b = 0$ for all $b \in A^*$. It follows that $x \in A^{**}$ and consequently $y \in \bigcap_{b \in A^*} b^{-1}A \subseteq A^{**}$. This proves the equality.

The following corollary provides a partial solution to the problem of determining the involutory ideals of commutative BCK-algebras proposed in [3]. The proof follows immediately from Propositions 3.3(1), 3.4 and 3.8.

COROLLARY 3.9. *Let A be a prime ideal of X for which $A^* \neq \{0\}$. Then A is an involutory ideal (that is $A^{**} = A$).*

Acknowledgements. The authors are grateful to the referee for useful suggestions and comments which led to an improvement of the paper. One of the authors (A.B. Thaheem) thanks KFUPM for providing excellent research facilities.

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Received June 24, 1993.