DEMONSTRATIO MATHEMATICA
Vol. XXVII No 3–4 1994

Ewa Rydzyńska

DIFFERENTIAL CALCULUS FOR QUANTIZED RANDOM VARIABLES IN PROBABILITY HEYTING ALGEBRAS

Dedicated to Professor Tadeusz Traczyk

0. Introduction

In this paper, the differential calculus for random variables defined on a Heyting algebra (or on a distributive lattice with pseudocomplementation) from [8], is developed. The case of quantized random variables is described. This theory can be applied in Bellert's space-time theory which is a cosmological theory connected with quantum physics. In [8], we made the considerations for an arbitrary complete Heyting algebra, but in the present paper we use (in Lemma 24) the specific properties of physical events, so we use the fact that our distributive lattice with pseudocomplementation is Bellert's space-time. Obviously, the fact used by us is only one of the properties of the space-time and it can be noted, in a simple way, using mathematical symbols. We will not do it because we regard it as uninteresting.

1. Bellert's space-time

The red shift effect is a well known empirical fact. Every present cosmological theory tries to explain it. Stanisław Bellert, in [2], [3], [4], was trying to explain this fact in a more natural way than it had been done before. Bellert derived the law of summation of radial distances

$$(1)\qquad D(a_1,a_3)=D(a_1,a_2)+D(a_2,a_3)-\frac{1}{k}D(a_1,a_2)D(a_2,a_3)$$

and derived the relationship between cosmic distance D (which we observe) and local distance x (traditional distance):

$$(2)\qquad D=k(1-e^{-kx})$$

where k is a constant whose empirical determined value is equal to $k = 1/cT$ (c = the light velocity, T = Hubble's constant). We will use such units of time and space-distance that $c = 1 = T = k$. Therefore we can denote the law of summation of cosmic distance in the following way:

$$D(a_1, a_3) = D(a_1, a_2) + D(a_2, a_3) - D(a_1, a_2)D(a_2, a_3) \tag{3}$$
$$:= D(a_1, a_2) \overset{B}{+} D(a_2, a_3) .$$

The operation $\overset{B}{+}$ defined above we call Bellert's sum.

Moreover, he assumed that the light velocity $c = 1$ is constant and hence there must exist the cosmic time τ which satisfies the law of summation:

$$\tau(a_1, a_3) = \tau(a_1, a_2) + \tau(a_2, a_3) - \tau(a_1, a_2)\tau(a_2, a_3) \tag{4}$$
$$:= \tau(a_1, a_2) \overset{B}{+} \tau(a_2, a_3)$$

and which depends on "normal" local time t as in formula:

$$\tau = 1 - e^{-t/T} . \tag{5}$$

In Bellert's theory, time is quantized, i.e. we must partition the time-axis into small, adjacent finite interval moments (see [5]).

It appears that a conditional probability has connections with time and space-distance in Bellert's theory.

Namely: the main axiom of conditional probability on Heyting algebra is the equation:

$$p(a_1, a_3) = p(a_2, a_3)p(a_1, a_2) \tag{6}$$

for $a_1 \leq a_2 \leq a_3$. If we now identify

$$p(a, b) \equiv 1 - \tau(a, b) \tag{7}$$

or

$$p(a, b) \equiv 1 - D(a, b) \tag{8}$$

then equation (6) is equivalent to equation (4) (or appropriately to equation (3)).

We proved in paper [9] (see also [5]), that Bellert's space-time with the order defined in a natural way is a distributive lattice with pseudocomplementation.

This algebraic construction of Bellert's space-time is as follows:

Construction 1. ([9]). V is the set of all events in the (empty) Universe described by Bellert's theory. If two events have the same time-interval (relative to the observer), we really cannot distinguish these. Therefore we will distinguish classes of events which have the same time-intervals and the same space-intervals. Let [d] denote the class of equivalency of these event which have the same time-interval and the same space-interval as event d. We call the set of all classes of events from V Bellert's space-time and mark with letter B.

We define relation "$\leq$" for classes $[d], [b] \in B$ in the following way: $[d] \leq [b]$ iff a certain event belonging to $[d]$ influences certain event from $[b]$.

Let us remove from the space-time V such events which observer S in the present moment T cannot know (even indirectly). We will denote V_T this part of space-time.

We define B_T as a set of classes of events from V_T with relation "$\leq$". Next we join together all events which happened in the first moment T_0 or earlier and call this class zero. Such space-time we denote B_0 and call Bellert's space-time of the observer.

THEOREM 2. ([9]). *The system* $(B_0, \sqcup, \sqcap, 0, 1)$ *defined above is a distributive lattice with* 0 *and* 1.

Remark 3. ([9] Remark 2.3). When we deal with certain classes of events, whose relations we can express using lattice joins and lattice meets, and we must examine the relations between these classes, then it suffices to examine the relation between their representative events lying on the same radius (for example on straight-line $\overline{SL}$). Hence: if events b and d lie on straight-line $\overline{SL}$ then $[b] \sqcup [d]$ is the class connected with the event lying on straight-line $\overline{SL}$ between the positions of events b and d which is a meeting of the light signals from b and d; and $[b] \sqcap [d]$ is the class connected with an event lying on the straight-line between the positions of events b and d, such that the signals sent from it reach b and d.

DEFINITION 4 ([9] construction 2.4 and Remark 3.1). Obviously, we de-fine $-0 := 1$, $-1 := 0$. We define $-d$ (for the other d) as the greatest event

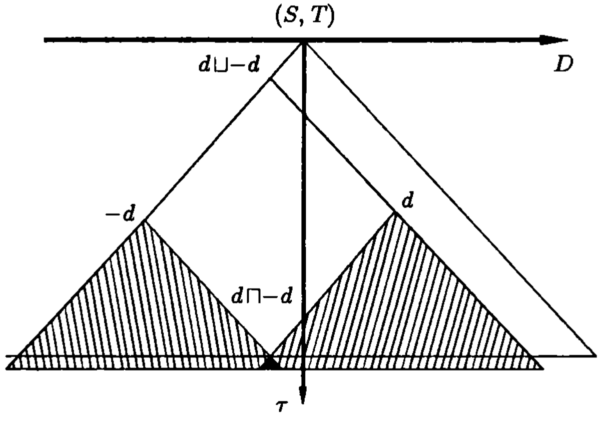

Fig. 1

with a past-cone disjoint with the past-cone of event d. The construction of $-d$ is shown in fig. 1. Obviously, we define $-[d] := [-d]$.

$-d$ can be found in the following way:

$-d$ is an event lying on straight-line $\overline{dS}$ on the reverse side of S than d and

1) a signal from $-d$ reaches the observer in the present moment, i.e. it reaches (S,T),

2) signals from a certain point betwen d and $-d$, sent in the beginning moment T_0, reach d and $-d$.

THEOREM 5. ([9]) Theorem 2.5.). *Bellert's space-time of the observer* $(B_0, \cup, \cap, -, 0, 1)$ *is a distributive lattice with pseudocomplementation.*

Remark 6. ([9] Remark 3.2). The interpretation of the mentioned operations is natural. The join of events can be interpreted as a direct effect of these events, and a meet of events as a direct cause of these events. Event $-d$ is an event whose world (or whose knowledge) is independent of d, in the sense that $-d$ does not know any fact which d knows, and d does not know any fact which $-d$ knows, i.e. there is no event in B_0 which can interact on both events d and $-d$. Moreover, it is the furthest event known by us satisfying this property.

We have shown that the space-time from Bellert's theory with these operations has a structure of a distributive lattice with pseudocomplementation. Simultaneously, conditional probability is a natural measure of time and space in this space-tine.

2. The definition of a derivative of a random variable

In this paper we will use some simple properties of Heyting algebras and distributive lattices with pseudocomplementation. They are as follows:

THEOREM 7 (see for example [1] p. 153). *We have for any elements a, b, c of a distributive lattice with pseudocomplementation*:

1) $a \cap (b \cup c) = (a \cap b) \cup (a \cap c)$, $a \cup (b \cap c) = (a \cup b) \cap (a \cup c)$. (*the distributivity*)
2) $a \cup b \geq a$, $a \cap b \leq a$.
3) $a \leq b \Rightarrow a \cap c \leq b \cap c$, $a \cup c \leq b \cup c$.
4) $a \leq b \Leftrightarrow a \cap b = a \Leftrightarrow a \cup b = b$.
5) $0 \cup a = a$, $0 \cap a = 0$, $1 \cup a = 1$, $1 \cap a = a$.
6) $-0 = 1$, $-1 = 0$.
7) $-a \cap a = 0$.
8) $a \leq b \Rightarrow -b \leq -a$.

Let A be Bellert's space-time of the observer. Such that A is a complete and completely distributive lattice with pseudocomplementation.

Definition 8 ([6] Definition 2.21). Let $X : A \to R$ be a random variable, $S \subseteq R$, G a set of intervals of which S consists. Let

$$h_X((\eta,\xi)) := \bigcup_{X(a)<\xi} a \cap - \bigcup_{X(b)\leq\eta} b,$$
$$h_X([\eta,\xi)) := \bigcup_{X(a)<\xi} a \cap - \bigcup_{X(b)<\eta} b,$$
$$h_X((\eta,\xi]) := \bigcup_{X(a)\leq\xi} a \cap - \bigcup_{X(b)\leq\eta} b,$$
$$h_X([\eta,\xi]) := \bigcup_{X(a)\leq\xi} a \cap - \bigcup_{X(b)<\eta} b,$$
$$h_X(S) := \bigcup_{E\in G} h_X(E).$$

$h_X : 2^R \to R$ will be called a spectral supermeasure.

Definition 9 ([6] §2). We call $p : A \to R$ which satisfies the following four axioms a probability on A:

(AI) $p(a) \geq 0$ for every $a \in A$,

(AII) $p(1) = 1$,

(AIII) $p(a_1 \cup \ldots \cup a_n) = \sum_{i=1}^{n} p(a_i)$ for $n \in N$ and $a_i \in A$ $(i = 1, \ldots, n)$ such that $a_i \cap a_j = 0$ for $i \neq j$,

(AIV) if $a \leq b$, then $p(a) \leq p(b)$.

From (AI)–(AIV) it follows that $p(0) = 0$.

Theorem 10. ([6] Definition 2.22 and 2.25). *A function $M_X : 2^R \to R$ defined as $M_X := p \circ h_X$ (where p is a probability on A and h_X is a spectral supermeasure) is a probability supermeasure, i.e. it satisfies the following conditions:*

(a1) $M_X(S) \geq 0$ *for every* $S \subseteq R$,

(a2) $M_X(\emptyset) = 0$, $M_X(R) = 1$,

(a3) $M_X(S_1 \cup \ldots \cup S_n) \geq \sum_{i=1}^{n} M_X(S_i)$ *when* $S_i \cap S_j = \emptyset$ *for* $i \neq j$,

(a4) *if* $E \subseteq F$, *then* $M_X(E) \leq M_X(F)$.

Definition 11 ([6] Definition 3.3). Let M be a probability supermeasure and let $f : D \to R$ be a step function ($D \subseteq R$). We denote by D_f the set of all sequences $\{d_i, D_i\}_{i=1}^{n}$ such that $n \in N$, $d_i \geq 0$, $D_i \cap D_j = \emptyset$ for $i \neq j$, and $f = \sum_{i=1}^{n} d_i \chi_{D_i}$. Then

$$\int f\, dM := \inf\Big\{ \sum_{i=1}^{n} d_i M(D_i) : \{d_i, D_i\}_{i-1}^{n} \in D_f \Big\}.$$

DEFINITION 12 ([6] Definitions 3.11 and 3.14). Let $f : R \to R_+$ be a bounded function and let M be a probability supermeasure. Then

$$\int f\,dM := \lim_{n\to\infty} \int f_n\,dM\,,$$

where $(f_n)_{n=1}^{\infty}$ is a sequence of step functions such that $f_n \searrow\searrow f$. Let $g : R \to R$ be a bounded function. Then

$$\int g\,dM := \int g^+\,dM - \int g^-\,dM\,,$$

where $g^+ := \max\{g, 0\}$, $g^- := \max\{-g, 0\}$.

Remark 13. Definition 12 is correct and the mentioned limit does not depend on $(f_n)_{n=1}^{\infty}$. So the defined integral satisfies all the most important properties of a classic Lebesque integral (see [6] Theorem 3.15).

DEFINITION 14 ([6] Definition 4.2). Let X be a bounded random variable. Then

$$\int X\,dp := \int \mathrm{id}\chi_{S_X}\,dM_X\,,$$

where id denotes the identity and

$$S_X := [\inf_{a\in A} X(a), \sup_{a\in A} X(a)]\,.$$

Remark 15. ([8]). Such an defined above integral is a number and has the character of a definite integral. We want to define a derivative of a random variable instead and we need, in our work, an indefinite integral. So we must define a new integral which has the character of an indefinite integral of a random variable.

DEFINITION 16 ([8] Definition 2.9). $M_{X/x}(E) := p(h_X(E) \cap x)$. We define integral $\int^x dp$ in the same way as integral $\int X\,dp$ when we put in the definition $M_{X/x}$ instead of M_X.

DEFINITION 17 ([8] Definition 2.11). Let $G(x) = \int^x X\,dp$. Then we say that X is a derivative of G and mark as $X(x) = G'(x)$ or $X(x) = \frac{dG(x)}{dp}$. Obviously, such a derivative does not have to be defined synonymously.

DEFINITION 18 (compare with [8]). The integral of a continuous bounded real function f with a domain W_Y : $\int_{W_Y}^{Y(x)} f(z)\,dz$ we define as

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(z_i^{n)})(z_i^{n)} - z_{i-1}^{n)}),\ z_i^{n)} - z_{i-1}^{n)} \le \underset{n\to\infty}{\searrow\searrow} 0,\ z_i^{n)} \in W_Y \cap [0, Y(x)]\,,$$

where $Y(x) \in R$, and the condition $z_i^{n)} - z_{i-1}^{n)} \le \underset{n\to\infty}{\searrow\searrow} 0$ symbolically means that: if the numbers from W_Y lie densely in a certain interval, then we have

$z_i^{n)} - z_{i-1}^{n)} \underset{n\to\infty}{\searrow\searrow} 0$ in this interval, and if it does not hold, then $z_i^{n)}$ and $z_{i-1}^{n)}$ are neighboring numbers of W_Y.

Remark 19 (compare with [8]). If W_Y is a set of values of random variable Y, then, obviously, this integral is independent of a choice of $z_i^{n)}$ and $\int_{W_Y}^{Y(x)} f(z)\,dz = \lim_{n\to\infty} \int_{W_Y}^{Y(x)} f(z)\,dz$, where $Y_n \searrow\searrow Y$ are step random variables (i.e. random variables which have finite numbers of values).

Remark 20 (compare with [8]). In the case when we have random variable Y whose values lie densely in interval E, then the above integral is exactly the Riemann integral of function f on $E \cap (-\infty, Y(x))$. In the case when Y has a finite number of values, this integral is equal to $\sum_{i=1}^{k} f(z_i)(z_i - z_{i-1})$, where $z_{i-1} < z_i$ and $z_k = Y(x)$.

DEFINITION 21 (compare with [8]). Let F be a real function and let it have the domain W. Then we call the integrable function f such that $F(y) = \int_W^y f(z)\,dz$, the derivative of function F.

Remark 22 (compare with [8]). The derivative of function F, which has domain W and whose values are dense in a certain interval, can be calculated as the classic derivative of a real function (compare with Remark 20).

The theory of a derivative of a random variable whose values lie densely, is described in [8]. There is also given the beginning of an analogous theory for quantized random variables, which will be developed, here. There is the description of the connection of these two theories for the purpose of receiving a differential calculus of continuously-quantized random variables, which we meet more often in physical problems in Bellert's theory. (It is because time is quantized in Bellert's theory – see [5] — and space is not quantized; and we usually use time and space together for a description of a physical situation).

3. Properties of a derivative of a quantized random variable

In this paragraph, we will develop our considerations from [8]. We will work out a differential calculus of quantized random variables (as time in Bellert's theory). Quantized random variables are, for example, in Bellert's theory, certain physical quantities measured in the same point in space but distributed in time.

DEFINITION 23. We call random variable G a quantized random variable, when it is defined only in the space-time points x_i $(i = 1, 2, \ldots)$ lying in the same point in space in such time-order that they form a chain: $x_1 > x_2 > \ldots$ A derivative of a quantized random variable is defined by the general definition, i.e. Definition 17.

LEMMA 24. *If, in Bellert's space-time, an event x_0 lies in the same place as an event x_k and, simultaneously,* $1 > x_0 \geq x_k$, *i.e. x_0 is later than x_k, then* $x_0 \cap -x_k \neq 0$.

Proof. Such x_0 and x_k are shown, in fig. 2, and $x_0 \cap -x_k$ is constructed, according to definition 4. It is easily seen in fig. 2 that $x_0 \cap -x_k \neq 0$. ■

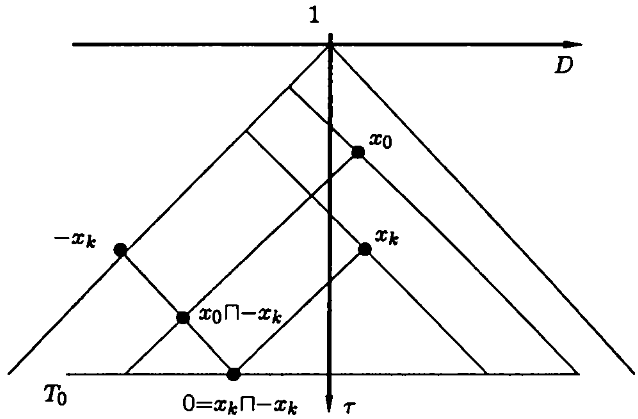

Fig. 2

LEMMA 25. $\bigcup_{G'(x) \leq G'(x_k)} x \cap x_k = x_k$ *and*

$$\bigcup_{G'(x) \leq G'(x_k)} x \cap - \bigcup_{G'(y) < G'(x_k)} y \cap x_k = - \bigcup_{G'(y) < G'(x_k)} y \cap x_k .$$

Proof. It is obvious that $G'(x_k) \leq G'(x_k)$, so it follows from Theorem 7.2) that $x_k \leq \bigcup_{G'(x) \leq G'(x_k)} x$. So, by Theorem 7.2), we have $x_k = x_k \cap x_k \leq \bigcup_{G'(x) \leq G'(x_k)} x \cap x_k \leq x_k$, so $\bigcup_{G'(x) \leq G'(x_k)} x \cap x_k = x_k$. The second part of the thesis follows from the first one in a simple way. ■

LEMMA 26. *By Definition 23, for chain* $x_1 > x_2 > \ldots$, *we have the property*:

if $x_i \geq x_k$ *and* $G'(x_m) > G'(x_i)$, *then* $h_{G'}(\{G'(x_m)\}) \cap x_k = 0$.

Proof. Let $x_i \geq x_k$ and $G'(x_m) > G'(x_i)$. Then we have by Theorem 7.2): $\bigcup_{G'(y) < G'(x_m)} y \geq x_i \geq x_k$, so by Theorem 7.8):

$$- \bigcup_{G'(y) < G'(x_m)} y \leq -x_k .$$

Therefore by Lemma 25, Theorem 7.3), 7) and Definition 8, we have

$$h_{G'}(\{G'(x_m)\}) \cap x_k := \bigcup_{G'(x) \leq G'(x_m)} x \cap - \bigcup_{G'(y) < G'(x_m)} y \cap x_k \leq -x_k \cap x_k = 0 . \ ■$$

LEMMA 27. *When the expression below exists, then, for quantized random variable G defined on the chain $x_1 > x_2 > \ldots$, we have*

$$G'(x_k) := \frac{G(x_k) - G(x_{j_k})}{p\Big(\bigcup_{G'(x) \le G'(x_k)} x \cap - \bigcup_{G'(y) \le G'(x_k)} y \cap x_k \Big)},$$

where x_{j_k} is the greatest element of our chain which is less than x_k and such that we have $G'(x_{j_k}) < G'(x_i)$ for $x_i \ge x_k$.

Proof. Let

$$G'(x_k) := \frac{G(x_k) - G(x_{j_k})}{p\Big(\bigcup_{G'(x) \le G'(x_k)} x \cap - \bigcup_{G'(y) \le G'(x_k)} y \cap x_k \Big)}.$$

Does $\int^{x_k} G'\, dp = G(x_k)$?

We have by the definition of the integral (Definitions 16, 14, 12, 11 and 8):

$$\int^{x_k} G'\, dp = \int \mathrm{id} \cdot \chi_{S_{G'}}\, dM_{G'/x_k} = \lim_{n \to \infty} \int g_n\, dM_{G'/x_k} =$$

$$= \lim_{n \to \infty} \sum_{m=0}^{n} G'(x_m) \cdot p(h_{G'}(\{G'(x_m)\}) \cap x_k) =$$

$$= \sum_{m \in M_k} G'(x_m) \cdot p\Big(\bigcup_{G'(x) \le G'(x_m)} x \cap - \bigcup_{G'(y) < G'(x_m)} y \cap x_k \Big),$$

where $m \in M_k$ iff $G'(x_m) < G'(x_i)$ for every $x_i > x_k$ from our chain. It follows because if we have, for example, $G'(x_m) > G'(x_i)$ and $x_i \ge x_k$, then, by Lemma 26, $h_{G'}(\{G'(x_m)\}) \cap x_k = 0$, so $p(h_{G'}(\{G'(x_m)\}) \cap x_k) = p(0) = 0$.

Now, we can assume that G' has different values for different x_m. Because if $G'(x_n) = G'(x_m)$ and $x_n < x_m$, then we do not have to consider event x_n at all, we can exlude it from the domain. (It is because when we calculate values $G'(x_m)$, then we will also know the values $G'(x_n)$, because it is equal to one of the known values $G'(x_m)$). Therefore we assume, in this place, that G' is of different values.

Obviously, we have by Theorem 7.1), 7) and 5): $(a \cup b) \cap -a = (a \cap -a) \cup (b \cap -a) = 0 \cup (b \cap -a) = b \cap -a$, so

$$\bigcup_{G'(x) \le G'(x_m)} x \cap - \bigcup_{G'(y) < G'(x_m)} y = \bigcup_{G'(x) = G'(x_m)} x \cap - \bigcup_{G'(y) < G'(x_m)} y.$$

Because of the fact that the value of G' are different, we have

$$\bigcup_{G'(x)=G'(x_m)} x = x_m$$

and it follows from the above, and from Theorem 7.4) that

$$\bigcup_{G'(x)\le G'(x_m)} x \cap -\bigcup_{G'(y)<G'(x_m)} y \cap x_k = x_m \cap x_k \cap -\bigcup_{G'(y)<G'(x_m)} y =$$
$$= x_m \cap -\bigcup_{G'(y)<G'(x_m)} y, \quad \text{for } x_k \ge x_m .$$

If $m \in M_k$, then we have $x_m \le x_k$. In fact, if $m \in M_k$, then, by the definition of M_k, we have $G'(x_m) \ne G'(x_i)$ for every $x_i > x_k$. Simultaneously obviously, $G'(x_m) = G'(x_m)$, so x_m cannot be one of these x_i which satisfies $x_i > x_k$; i.e. $x_m > x_k$ cannot hold. Since x_m and x_k are elements of the same chain, we have $x_m \le x_k$.

Hence, we have $x_m \le x_k$ for every $m \in M_k$, and therefore

$$\bigcup_{G'(x)\le G'(x_m)} x \cap -\bigcup_{G'(y)<G'(x_m)} y \cap x_k = x_m \cap -\bigcup_{G'(y)<G'(x_m)} y .$$

We have, obviously, $x_m \le x_m$, so $\bigcup_{G'(x)\le G'(x_m)} x \cap -\bigcup_{G'(y)<G'(x_m)} y \cap x_m = x_m \cap -\bigcup_{G'(y)<G'(x_m)} y$.

Now, we return to our integral. We obtain (by the formula for $G'(x_m)$ at the beginning of the proof):

$$\int^{x_k} G'\,dp = \sum_{m\in M_k} G'(x_m)\cdot \Big(\bigcup_{G'(x)\le G'(x_m)} x \cap -\bigcup_{G'(y)<G'(x_m)} y \cap x_k\Big) =$$
$$= \sum_{m\in M_k} \frac{(G(x_m)-G(x_{j_m}))\cdot p(-\bigcup_{G'(x)\le G'(x_m)} x \cap -\bigcup_{G'(y)<G'(x_m)} y \cap x_k)}{p(\bigcup_{G'(x)\le G'(x_m)} x \cap -\bigcup_{G'(y)<G'(x_m)} y \cap x_m)} =$$
$$= \sum_{m\in M_k} \frac{(G(x_m)-G(x_{j_m}))\cdot p((-\bigcup_{G'(y)<G'(x_m)} y \cap x_m)}{p(-\bigcup_{G'(y)<G'(x_m)} y \cap x_m)} =$$
$$= \sum_{m\in M_k} (G(x_m)-G(x_{j_m})) = G(x_k), \quad \text{q.e.d.} \ \blacksquare$$

LEMMA 28. *If a quantized random variable G defined on $x_1 > x_2 > \dots$ is bounded, then we can define G on point x_0, where $x_0 > x_k$ for $k = 1, 2, \dots$, such that $\bigcup_{G'(y)<G'(x_m)} y = x_0$ for $k = 1, 2, \dots$.*

Proof. Let us take a certain $x_0 > x_k$ for $k = 1, 2, \dots$. Simultaneously, we take x_0 such that $p(x_0 \cap -x_k) \ne 0$ for $k = 1, 2, \dots$. It is a natural condition in Bellert's space-time, and it holds in every natural situation in this space-time. (We see this when we look at Lemma 24).

1) Now, we define $G(x_0)$ such that

$$\sup_{k,m=1,2,\ldots} \frac{G(x_0)-G(x_m)}{p(x_0\cap -x_k)} < \inf_{k=1,2,\ldots} G'(x_k)\,.$$

We can easily do this for the known values $G(x_m)$, $G'(x_k)$ and $p(x_0 \cap -x_k)$ $(k, m = 1, 2, \ldots)$ and for the bounded function G, because of Lemma 27.

Now, we will prove that such G defined above satisfies the property:

$$\text{(A)} \qquad G'(x_0) < G'(x_k) \quad \text{for } k = 1, 2, \ldots\,.$$

Let us assume, for this purpose, that condition (A) does not hold. So we have

$$\text{(B)} \qquad G'(x_k) \le G'(x_0) \quad \text{for a certain } k > 0\,.$$

We have, obviously, $\bigcup_{G'(y)<G'(x_0)} y = x_k$, where x_k is a certain element of our chain on which G is defined, and it is not x_0. Hence we have (by Lemma 25 and Lemma 24):

$$\bigcup_{G'(x)\le G'(x_0)} x \cap - \bigcup_{G'(y)<G'(x_0)} y \cap x_0 = x_0 \cap - \bigcup_{G'(y)<G'(x_0)} y = x_0 \cap -x_k \ne 0\,.$$

Then, it follows from Lemma 27 and from our assumption connected with the definition of $G(x_0)$ that

$$G'(x_0) = \frac{G(x_0)-G(x_{j_0})}{p(x_0\cap -x_k)} \le \sup_{k,m=1,2,\ldots} \frac{G(x_0)-G(x_m)}{p(x_0\cap -x_k)} <$$
$$< \inf_{k=1,2,\ldots} G'(x_k) \le G'(x_k) \quad \text{for } k > 0\,,$$

and it is contradictory with (B).

We obtained a contradiction, so we proved that assumption (B) is contradictory, i.e. that G (defined by us on x_0) satisfies condition (A).

2) Now, let us assume that we defined G on the event x_0 such that condition (A) holds.

Let us take a certain fixed $k > 0$. For this k, we have by (A): $G'(x_0) < G'(x_k)$. So x_0 is the one of these y's for which $G'(y) < G'(x_k)$, and therefore we have by Theorem 7.2): $\bigcup_{G'(y)\le G'(x_k)} y \ge x_0$. But, obviously, all y's considered by us belong to the domain of G, so we have $y \le x_0$ for every such y. Hence, and by Theorem 7.3), we have $\bigcup_{G'(y)\le G'(x_k)} y \le x_0$.

In this way, we showed that $\bigcup_{G'(y)\le G'(x_k)} y = x_0$ for any $k > 0$. Now, the thesis follows from Lemma 25. ∎

THEOREM 29. *For a quantized bounded random variable G defined on chain $1 > x_1 > x_2 > \ldots$, we can define G on $x_0 > x_1$ in such a way that,* *for every $k > 0$, we have*

$$G'(x_k) = \frac{G(x_k) - G(x_{j_k})}{p(x_0 \cap -x_k)},$$

where x_{j_k} is the greatest element of our chain which is less than x_k and for which the condition $G'(x_{j_k}) < G'(x_i)$ holds for every $x_i \geq x_k$.

Proof. This follows from Lemmas 27, 28 and 25. ∎

THEOREM 30. *When we have G as in Theorem 29, then*

$$G'(x_k) = \frac{G(x_k) - G(x_{k+1})}{p(x_0 \cap -x_k)} \quad \text{for } k = 1, 2, \ldots .$$

Proof. Let the assumption of theorem hold. Let us take a fixed $k > 0$ and an arbitrary $m \geq k$. We have $x_m \leq x_k$ and then, from Theorem 7.3), Lemmas 28, 25 and 24 it follows that $h_{G'}(\{G'(x_m)\}) \cap x_k \geq h_{G'}(\{G'(x_m)\}) \cap x_m = -x_0 \cap x_m > 0$. Hence $h_{G'}(\{G'(x_m)\}) \cap x_k \neq 0$. If we now, conclude from Lemma 26 that the situation cannot exist such that $x_i \geq x_k$ and simultaneously $G'(x_m) > G'(x_i)$. Therefore, for every $x_i \geq x_k$, we have $G'(x_m) \leq G'(x_i)$. If we assume, as in the proof of Lemma 27, that G' has all values different, and when we remember the definition of M_k from the proof of Lemma 27, then we can conclude that $m \in M_k$.

Therefore $M_k = \{k, k+1, k+2, \ldots\}$, q.e.d. ∎

THEOREM 31. *For an "appropriately defined" (for example, as in Theorem 29) quantized bounded random variable G and for a differentiable bounded real function F, if G' is a derivative of G, then $F'(G(x_k)) \cdot G'(x_k)$ is a derivative of $F(G(\cdot))$ in the point x_k, for $k = 1, 2, \ldots$.*

Proof. Let $f(X(x_i)) := F'(X(x_i))$, i.e. $F(X(x_i)) = \int^{X(x_i)} f(z)\,dz$. Such defined $F(G(\cdot))$ is also a quantized bouded random variable, so we have by Theorem 30 and Remark 20:

$$(F(G(x_k)))' =$$

$$= \frac{F(G(x_k)) - F(G(x_{k+1}))}{p(x_k \cap -x_0)} = \frac{\int^{G(x_k)} f(z)\,dz - \int^{G(x_{k+1})} f(z)\,dz}{p(x_k \cap -x_0)} =$$

$$= \frac{\sum_{i=k} f(G(x_i))(G(x_i) - G(x_{i+1})) - \sum_{i=k+1} f(G(x_i))(G(x_i) - G(x_{i+1}))}{p(x_k \cap -x_0)} =$$

$$= \frac{f(G(x_k))(G(x_k) - G(x_{k+1}))}{p(x_k \cap -x_0)} = F'(G(x_k)) \cdot G'(x_k), \quad \text{q.e.d. } ∎$$

4. Final conclusion

COROLLARY 32. *The formulas from Theorem* 30 *and* 31 *give us the possibility of calculating of a derivative in simple concrete physical situations in Bellert's space-time. We see that a derivative of a quantized random variable can be calculated in a simple way and a derivative of a superposition of functions can be calculated in the classical way.*

It is described in [7], how we can apply this mathematical theory in physics. When we look at [7], we can easily notice that the differential calculus for random variables on Heyting algebras, described here and in [8], has the same role in Bellert's theory as the classical differential calculus in classical physics.

COROLLARY 33. *Since we essentially use, in Theorems* 30 *and* 31 *which give a possibility of concrete calculations in physical problems in Bellert's theory, and in Lemma* 24, *Lemma* 28 *and Theorem* 29 *which lead to the two theorems mentioned, the assumption from Lemma* 24 *that* $x_0 < 1 = (S,T)$, *so we can obtain concrete results only for* x_k *which are situated at a certain concrete and finite space-time distance from the observer* (S,T). *It is because random variable* G *in our differential calculus can have a real physical meaning only for moments* x_k *such that* $x_k \leq x_1 < x_0 < 1 = (S,T)$. *For event* x_0 *which is less than* 1 *(so* x_0 *is not the event of observation) the value of random variable* G *is defined artifically and it has not a physical meaning. Therefore we cannot explore any quantized parameter in the vicinity of the observation, in Bellert's space-time. It is probably a kind of uncertainty connected with the quantum nature of the random variable and of the space-time.*

References

[1] R. Balbes, P. Dwinger; *Distributive Lattices*. University of Missouri Press 1974.

[2] S. Bellert; *On a new hypothesis concerning the red shift*, Astroph. Space Sci. 3 (1969), 268–282.

[3] S. Bellert; *On the cosmological red shift*, Astroph. Space Sci. 7 (1970), 211–230.

[4] S. Bellert; *Does the Speed of Light Decrease with time?*, Astroph. Space Sci. 47 (1977), 263–276.

[5] E. Rydzyńska; *Radial hyphothesis*, Astroph. Space Sci. 137 (1987), 183–187.

[6] E. Rydzyńska; *Random variables on Boolean and Heyting algebras and their numerical chracteristics*, Dissertationes Math. 247, PWN, Warszawa 1990.

[7] E. Rydzyńska; *Bellert's general relativity theory*, Astroph. Space Sci. 173 (1990), 241–249.

[8] E. Rydzyńska; *Differential calculus in probability Heyting algebras*, Demonstratio Math., 25 (1992) 887–904.
[9] E. Rydzyńska; *Bellert's Space-time as a Distributive Lattice with Pseudocomplementation. A Mathematical Apparatus of Bellert's Theory*, Physics Essays, submitted.

Author's address:
Francuska 34 m. 3
03-905 WARSZAWA, POLAND

Received February 2nd, 1992; revised version May 31, 1993.